

Research Article

A Complex Integrable Hierarchy and Its Hamiltonian Structure for Integrable Couplings of WKI Soliton Hierarchy

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We generate complex integrable couplings from zero curvature equations associated with matrix spectral problems in this paper. A direct application to the WKI spectral problem leads to a novel soliton equation hierarchy of integrable coupling system; then we consider the Hamiltonian structure of the integrable coupling system. We select the \bar{U} , \bar{V} and generate the nonlinear composite parts, which generate new extended WKI integrable couplings. It is also indicated that the method of block matrix is an efficient and straightforward way to construct the integrable coupling system.

1. Introduction

In what follows, we say that a system of partial differential equations (PDEs) or ordinary differential equations (ODEs) is integrable if it has infinitely many symmetries which are functionally independent on the jet spaces and Liouville integrable if it is Hamiltonian and possesses infinitely many conserved densities being functionally independent on the jet spaces. However, how to search for the new equation hierarchy and its Hamiltonian structure is an important work.

Integrable coupling system is an interesting and important topic in soliton theory [1–23]. Many coupled equations appear in practical problems, such as biochemistry, physics, and mechanics. For example, in order to systematically describe a kind of biochemistry model, Prigogine and Lefever proposed a coupled mathematical model in [1], which describes a biology-chemistry model; the well-known shallow water wave mathematics model [2]; and coupled KdV model [3], so that many integrable coupled mathematical models can solve practical problems [4].

A few ways to construct integrable couplings are presented by using perturbations [5, 6], enlarging spectral problems [7, 8], and creating new loop algebras [9, 10]. The study of integrable couplings not only generalizes the symmetry problem, but also provides clues towards complete classification of integrable systems. The first point has been clearly presented and analyzed carefully in [5, 6, 11, 12].

Professors Ma and Fuchssteiner first proposed perturbation method for establishing integrable couplings [5]. Professors Y. Zhang and H. Zhang presented the enlarged Lie algebra method to obtain integrable couplings [9]. On the other hand, the theory of integrable couplings brings other interesting results such as Lax pairs of block form and several spectral parameters [13, 14], integrable constrained flows with higher multiplicity [15], local bi-Hamiltonian structures in higher dimensions [11], and hereditary recursion operators of higher order [5, 16]. On the other hand, Ma et al. once used semidirect sums of Lie algebras to obtain the discrete integrable coupling system in [17].

The integrable coupling system is presented by using Virasoro symmetry algebra. In details, let $u_t = K(u)$ be a known integrable system; the following system $\{u_t = K(u), v_t = S(u, v)\}$ is called integrable coupling of the system $u_t = K(u)$, if $v_t = S(u, v)$ is also integrable and $S(u, v)$ contains explicitly u or u -derivatives with respect to x .

In recent years, it is becoming increasingly important to search for a new method to construct the integrable couplings of soliton equation hierarchy. As everyone knows that Tu method is a power tool to generate the soliton hierarchy and Hamiltonian functions. In this paper, we will use Ma's constructing method to produce new miraculous integrable couplings of WKI soliton hierarchy and its Hamiltonian structure with Lax pairs \bar{U} and \bar{V} [6].

2. The WKI Soliton Hierarchy and Its Hamiltonian Structure

The traditional spatial spectral problem for the WKI hierarchy is given with $g = \text{sl}(2, R)$ [19]:

$$\begin{aligned} \psi_x &= [U, \psi], & \lambda_t &= 0, \\ U &= U[u, \lambda] = \begin{pmatrix} -i\lambda & p\lambda \\ q\lambda & i\lambda \end{pmatrix}, & V &= \begin{pmatrix} -i\lambda a & \lambda pa + b_x \\ \lambda qa + c_x & i\lambda a \end{pmatrix}. \end{aligned} \tag{1}$$

We solve the stationary zero-curvature equation

$$V_x = [U, V] \tag{2}$$

and acquire the following system:

$$\begin{aligned} -ia_x &= pc_x - qb_x, \\ \lambda pa + b_x &= -2i\lambda b, \\ \lambda qa + c_x &= 2i\lambda c. \end{aligned} \tag{3}$$

Letting $a = \sum_{m=1}^{\infty} a_m \lambda^{-m}$, $b = \sum_{m=1}^{\infty} b_m \lambda^{-m}$, and $c = \sum_{m=1}^{\infty} c_m \lambda^{-m}$, then we can obtain the recursive equations:

$$\begin{aligned} -ia_{mx} &= pc_{mx} - qb_{mx}, \\ pa_{m+1} + b_{mx} &= -2ib_{m+1}, \\ qa_{m+1} + c_{mx} &= 2ic_{m+1}. \end{aligned} \tag{4}$$

According to system (4), a new system is given as follows:

$$\begin{aligned} a_{m+1} &= \frac{1}{2\sqrt{1-pq}} \partial^{-1} \left(\frac{q}{\sqrt{1-pq}} b_{mxx} + \frac{p}{\sqrt{1-pq}} c_{mxx} \right), \\ b_{m+1} &= \frac{ip}{2} a_{m+1} + \frac{i}{2} b_{mx}, \\ c_{m+1} &= -\frac{iq}{2} a_{m+1} - \frac{i}{2} c_{mx}, \end{aligned} \tag{5}$$

and the initial values satisfy the contents:

$$\begin{aligned} (2-2pq)a_{0x} - (pq)_x a_0 &= 0, & a_0 &= \frac{2}{\sqrt{pq}}, \\ b_0 &= \frac{ip}{\sqrt{pq}}, & c_0 &= \frac{-iq}{\sqrt{pq}}, \\ a_1 &= \frac{pq_x - p_x q}{2(1-pq)^{3/2}} i, & b_1 &= \frac{p_x}{2(1-pq)^{3/2}}, \\ c_1 &= \frac{-q_x}{2(1-pq)^{3/2}}. \end{aligned} \tag{6}$$

In order to obtain the Hamiltonian function of the WKI soliton hierarchy, we use the trace identity [8]:

$$\frac{\delta}{\delta u_i} \left\langle V, \frac{\partial U}{\partial \lambda} \right\rangle = \left(\lambda^{-r} \frac{\partial}{\partial \lambda} \lambda^r \right) \left\langle V, \frac{\partial U}{\partial u_i} \right\rangle. \tag{7}$$

Through the calculating, we have

$$\begin{aligned} \left\langle V, \frac{\partial U}{\partial \lambda} \right\rangle &= \text{tr} \left(V, \frac{\partial U}{\partial \lambda} \right) \\ &= -2\lambda a + 2\lambda pqa + qb_x + pc_x, \\ \left\langle V, \frac{\partial U}{\partial q} \right\rangle &= \text{tr} \left(V \frac{\partial U}{\partial q} \right) = \lambda^2 qa + \lambda c_x, \\ \left\langle V, \frac{\partial U}{\partial r} \right\rangle &= \lambda^2 pa + \lambda b_x. \end{aligned} \tag{8}$$

Therefore we obtain this hierarchy by using the trace identity (7):

$$\left(\begin{matrix} \delta \\ \delta p \\ \delta \\ \delta q \end{matrix} \right) (-ia_m + qb_m - pc_m) = (r-m) \begin{pmatrix} -c_m \\ b_m \end{pmatrix}. \tag{9}$$

When $m = 0$, $r = 1$ in (9), we obtain the Hamiltonian structure

$$\begin{pmatrix} -c_m \\ b_m \end{pmatrix} = H_m \begin{pmatrix} \delta \\ \delta p \\ \delta \\ \delta q \end{pmatrix}, \tag{10}$$

with

$$H_m = \frac{ia_m - qb_m + pc_m}{m-1}. \tag{11}$$

However $m = 1$; the H_1 is meaningless, so we define H_1 from (11):

$$H_1 = \frac{pq_x - p_x q}{2pq} \left(1 - \frac{1}{\sqrt{1-pq}} \right). \tag{12}$$

The soliton WKI equation hierarchy is given rise to as follows:

$$\begin{aligned} u_t &= \begin{pmatrix} p_t \\ q_t \end{pmatrix} = \begin{pmatrix} b_{mxx} \\ c_{mxx} \end{pmatrix} = J \begin{pmatrix} -c_m \\ b_m \end{pmatrix}, \\ J &= \begin{pmatrix} 0 & \partial^2 \\ -\partial^2 & 0 \end{pmatrix}, \end{aligned} \tag{13}$$

and the recursive operator L and recurrence relation are presented:

$$\begin{aligned} \begin{pmatrix} -c_{m+1} \\ b_{m+1} \end{pmatrix} &= L \begin{pmatrix} -c_m \\ b_m \end{pmatrix}, \\ L &= \begin{pmatrix} -\frac{iq}{4\sqrt{1-pq}}\partial^{-1} - \frac{p}{\sqrt{1-pq}}\partial^2 - \frac{i}{2}\partial & \frac{iq}{4\sqrt{1-pq}}\partial^{-1} - \frac{q}{\sqrt{1-pq}}\partial^2 \\ -\frac{ip}{4\sqrt{1-pq}}\partial^{-1} - \frac{p}{\sqrt{1-pq}}\partial^2 & \frac{ip}{4\sqrt{1-pq}}\partial^{-1} - \frac{q}{\sqrt{1-pq}}\partial^2 + \frac{i}{2}\partial \end{pmatrix}. \end{aligned} \tag{14}$$

Specially, we consider a case of the WKI equation and its Hamiltonian function:

$$u_t = J \begin{pmatrix} -c_m \\ b_m \end{pmatrix} = JL^m \begin{pmatrix} \frac{iq}{\sqrt{pq}} \\ \frac{ip}{\sqrt{pq}} \end{pmatrix} = J \frac{\delta}{\delta u_i} H_m. \tag{15}$$

3. The Integrable Couplings and Its Hamiltonian Structure

Integrable coupling system is an interesting and important topic in soliton theory. The theory of integrable couplings

brings other interesting results such as Lax pairs of block form and several spectral parameters, integrable constrained flows with higher multiplicity, local bi-Hamiltonian structures in higher dimensions, and hereditary recursion operators of higher order.

Based on the theory of Ma's design of the integrable couplings, a new integrable coupling of WKI soliton hierarchy and its Hamiltonian structure are obtained. We select \bar{U} and \bar{V} as follows:

$$\begin{aligned} \bar{U}(\bar{u}, \lambda) &= \begin{pmatrix} -2i\lambda & p\lambda & 0 & r\lambda & 0 \\ 2q\lambda & 0 & 2p\lambda & s\lambda & r\lambda \\ 0 & q\lambda & 2i\lambda & 0 & s\lambda \\ 0 & 0 & 0 & -i\lambda & p\lambda \\ 0 & 0 & 0 & q\lambda & i\lambda \end{pmatrix}, \\ \bar{u} &= \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}, \end{aligned} \tag{16}$$

$$\bar{V} = \begin{pmatrix} -2i\lambda a & p\lambda a + b_x & 0 & f\lambda & 0 \\ 2\lambda q a + 2c_x & 0 & 2p\lambda a + 2b_x & g\lambda & f\lambda \\ 0 & q\lambda a + c_x & 2i\lambda a & 0 & g\lambda \\ 0 & 0 & 0 & -2i\lambda a & p\lambda a + b_x \\ 0 & 0 & 0 & q\lambda a + c_x & i\lambda a \end{pmatrix}.$$

From the stationary zero curvature equation

$$\bar{V}_x = [\bar{U}, \bar{V}], \tag{17}$$

it gives rise to

$$\begin{aligned} -ia_x &= pc_x - qb_x, \\ \lambda pa + b_x &= -2i\lambda b, \\ \lambda qa + c_x &= 2i\lambda c, \end{aligned}$$

$$\begin{aligned} f_x &= -ir\lambda a - ps\lambda a - sb_x - i\lambda f + p\lambda g, \\ g_x &= is\lambda a - qr\lambda a - rc_x + q\lambda f + i\lambda g. \end{aligned} \tag{18}$$

Setting

$$\begin{aligned} a &= \sum_{m=1}^{\infty} a_m \lambda^{-m}, \quad b = \sum_{m=1}^{\infty} b_m \lambda^{-m}, \quad c = \sum_{m=1}^{\infty} c_m \lambda^{-m}, \\ f &= \sum_{m=1}^{\infty} f_m \lambda^{-m}, \quad g = \sum_{m=1}^{\infty} g_m \lambda^{-m} \end{aligned} \tag{19}$$

by using of the stationary zero curvature equation (17), we obtain the extended recurrence relations

$$\begin{aligned} -ia_{mx} &= pc_{mx} - qb_{mx}, \\ pa_{m+1} + b_{mx} &= -2ib_{m+1}, \\ qa_{m+1} + c_{mx} &= 2ic_{m+1}, \end{aligned} \tag{20}$$

$$\begin{aligned} f_{mx} &= -ira_{m+1} - psa_{m+1} - sb_{mx} - if_{m+1} + pg_{m+1}, \\ g_{mx} &= isa_{m+1} - qra_{m+1} - rc_{mx} + qf_{m+1} + ig_{m+1}, \end{aligned}$$

or

$$\begin{aligned} a_{m+1} &= \frac{1}{2\sqrt{1-pq}} \partial^{-1} \left(\frac{q}{\sqrt{1-pq}} b_{mxx} + \frac{p}{\sqrt{1-pq}} c_{mxx} \right), \\ b_{m+1} &= \frac{ip}{2} a_{m+1} + \frac{i}{2} b_{mx}, \\ c_{m+1} &= -\frac{iq}{2} a_{m+1} - \frac{i}{2} c_{mx}, \\ f_{m+1} &= -ra_{m+1} + ipsa_{m+1} + isb_{mx} + if_{mx} - ipg_{m+1}, \\ g_{m+1} &= -ig_{mx} - sa_{m+1} - iqra_{m+1} - irc_{mx} + iqf_{m+1}. \end{aligned} \tag{21}$$

And the initial value is given as follows:

$$\begin{aligned} (2-2pq)a_{0x} - (pq)_x a_0 &= 0, \\ a_0 &= \frac{2}{\sqrt{pq}}, \quad b_0 = \frac{ip}{\sqrt{pq}}, \\ c_0 &= \frac{-iq}{\sqrt{pq}}, \quad f_0 = \frac{4ips - 2r - 2pqr}{(1-pq)^{3/2}}, \\ g_0 &= \frac{-4iqr - 2s - 2pqs}{(1-pq)^{3/2}}, \quad a_1 = \frac{pq_x - P_x q}{2(1-pq)^{3/2}} i, \\ b_1 &= \frac{P_x}{2(1-pq)^{3/2}}, \quad c_1 = \frac{-q_x}{2(1-pq)^{3/2}}, \\ f_1 &= (-10ipp_x q^2 r + 3pp_x qs - 10p^2 p_x q^2 s + 11ip_x q r \\ &\quad - 2ip^2 qq_x r + 9p^2 q_x s - 10p^3 q^3 q_x s - ipq_x r \\ &\quad - 8ipq_x r_x - 4ps + 4p^2 qs_x + 10p_x s + 4ir_x \\ &\quad + 4ip^2 q^2 r_x + 4p^2 qs + 4p^3 q^2 s_x) \frac{1}{2(1-pq)^{5/2}}, \\ g_1 &= (p_x q^2 r + 2ip_x qs + ipp_x q^2 s - 3p^2 p_x q^2 r - 6pq_x s \\ &\quad + 4ip^2 qq_x s - p^2 q^2 q_x r + 6q_x r + 4ipq_s x \\ &\quad + 4pq^2 r_x + 2qr_x - is_x - 2ip^2 q^2 s_x \\ &\quad + 2pp_x q^3 r + 2p^2 q^3 r_x) \frac{1}{2(1-pq)^{5/2}}. \end{aligned} \tag{22}$$

The Hamiltonian structures of the integrable couplings are obtained by using the quadratic identity. In comparison with the previous theoretical investigation, we regard our investigation as the generalization of the integrable couplings and Hamiltonian structures of the WKI hierarchy in soliton theory. Then we use the variational identity [6–10], which can generate the coupling Hamiltonian structure. The variational identity is considered as follows [6]:

$$\begin{aligned} \frac{\delta}{\delta u_i} \int \left\langle \bar{V}, \frac{\partial \bar{U}}{\partial \lambda} \right\rangle d_x &= \left(\lambda^{-r} \frac{\partial}{\partial \lambda} \lambda^r \right) \left\langle \bar{V}, \frac{\partial \bar{U}}{\partial u_i} \right\rangle, \\ r &= \frac{\lambda}{2} \frac{d}{d\lambda} \ln \left| \text{tr} \left(\bar{V}^2 \right) \right|, \end{aligned} \tag{23}$$

where the operation relations are defined by

$$\begin{aligned} M(a, b, c, f, g) &= \begin{pmatrix} 2a & b & 0 & f & g \\ 2c & 0 & 2b & g & f \\ 0 & c & -2a & 0 & g \\ 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & c & -a \end{pmatrix}, \\ M(a', b', c', f', g') &= \begin{pmatrix} 2a' & b' & 0 & f' & g' \\ 2c' & 0 & 2b' & g' & f' \\ 0 & c' & -2a' & 0 & g' \\ 0 & 0 & 0 & a' & b' \\ 0 & 0 & 0 & c' & -a' \end{pmatrix}, \end{aligned} \tag{24}$$

and the \langle, \rangle is defined as follows:

$$\begin{aligned} \langle M(a, b, c, f, g), M(a', b', c', f', g') \rangle &= (a, b, c, f, g) F(a', b', c', f', g')^T \\ &= \eta_1 \left(aa' + \frac{1}{2} bc' + \frac{1}{2} cb' \right) \\ &\quad + \eta_2 (bg' - cf' - fc' + gb'), \end{aligned} \tag{25}$$

and the transformation matrix is presented:

$$F = \begin{pmatrix} \eta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\eta_1 & 0 & \eta_2 \\ 0 & \frac{1}{2}\eta_1 & 0 & -\eta_2 & 0 \\ 0 & 0 & -\eta_2 & 0 & 0 \\ 0 & \eta_2 & 0 & 0 & 0 \end{pmatrix}. \tag{26}$$

According to (24), (25), and (26), we obtain the equations

$$\begin{aligned} \left\langle \bar{V}, \frac{\partial \bar{U}}{\partial \lambda} \right\rangle &= \eta_1 (-\lambda a - iq\lambda b + ip\lambda c) \\ &\quad + \eta_2 (p\lambda g - q\lambda f - 2is\lambda b - 2ir\lambda c), \\ \left\langle \bar{V}, \frac{\partial \bar{U}}{\partial p} \right\rangle &= \eta_1 (i\lambda^2 c) + \eta_2 (\lambda^2 g), \\ \left\langle \bar{V}, \frac{\partial \bar{U}}{\partial q} \right\rangle &= -\eta_1 (i\lambda^2 b) - \eta_2 (\lambda^2 f), \\ \left\langle \bar{V}, \frac{\partial \bar{U}}{\partial r} \right\rangle &= -\eta_2 (2i\lambda^2 c), \\ \left\langle \bar{V}, \frac{\partial \bar{U}}{\partial s} \right\rangle &= -\eta_2 (2i\lambda^2 b). \end{aligned} \tag{27}$$

Based on the variational identity, we obtain the following system:

$$\begin{aligned} &\begin{pmatrix} \frac{\delta}{\delta p} \\ \frac{\delta}{\delta q} \\ \frac{\delta}{\delta r} \\ \frac{\delta}{\delta s} \end{pmatrix} \int [\eta_1 (-a_m - iq b_m + ip c_m) \\ &\quad + \eta_2 (p g_m - q f_m - 2is b_m - 2ir c_m)] d_x \\ &= (r - m + 1) \begin{pmatrix} \eta_1 (i c_m) + \eta_2 g_m \\ -\eta_1 (i b_m) - \eta_2 f_m \\ -\eta_2 (2i c_m) \\ -\eta_2 (2i b_m) \end{pmatrix}, \end{aligned} \tag{28}$$

in which we calculate r by

$$\begin{aligned} r &= \frac{\lambda}{2} \frac{d}{d\lambda} \ln |\text{tr}(\bar{V}^2)| \\ &= \frac{\eta_1 (-\lambda_2 a_2 + 4\lambda^2 bc) + \eta_2 (-4i\lambda gb - 4i\lambda fc)}{\eta_1 (-\lambda_2 a_2 + 4\lambda^2 bc) + \eta_2 (-4i\lambda gb - 4i\lambda fc)} \\ &= 1. \end{aligned} \tag{29}$$

The Hamiltonian function of coupling WKI equation hierarchy is presented:

$$\begin{aligned} \frac{\delta}{\delta u} \bar{H}_m &= \begin{pmatrix} i\eta_1 c_m + \eta_2 g_m \\ -i\eta_1 b_m - \eta_2 f_m \\ -2i\eta_2 c_m \\ -2i\eta_2 b_m \end{pmatrix}, \\ \bar{H}_m &= \int [(\eta_1 (-a_m - iq b_m + ip c_m) \\ &\quad + \eta_2 (p g_m - q f_m - 2is b_m - 2ir c_m)) \\ &\quad \times (2 - m)^{-1}] d_x. \end{aligned} \tag{30}$$

Based on the Tu method, it gives rise to the integrable couplings of WKI equation hierarchy:

$$\begin{aligned} \bar{U}_t &= \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}_t = \begin{pmatrix} b_{mxx} \\ c_{mxx} \\ f_{mx} + sb_{mx} \\ g_{mx} + rc_{mx} \end{pmatrix} = \bar{J} \begin{pmatrix} -c_m \\ b_m \\ -g_m \\ f_m \end{pmatrix}, \\ \bar{J} &= \begin{pmatrix} 0 & \partial^2 & 0 & 0 \\ -\partial^2 & 0 & 0 & 0 \\ 0 & s\partial & 0 & \partial \\ -r\partial & 0 & -\partial & 0 \end{pmatrix}, \\ \bar{U}_t &= \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}_t = \bar{J} \begin{pmatrix} -c_m \\ b_m \\ -g_m \\ f_m \end{pmatrix} = \bar{J} M \frac{\delta}{\delta u} \bar{H}_m, \end{aligned} \tag{31}$$

with

$$\begin{aligned} M \frac{\delta}{\delta u} \bar{H}_m &= \begin{pmatrix} -c_m \\ b_m \\ -g_m \\ f_m \end{pmatrix}, \\ M &= \begin{pmatrix} 0 & 0 & -\frac{i}{2\eta_2} & 0 \\ 0 & 0 & 0 & \frac{i}{2\eta_2} \\ -\frac{1}{\eta_2} & 0 & -\frac{\eta_1}{2\eta_2} & 0 \\ 0 & -\frac{1}{\eta_2} & 0 & \frac{\eta_1}{2\eta_2} \end{pmatrix}, \end{aligned} \tag{32}$$

where \bar{J} is Hamiltonian operator.

According to (20), the recursive operator \bar{L} and the recursive relation are presented as follows:

$$\begin{aligned} \begin{pmatrix} -c_{m+1} \\ b_{m+1} \\ -g_{m+1} \\ f_{m+1} \end{pmatrix} &= \bar{L} \begin{pmatrix} -c_m \\ b_m \\ -g_m \\ f_m \end{pmatrix}, \\ \bar{L} &= \begin{pmatrix} \bar{L}_{11} & \bar{L}_{12} & 0 & 0 \\ \bar{L}_{21} & \bar{L}_{22} & 0 & 0 \\ \bar{L}_{31} & \bar{L}_{32} & -\frac{i\partial}{1-pq} & \frac{q\partial}{1-pq} \\ \bar{L}_{41} & \bar{L}_{42} & \frac{p\partial}{1-pq} & \frac{i\partial}{1-pq} \end{pmatrix}, \end{aligned} \tag{33}$$

with

$$\begin{aligned} \bar{L}_{11} &= -\frac{iq}{4\sqrt{1-pq}} \partial^{-1} \frac{p}{\sqrt{1-pq}} \partial^2 - \frac{i}{2} \partial, \\ \bar{L}_{12} &= \frac{iq}{4\sqrt{1-pq}} \partial^{-1} \frac{q}{\sqrt{1-pq}} \partial^2, \end{aligned}$$

$$\begin{aligned}
\overline{L}_{21} &= -\frac{ip}{4\sqrt{1-pq}}\partial^{-1}\frac{p}{\sqrt{1-pq}}\partial^2, \\
\overline{L}_{22} &= \frac{ip}{4\sqrt{1-pq}}\partial^{-1}\frac{q}{\sqrt{1-pq}}\partial^2 + \frac{i}{2}\partial, \\
\overline{L}_{31} &= \frac{1}{1-pq}\left[(-s-2iqr-pqs)\frac{1}{2\sqrt{1-pq}}\partial^{-1}\right. \\
&\quad \left.\times\frac{p}{\sqrt{1-pq}}\partial^2 - ir\partial\right], \\
\overline{L}_{32} &= \frac{1}{1-pq}\left[(s+2iqr+pqs)\frac{1}{2\sqrt{1-pq}}\partial^{-1}\right. \\
&\quad \left.\times\frac{q}{\sqrt{1-pq}}\partial^2 + qs\partial\right], \\
\overline{L}_{41} &= \frac{1}{1-pq}\left[(r-2ips+pqr)\frac{1}{2\sqrt{1-pq}}\partial^{-1}\right. \\
&\quad \left.\times\frac{p}{\sqrt{1-pq}}\partial^2 + pr\partial\right], \\
\overline{L}_{42} &= \frac{1}{1-pq}\left[(-r+2iqs-pqr)\frac{1}{2\sqrt{1-pq}}\partial^{-1}\right. \\
&\quad \left.\times\frac{q}{\sqrt{1-pq}}\partial^2 + is\partial\right].
\end{aligned} \tag{34}$$

A complex integrable coupling and its Hamiltonian structure of WKI hierarchy are generated as follows:

$$\begin{aligned}
\overline{U}_t &= \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}_t = \overline{J} \begin{pmatrix} -c_m \\ b_m \\ -g_m \\ f_m \end{pmatrix} = \overline{J}\overline{L} \begin{pmatrix} -c_{m-1} \\ b_{m-1} \\ -g_{m-1} \\ f_{m-1} \end{pmatrix} \\
&= \overline{J}\overline{L}^m \begin{pmatrix} -c_0 \\ b_0 \\ -g_0 \\ f_0 \end{pmatrix} = \overline{J}\overline{L}^m \begin{pmatrix} \frac{iq}{\sqrt{pq}} \\ \frac{ip}{\sqrt{pq}} \\ \frac{4iqr+2s+2pqs}{(1-pq)^{3/2}} \\ \frac{4ips-2r-2pqr}{(1-pq)^{3/2}} \end{pmatrix}.
\end{aligned} \tag{35}$$

The hierarchy (35) is derived from the zero-curvature equation; therefore it is integrable in the Lax sense. Again taking $r = s = 0$, \overline{J} and \overline{L} are reduced to J and L , respectively. According to the definition of integrable coupling, we find that (35) is the integrable coupling system of the WKI equation hierarchy (15).

The WKI integrable equation hierarchy with variable spectral parameters is presented. Then, a technically feasible way to construct integrable couplings of soliton equations has been presented by using the Lie algebras of block matrix,

and the Hamiltonian structures of the integrable coupling of WKI hierarchy are obtained by using the variational identity. A kind of higher-dimensional Lie algebras is given, which can be used to directly construct integrable couplings of the soliton integrable systems. We have derived a new integrable coupling and its Hamiltonian structure of the WKI equation hierarchy.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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