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Research Article

On Two-Dimensional Quaternion Wigner-Ville Distribution

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We present the two-dimensional quaternion Wigner-Ville distribution (QWVD). The transform is constructed by substituting the Fourier transform kernel with the quaternion Fourier transform (QFT) kernel in the classical Wigner-Ville distribution definition. Based on the properties of quaternions and the QFT kernel we obtain three types of the QWVD. We discuss some useful properties of various definitions for the QWVD, which are extensions of the classical Wigner-Ville distribution properties.

1. Introduction

The classical Wigner-Ville distribution (WVD) or Wigner-Ville transform (WVT) is an important tool in the time-frequency signal analysis. It was first introduced by Eugene Wigner in his calculation of the quantum corrections of classical statistical mechanics. It was independently derived again by J. Ville in 1948 as a quadratic representation of the local time-frequency energy of a signal. In [1–3], the authors introduced the WVT and established some important properties of the WVT. The transform is then extended to the linear canonical transform (LCT) domain by replacing the kernel of the classical Fourier transform (FT) with the kernel of the LCT in the WVD domain [4].

As a generalization of the real and complex Fourier transform (FT), the quaternion Fourier transform (QFT) has been of interest to researchers for some years. A number of useful properties of the QFT have been found including shift, modulation, convolution, correlation, differentiation, energy conservation, uncertainty principle, and so on. Due to the noncommutative property of quaternion multiplication, there are three different types of two-dimensional QFTs. These three QFTs are so-called a left-sided QFT, a right-sided QFT, and a two-sided QFT, respectively (see, e.g., [5–9]). In [10, 11], special properties of the asymptotic behaviour of the right-sided QFT are discussed and generalization of the classical Bohner-Millos theorems to the framework of quaternion analysis is established. Many generalized transforms are closely related to the QFTs, for example, the quaternion wavelet transform, fractional quaternion Fourier transform, quaternion linear canonical transform, and quaternionic windowed Fourier transform [12–18]. Based on the QFTs, one also may extend the WVD to the quaternion algebra while enjoying similar properties as in the classical case.

Therefore, the main purpose of this paper is to propose a generalization of the classical WVD to quaternion algebra, which we call the *quaternion Wigner-Ville distribution (QWVD)*. Our generalization is constructed by substituting the kernel of the FT with the kernel of the QFT in the classical WVT definition. Due to the non-commutative rule of quaternions and the QFT kernel we obtain the definition of different types of the QWVD. We then derive some important properties of the QWVD such as shift, reconstruction formula, modulation, and orthogonality relation in detail. We present an example to show the difference between the QWVD and the WVD.

The organization of the paper is as follows. The remainder of this section introduces some notations and briefly recalls some general definitions and basic properties of quaternion algebra and quaternion Fourier transform. In Section 3, we provide the basic ideas for the construction of the two-sided QWVD and derive several its important properties using the two-sided QFT. The construction of the right-sided QWVD is provided in Section 4. In Section 5 we introduce quaternion ambiguity function (QAF); its important properties are also discussed in this section.

2. Basics

2.1. Quaternion Algebra. The quaternion algebra was formally introduced by Hamilton in 1843, and it is a generalization of complex numbers. The quaternion algebra over $\mathbb{R}$, denoted by $\mathbb{H}$, is an associative non-commutative four-dimensional algebra:

$$\mathbb{H} = \{q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \mid q_0, q_1, q_2, q_3 \in \mathbb{R}\}, \quad (1)$$

which obey Hamilton's multiplication rules

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \qquad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \qquad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \\ \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \quad (2)$$

The quaternion conjugate of a quaternion q is given by

$$\bar{q} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3, \quad q_0, q_1, q_2, q_3 \in \mathbb{R}, \quad (3)$$

and it is an anti-involution; that is,

$$\overline{qp} = \bar{p}\,\bar{q}. \quad (4)$$

From (3) we obtain the norm of $q \in \mathbb{H}$ defined as

$$|q| = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}. \quad (5)$$

It is not difficult to see that

$$|qp| = |q|\,|p|, \quad \forall p, q \in \mathbb{H}. \quad (6)$$

Using the conjugate (3) and the modulus of q, we can define the inverse of $q \in \mathbb{H} \setminus \{0\}$ as

$$q^{-1} = \frac{\bar{q}}{|q|^2} \quad (7)$$

which shows that $\mathbb{H}$ is a normed division algebra.

We use parenthesis $(,)$ to denote the inner product of two quaternion functions, $f, g : \mathbb{R}^2 \to \mathbb{H}$, as follows:

$$(f, g)_{L^2(\mathbb{R}^2;\mathbb{H})} = \int_{\mathbb{R}^2} f(\mathbf{x})\,\overline{g(\mathbf{x})}\,d^2\mathbf{x}. \quad (8)$$

When $f = g$ we obtain the associated norm

$$\|f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})} = \int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d^2\mathbf{x}. \quad (9)$$

As a consequence of the inner product (8) we obtain the *quaternion Cauchy-Schwarz* inequality

$$\left|\int_{\mathbb{R}^2} f\bar{g}\,d^2\mathbf{x}\right| \le \left(\int_{\mathbb{R}^2} |f|^2 d^2\mathbf{x}\right)^{1/2} \left(\int_{\mathbb{R}^2} |g|^2 d^2\mathbf{x}\right)^{1/2}, \\ \forall f, g \in L^2(\mathbb{R}^2; \mathbb{H}). \quad (10)$$

2.2. Quaternion Fourier Transform (QFT). The quaternion (or hypercomplex) Fourier transform is defined similar to the classical FT of the 2D functions. The noncommutative property of quaternion multiplication allows us to have three different definitions of the QFT. In the following we briefly introduce the two-sided QFT and the right-sided QFT. For more details we refer the reader to [7, 8, 19].

Definition 1 (two-sided QFT). The two-sided QFT of $f \in L^2(\mathbb{R}^2; \mathbb{H})$ is the transform $\mathcal{F}_q^D\{f\} \in L^2(\mathbb{R}^2, \mathbb{H})$ given by the integral

$$\mathcal{F}_q^D\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1 x_1} f(\mathbf{x})\, e^{-\mathbf{j}\omega_2 x_2} d^2\mathbf{x}, \quad (11)$$

where $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$, $\boldsymbol{\omega} = \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2$, and the quaternion exponential product $e^{-\mathbf{i}\omega_1 x_1} e^{-\mathbf{j}\omega_2 x_2}$ is called the quaternion Fourier kernel.

Theorem 2 (inverse two-sided QFT). *Suppose that $f \in L^2(\mathbb{R}^2; \mathbb{H})$ and $\mathcal{F}_q^D\{f\} \in L^1(\mathbb{R}^2; \mathbb{H})$. Then the two-sided QFT of f is an invertible transform and its inverse is given by*

$$f(\mathbf{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{\mathbf{i}\omega_1 x_1} \mathcal{F}_q^D\{f\}(\boldsymbol{\omega})\, e^{\mathbf{j}\omega_2 x_2} d^2\boldsymbol{\omega}, \quad (12)$$

where the quaternion exponential product $e^{\mathbf{j}\omega_2 x_2} e^{\mathbf{i}\omega_1 x_1}$ is called the inverse two-sided quaternion Fourier kernel.

Notice that if $\mathcal{F}_q^D\{f\}(\boldsymbol{\omega}) = 1$ from the inverse of the two-sided QFT we obtain quaternion Dirac's delta function; that is,

$$\delta(\mathbf{x}) = \delta(x_1)\,\delta(x_2) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{\mathbf{i}\omega_1 x_1} e^{\mathbf{j}\omega_2 x_2} d^2\boldsymbol{\omega}. \quad (13)$$

We have known that Parseval's formula is not valid for the two-sided QFT, but a special case of Parseval's formula called the Plancherel formula remains valid; that is,

$$\|f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})} = \frac{1}{(2\pi)^2} \left\|\mathcal{F}_q^D\{f\}\right\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}. \quad (14)$$

We introduce Parseval's formula for the right sided QFT as

$$(f, g)_{L^2(\mathbb{R}^2;\mathbb{H})} = \frac{1}{(2\pi)^2} \left(\mathcal{F}_q^R\{f\}, \mathcal{F}_q^R\{g\}\right)_{L^2(\mathbb{R}^2;\mathbb{H})}, \quad (15)$$

where the right-sided QFT is defined by for every $f \in L^2(\mathbb{R}^2; \mathbb{H})$

$$\mathcal{F}_q^R\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} f(\mathbf{x})\, e^{-\mathbf{i}\omega_1 x_1} e^{-\mathbf{j}\omega_2 x_2} d^2\mathbf{x}. \quad (16)$$

2.3. Fundamental Operators. Before we discuss the QWVD we need to introduce some notations, which will be used in the next section. For a quaternion function $f \in L^2(\mathbb{R}^2; \mathbb{H})$ we define the translation, modulation, and dilation as follows:

$$T_{\mathbf{a}} f(\mathbf{x}) = f(\mathbf{x} - \mathbf{a}), \qquad M_{\boldsymbol{\omega}_0} f(\mathbf{x}) = e^{\mathbf{i}u_0 x_1} f(\mathbf{x})\, e^{\mathbf{j}v_0 x_2}, \\ \mathcal{D}_c f(\mathbf{x}) = \frac{1}{c} f\left(\frac{\mathbf{x}}{c}\right), \quad (17)$$

where $\boldsymbol{\omega}_0 = u_0\mathbf{e}_1 + v_0\mathbf{e}_2$ and $c \in \mathbb{R}^+$. The composition of the translation and modulation is called the time-frequency shift; that is,

$$M_{\boldsymbol{\omega}_0} T_{\mathbf{a}} f(\mathbf{x}) = e^{\mathbf{i}u_0 x_1} f(\mathbf{x}-\mathbf{a}) e^{\mathbf{j}v_0 x_2}, \quad \mathbf{a} \in \mathbb{R}^2. \tag{18}$$

Just as the classical case (see [3]), we obtain the canonical commutation relations

$$T_{\mathbf{a}} M_{\boldsymbol{\omega}_0} f = e^{-\mathbf{i}u_0 a_1} M_{\boldsymbol{\omega}_0} T_{\mathbf{a}} f e^{-\mathbf{j}v_0 a_2}. \tag{19}$$

3. Quaternion Wigner-Ville Distribution (QWVD)

In this section, we introduce the 2D quaternion Wigner-Ville distribution (QWVD). We investigate several basic properties of the QWVD which are important for signal representation in signal processing.

3.1. Definition of 2-D QWVD. Based on the properties of quaternions and the definition of the classical Wigner-Ville transform associated with the Fourier transform, we obtain a definition of the QWVD by replacing the kernel of the FT with the kernel of the two-sided QFT in the classical WVD definition as follows.

Definition 3. The cross two-sided quaternion Wigner-Ville distribution of two-dimensional functions (or signals) $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$ is given by

$$\mathscr{W}_{f,g}^{D}(\mathbf{t}, \boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\tau_1} f\left(\mathbf{t} + \frac{\boldsymbol{\tau}}{2}\right) \overline{g}\left(\mathbf{t} - \frac{\boldsymbol{\tau}}{2}\right) e^{-\mathbf{j}\omega_2\tau_2} d^2\boldsymbol{\tau}, \tag{20}$$

provided the integral exists.

It should be remembered that the kernel of the cross two-sided QWVD in (20) does not commute with quaternion functions f and g so that several properties of the WVD are not valid in the cross two-sided QWVD.

By making the change of variables $\mathbf{t} + (\boldsymbol{\tau}/2) = \mathbf{x}$, (20) can be written in the form

$$\begin{aligned} &\mathscr{W}_{f,g}^{D}(\mathbf{t}, \boldsymbol{\omega}) \\ &= 4\int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1(2x_1-2t_1)} f(\mathbf{x}) \overline{g}(2\mathbf{t}-\mathbf{x}) e^{-\mathbf{j}\omega_2(2x_2-2t_2)} d^2\mathbf{x} \\ &= 4e^{\mathbf{i}2\omega_1 t_1} \int_{\mathbb{R}^2} e^{-\mathbf{i}2\omega_1 x_1} f(\mathbf{x}) \overline{g}(2\mathbf{t}-\mathbf{x}) e^{-\mathbf{j}2\omega_2 x_2} d^2\mathbf{x} e^{\mathbf{j}2\omega_2 t_2} \\ &= 4e^{\mathbf{i}2\omega_1 t_1} \int_{\mathbb{R}^2} e^{-\mathbf{i}2\omega_1 x_1} f(\mathbf{x}) \overline{g}(-(\mathbf{x}-2\mathbf{t})) e^{-\mathbf{j}2\omega_2 x_2} d^2\mathbf{x} e^{\mathbf{j}2\omega_2 t_2}. \end{aligned} \tag{21}$$

We see that the above expression gives an equivalent definition of $\mathscr{W}_{f,g}^{D}(\mathbf{t}, \boldsymbol{\omega})$ in the form

$$\mathscr{W}_{f,g}^{D}(\mathbf{t}, \boldsymbol{\omega}) = 4e^{\mathbf{i}2\omega_1 t_1} Q_{Ig} f(2\mathbf{t}, 2\boldsymbol{\omega}) e^{\mathbf{j}2\omega_2 t_2}, \tag{22}$$

where $Ig(\mathbf{x}) = g(-\mathbf{x})$. Here $Q_{Ig} f(2\mathbf{t}, 2\boldsymbol{\omega})$ is the windowed quaternionic Fourier transform which was recently proposed by Fu et al. [13]. Furthermore, if we write $R_t(\boldsymbol{\tau}) = f(\mathbf{t} + (\boldsymbol{\tau}/2))\overline{g}(\mathbf{t} - (\boldsymbol{\tau}/2))$, we immediately obtain

$$\begin{aligned} \mathscr{W}_{f,g}^{D}(\mathbf{t}, \boldsymbol{\omega}) &= \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\tau_1} R_t(\boldsymbol{\tau}) e^{-\mathbf{j}\omega_2\tau_2} d^2\boldsymbol{\tau} \\ &= \mathscr{F}_q^{D}\{R_t\}(\boldsymbol{\omega}), \end{aligned} \tag{23}$$

which tells us that the cross two-sided QWVD is in fact the two-sided QFT of the function $R_t(\boldsymbol{\tau}) = f(\mathbf{t}+(\boldsymbol{\tau}/2))\overline{g}(\mathbf{t}-(\boldsymbol{\tau}/2))$ with respect to $\boldsymbol{\tau}$. This fact is very important in proving Moyal's formula for the two-sided QWVD.

Lemma 4. *Let $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$ be two quaternion-valued functions. Then $\mathscr{W}_{f,g}^{D}(\mathbf{t}, \boldsymbol{\omega})$ is bounded on $L^2(\mathbb{R}^2; \mathbb{H})$; that is,*

$$\left|\mathscr{W}_{f,g}^{D}(\mathbf{t}, \boldsymbol{\omega})\right| \leq 4\|f\|_{L^2(\mathbb{R}^2;\mathbb{H})}\|g\|_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{24}$$

In particular, if $f = g$, the above expression reduces to

$$\left|\mathscr{W}_{f}^{D}(\mathbf{t}, \boldsymbol{\omega})\right| \leq 4\mathscr{W}_{f}^{D}(\mathbf{0}, \mathbf{0}). \tag{25}$$

Proof. By application of the quaternion Cauchy-Schwarz inequality (10) we easily obtain

$$\begin{aligned} \left|\mathscr{W}_{f,g}^{D}(\mathbf{t}, \boldsymbol{\omega})\right|^2 &= \left|\int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\tau_1} f\left(\mathbf{t} + \frac{\boldsymbol{\tau}}{2}\right) \overline{g}\left(\mathbf{t} - \frac{\boldsymbol{\tau}}{2}\right) e^{-\mathbf{j}\omega_2\tau_2} d^2\boldsymbol{\tau}\right|^2 \\ &\leq \int_{\mathbb{R}^2} \left|e^{-\mathbf{i}\omega_1\tau_1} f\left(\mathbf{t} + \frac{\boldsymbol{\tau}}{2}\right) \overline{g}\left(\mathbf{t} - \frac{\boldsymbol{\tau}}{2}\right) e^{-\mathbf{j}\omega_2\tau_2}\right|^2 d^2\boldsymbol{\tau} \\ &= \int_{\mathbb{R}^2} \left|f\left(\mathbf{t} + \frac{\boldsymbol{\tau}}{2}\right) \overline{g}\left(\mathbf{t} - \frac{\boldsymbol{\tau}}{2}\right)\right|^2 d^2\boldsymbol{\tau} \\ &\leq \int_{\mathbb{R}^2} \left|f\left(\mathbf{t} + \frac{\boldsymbol{\tau}}{2}\right)\right|^2 d^2\boldsymbol{\tau} \int_{\mathbb{R}^2} \left|\overline{g}\left(\mathbf{t} - \frac{\boldsymbol{\tau}}{2}\right)\right|^2 d^2\boldsymbol{\tau} \\ &= 16 \int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d^2\mathbf{x} \int_{\mathbb{R}^2} |\overline{g}(\mathbf{y})|^2 d^2\mathbf{y}. \end{aligned} \tag{26}$$

It means that for all $\mathbf{t}$ and $\boldsymbol{\omega}$ we have

$$\left|\mathscr{W}_{f,g}^{D}(\mathbf{t}, \boldsymbol{\omega})\right| \leq 4\|f\|_{L^2(\mathbb{R}^2;\mathbb{H})}\|g\|_{L^2(\mathbb{R}^2;\mathbb{H})}, \tag{27}$$

which was to be proved. □

The following theorem shows that the cross two-sided QWVD is invertible; that is, the original quaternion signal f can be uniquely determined in terms of its cross two-sided QWVD within a constant factor.

Theorem 5 (reconstruction formula for the two-sided QWVD). *The inverse transform of the cross two-sided QWVD of the signal $f \in L^2(\mathbb{R}^2; \mathbb{H})$ is given by*

$$f(t) = \frac{1}{(2\pi)^2 \overline{g}(\mathbf{0})} \int_{\mathbb{R}^2} e^{\mathbf{i}\omega_1 t_1} \mathscr{W}_{f,g}^{D}\left(\frac{\mathbf{t}}{2}, \boldsymbol{\omega}\right) e^{\mathbf{j}\omega_2 t_2} d^2\boldsymbol{\omega}, \tag{28}$$

provided $\overline{g}(\mathbf{0}) \neq \mathbf{0}$.

Proof. From the definition of the cross two-sided QWVD, we know that

$$\mathscr{W}_{f,g}^{D}(\mathbf{t},\boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\tau_1} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)e^{-\mathbf{j}\omega_2\tau_2}d^2\boldsymbol{\tau}. \tag{29}$$

Indeed, from the inverse of the two-sided QFT (12), it follows that

$$f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{\mathbf{i}\omega_1\tau_1}\mathscr{W}_{f,g}^{D}(\mathbf{t},\boldsymbol{\omega})e^{\mathbf{j}\omega_2\tau_2}d^2\boldsymbol{\omega}. \tag{30}$$

Letting $\boldsymbol{\tau}/2 = \mathbf{t}$, the above expression will lead to

$$f(2\mathbf{t})\overline{g}(\mathbf{0}) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{2\mathbf{i}\omega_1 t_1}\mathscr{W}_{f,g}^{D}(\mathbf{t},\boldsymbol{\omega})e^{2\mathbf{j}\omega_2 t_2}d^2\boldsymbol{\omega}, \tag{31}$$

and the final result can be obtained by letting $2\mathbf{t} = \mathbf{s}$; that is,

$$f(s) = \frac{1}{(2\pi)^2\overline{g}(\mathbf{0})}\int_{\mathbb{R}^2} e^{\mathbf{i}\omega_1 s_1}\mathscr{W}_{f,g}^{D}\left(\frac{\mathbf{s}}{2},\boldsymbol{\omega}\right)e^{\mathbf{j}\omega_2 s_2}d^2\boldsymbol{\omega}, \tag{32}$$

which completes the proof. □

Lemma 6. *For any t and ω, if $\|f\|_{L^2(\mathbb{R}^2;\mathbb{H})} = 1$ and $\|g\|_{L^2(\mathbb{R}^2;\mathbb{H})} = 1$, then we obtain*

$$\iint_{\mathbb{R}^2}\left|\mathscr{W}_{f,g}^{D}(\mathbf{t},\boldsymbol{\omega})\right|^2 d^2\mathbf{t}\,d^2\boldsymbol{\omega} = (4\pi)^2. \tag{33}$$

Proof. Since $\mathscr{W}_{f,g}^{D}(\mathbf{t},\boldsymbol{\omega})$ is the two-sided QFT of $R_t(\boldsymbol{\tau})$, (14) yields

$$\begin{aligned}\frac{1}{(2\pi)^2}&\int_{\mathbb{R}^2}\left|\mathscr{W}_{f,g}^{D}(\mathbf{t},\boldsymbol{\omega})\right|^2 d^2\boldsymbol{\omega}\\ &= \int_{\mathbb{R}^2}\left|R_t(\boldsymbol{\tau})\right| d^2\boldsymbol{\tau}\\ &= \int_{\mathbb{R}^2}\left|f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\right|^2 d^2\boldsymbol{\tau}.\end{aligned} \tag{34}$$

Using the change of variables $\mathbf{x} = \mathbf{t} + (\boldsymbol{\tau}/2)$ and $\mathbf{y} = \mathbf{t} - (\boldsymbol{\tau}/2)$ and then integrating (34) with respect to $d^2\mathbf{t}$ we immediately get

$$\begin{aligned}\frac{1}{(2\pi)^2}&\iint_{\mathbb{R}^2}\left|\mathscr{W}_{f,g}^{D}(\mathbf{t},\boldsymbol{\omega})\right|^2 d^2\mathbf{t}\,d^2\boldsymbol{\omega}\\ &= \iint_{\mathbb{R}^2}\left|f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\right|^2 d^2\mathbf{t}\,d^2\boldsymbol{\tau}\\ &= \iint_{\mathbb{R}^2}\left|f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\right|^2\left|\overline{g}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\right|^2 d^2\mathbf{t}\,d^2\boldsymbol{\tau}\\ &= 4\iint_{\mathbb{R}^2}|f(\mathbf{x})|^2|\overline{g}(\mathbf{y})|^2 d^2\mathbf{y}\,d^2\mathbf{t}.\end{aligned} \tag{35}$$

This finishes the proof. □

Remark 7. Equation (33) is known as the radar uncertainty principle in the cross two-sided QWVD domain. It shows that the quaternion function $\mathscr{W}_{f,g}^{D}(\mathbf{t},\boldsymbol{\omega})$ cannot be concentrated arbitrarily close to the origin.

We obtain the following results which correspond to classical WVD properties (compare to [1, 2]).

Lemma 8. *For $f, g \in L^2(\mathbb{R}^2;\mathbb{H})$, the two-sided QFT of (20) with respect to $\boldsymbol{\omega}$ can be represented in the form*

$$\begin{aligned}\mathscr{F}_q\left\{\mathscr{W}_{f,g}^{D}\right\}(\mathbf{t},\boldsymbol{\sigma}) &= \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\sigma_1}\mathscr{W}_{f,g}^{D}(\mathbf{t},\boldsymbol{\omega})e^{-\mathbf{j}\omega_2\sigma_2}d^2\boldsymbol{\omega}\\ &= (2\pi)^2 f\left(\mathbf{t}-\frac{\boldsymbol{\sigma}}{2}\right)\overline{g}\left(\mathbf{t}+\frac{\boldsymbol{\sigma}}{2}\right).\end{aligned} \tag{36}$$

Proof. A simple computation gives for every $f, g \in L^2(\mathbb{R}^2;\mathbb{H})$

$$\begin{aligned}\mathscr{F}_q&\left\{\mathscr{W}_{f,g}^{D}\right\}(\mathbf{t},\boldsymbol{\sigma})\\ &\overset{(11)}{=} \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\sigma_1}\mathscr{W}_{f,g}(\mathbf{t},\boldsymbol{\omega})e^{-\mathbf{j}\omega_2\sigma_2}d^2\boldsymbol{\omega}\\ &\overset{(23)}{=} \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\sigma_1}\int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\tau_1}R_t(\boldsymbol{\tau})e^{-\mathbf{j}\omega_2\tau_2}d^2\boldsymbol{\tau}e^{-\mathbf{j}\omega_2\sigma_2}d^2\boldsymbol{\omega}\\ &= \iint_{\mathbb{R}^2} e^{\mathbf{i}\omega_1(-\tau_1-\sigma_1)}R_t(\boldsymbol{\tau})e^{\mathbf{j}\omega_2(-\tau_2-\sigma_2)}d^2\boldsymbol{\omega}\,d^2\boldsymbol{\tau}\\ &= (2\pi)^2\int_{\mathbb{R}^2}\delta(-\tau_1-\sigma_1)R_t(\boldsymbol{\tau})\delta(-\tau_2-\sigma_2)d^2\boldsymbol{\tau}\\ &= (2\pi)^2 R_t(-\boldsymbol{\sigma}) = (2\pi)^2 f\left(\mathbf{t}-\frac{\boldsymbol{\sigma}}{2}\right)\overline{g}\left(\mathbf{t}+\frac{\boldsymbol{\sigma}}{2}\right),\end{aligned} \tag{37}$$

where as usual δ denotes the Dirac delta function (13). □

It is not difficult to see that for $f = g$ we get the auto two-sided quaternion Wigner-Ville distribution defined by

$$\begin{aligned}\mathscr{W}_{f,f}^{D}(\mathbf{t},\boldsymbol{\omega}) &= \mathscr{W}_{f}^{D}(\mathbf{t},\boldsymbol{\omega})\\ &= \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\tau_1} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{f}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)e^{-\mathbf{j}\omega_2\tau_2}d^2\boldsymbol{\tau}.\end{aligned} \tag{38}$$

Both the cross two-sided quaternion Wigner-Ville distribution and the auto two-sided Wigner-Ville distribution are often so-called the two-sided Wigner-Ville distribution or two-sided quaternion Wigner-Ville transform (QWVT). If f is a real function, then the change of variables $\boldsymbol{\tau} = -\mathbf{x}$ to (38) yields

$$\begin{aligned}\mathscr{W}_{f}^{D}(\mathbf{t},\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} e^{\mathbf{i}\omega_1 x_1} f\left(\mathbf{t}-\frac{\mathbf{x}}{2}\right)\overline{f}\left(\mathbf{t}+\frac{\mathbf{x}}{2}\right)e^{\mathbf{j}\omega_2 x_2}d^2\mathbf{x}\\ &= \int_{\mathbb{R}^2} e^{\mathbf{i}\omega_1 x_1} f\left(\mathbf{t}+\frac{\mathbf{x}}{2}\right)\overline{f}\left(\mathbf{t}-\frac{\mathbf{x}}{2}\right)e^{\mathbf{j}\omega_2 x_2}d^2\mathbf{x}\\ &= \mathscr{W}_{f}^{D}(\mathbf{t},-\boldsymbol{\omega}).\end{aligned} \tag{39}$$

Lemma 9 (time energy density). *If $f \in L^2(\mathbb{R}^2;\mathbb{H})$, then the two-sided QWVD satisfies the time energy density as*

$$\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}\mathscr{W}_f^D(\mathbf{t},\boldsymbol{\omega})\,d^2\mathbf{t}\,d^2\boldsymbol{\omega}=\|f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}. \quad (40)$$

Proof. A direct calculation shows that

$$\begin{aligned}\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}e^{-\mathbf{i}\omega_1\sigma_1}\mathscr{W}_{f,g}^D(\mathbf{t},\boldsymbol{\omega})\,e^{-\mathbf{j}\omega_2\sigma_2}d^2\boldsymbol{\omega}\\ =f\left(\mathbf{t}-\frac{\boldsymbol{\sigma}}{2}\right)\overline{g}\left(\mathbf{t}+\frac{\boldsymbol{\sigma}}{2}\right).\end{aligned} \quad (41)$$

Changing variables $\mathbf{t}+(\boldsymbol{\sigma}/2)=\mathbf{s}$ and $\mathbf{t}-(\boldsymbol{\sigma}/2)=\mathbf{z}$ in the above expression, we immediately get

$$\begin{aligned}\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}e^{-\mathbf{i}\omega_1 2(s_1-z_1)}\mathscr{W}_{f,g}^D\left(\frac{\mathbf{s}+\mathbf{z}}{2},\boldsymbol{\omega}\right)e^{-\mathbf{j}\omega_2 2(s_2-z_2)}d^2\boldsymbol{\omega}\\ =f(\mathbf{z})\,\overline{g}(\mathbf{s}).\end{aligned} \quad (42)$$

In particular, if we substitute $\mathbf{s}=\mathbf{z}=\mathbf{t}$ in (42), we obtain

$$\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\mathscr{W}_{f,g}^D(\mathbf{t},\boldsymbol{\omega})\,d^2\boldsymbol{\omega}=f(\mathbf{t})\,\overline{g}(\mathbf{t}). \quad (43)$$

If $f=g$, then integrating (43) with respect to $d^2\mathbf{t}$ gives the final result

$$\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}\mathscr{W}_f^D(\mathbf{t},\boldsymbol{\omega})\,d^2\boldsymbol{\omega}\,d^2\mathbf{t}=\int_{\mathbb{R}^2}|f(\mathbf{t})|^2d^2\mathbf{t}. \quad (44)$$

This gives the desired result. □

Remark 10. Unlike classical case, we cannot establish the frequency energy density, because the Parseval formula does not hold for the two-sided QFT.

For an illustrative purpose, we consider an example of the two-sided QWVD.

Example 11. Given a Gaussian signal defined by

$$f(\mathbf{t})=e^{-(t_1^2+t_2^2)}, \quad (45)$$

find the two-sided QWVD of a pure sine wave $g(\mathbf{t})=e^{\mathbf{i}u_0t_1}e^{\mathbf{j}v_0t_2}$.

From Definition 3 we obtain

$$\begin{aligned}&\mathscr{W}_{f,g}^D(\mathbf{t},\boldsymbol{\omega})\\ &=\int_{\mathbb{R}^2}e^{-\mathbf{i}\omega_1\tau_1}f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)e^{-\mathbf{j}\omega_2\tau_2}\,d^2\boldsymbol{\tau}\\ &=\int_{\mathbb{R}^2}e^{-\mathbf{i}\omega_1\tau_1}e^{-(t_1+\tau_1/2)^2}e^{-(t_2+\tau_2/2)^2}e^{-\mathbf{i}u_0(t_1-\tau_1/2)}\\ &\qquad\times e^{-\mathbf{j}v_0(t_2-\tau_2/2)}e^{-\mathbf{j}\omega_2\tau_2}\,d^2\boldsymbol{\tau}\\ &=e^{-(t_1^2+t_2^2)}e^{-\mathbf{i}u_0t_1}\int_{\mathbb{R}^2}e^{-((\tau_1^2/4)+(\tau_2^2/4))}e^{-\mathbf{i}\omega_1\tau_1}e^{-t_1\tau_1}e^{\mathbf{i}u_0\tau_1/2}\\ &\qquad\times e^{-\mathbf{j}\omega_2\tau_2}e^{-t_2\tau_2}e^{\mathbf{j}v_0\tau_2/2}\,d^2\boldsymbol{\tau}e^{-\mathbf{j}v_0t_2}\\ &=e^{-(t_1^2+t_2^2)}e^{-\mathbf{i}u_0t_1}\int_{\mathbb{R}^2}e^{-\tau_1^2/4-t_1\tau_1+\mathbf{i}u_0\tau_1/2-\mathbf{i}\omega_1\tau_1}\\ &\qquad\times e^{-\tau_2^2/4-t_2\tau_2+\mathbf{j}v_0\tau_2/2-\mathbf{j}\omega_2\tau_2}\,d^2\boldsymbol{\tau}e^{-\mathbf{j}v_0t_2}\\ &=e^{-(t_1^2+t_2^2)}\sqrt{4\pi}e^{-(\omega_1+t_1-u_0/2)}e^{-\mathbf{i}u_0t_1}\sqrt{4\pi}e^{-(\omega_2+t_2-v_0/2)}e^{-\mathbf{j}v_0t_2}.\end{aligned} \quad (46)$$

3.2. Basic Properties of Two-Sided QWVD. The following propositions describe the elementary properties of the two-sided QWVD and their proofs in detail. We find that most of the expected properties of the classical WVD are still valid with some modifications in this case.

Proposition 12 (shift). *Let $f,g\in L^2(\mathbb{R}^2;\mathbb{H})$ be quaternion signals. If we time shift the signals by $\mathbf{a}$, then we obtain*

$$\mathscr{W}_{T_\mathbf{a}f,T_\mathbf{a}g}^D(\mathbf{t},\boldsymbol{\omega})=\mathscr{W}_{f,g}(\mathbf{t}-\mathbf{a},\boldsymbol{\omega}). \quad (47)$$

Proposition 13 (nonlinearity). *Let $f,g\in L^2(\mathbb{R}^2;\mathbb{H})$ be quaternion signals. The two-sided QWVD is nonlinear. That is*

$$\begin{aligned}\mathscr{W}_{f+g}^D(\mathbf{t},\boldsymbol{\omega})&=\mathscr{W}_f^D(\mathbf{t},\boldsymbol{\omega})+\mathscr{W}_{f,g}^D(\mathbf{t},\boldsymbol{\omega})\\ &\quad+\mathscr{W}_{g,f}^D(\mathbf{t},\boldsymbol{\omega})+\mathscr{W}_g^D(\mathbf{t},\boldsymbol{\omega}).\end{aligned} \quad (48)$$

Proof. Applying (20) and the basic properties of quaternions we get

$$\begin{aligned}&\mathscr{W}_{f+g}^D(\mathbf{t},\boldsymbol{\omega})\\ &\quad=\int_{\mathbb{R}^2}e^{-\mathbf{i}\omega_1\tau_1}\left(f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)+g\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\right)\\ &\qquad\times\overline{\left(f\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)+g\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\right)}e^{-\mathbf{j}\omega_2\tau_2}\,d^2\boldsymbol{\tau}\\ &\quad=\int_{\mathbb{R}^2}e^{-\mathbf{i}\omega_1\tau_1}\left(f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)+g\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\right)\\ &\qquad\times\left(\overline{f}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)+\overline{g}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\right)e^{-\mathbf{j}\omega_2\tau_2}\,d^2\boldsymbol{\tau}\end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\tau_1}\left(f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{f}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\right.\\
&\qquad + f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right) + g\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\\
&\qquad \times \overline{f}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\\
&\qquad \left. + g\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\right) e^{-\mathbf{j}\omega_2\tau_2}\, d^2\boldsymbol{\tau}\\
&= \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\tau_1} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{f}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right) e^{-\mathbf{j}\omega_2\tau_2}\, d^2\boldsymbol{\tau}\\
&\quad + \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\tau_1} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right) e^{-\mathbf{j}\omega_2\tau_2}\, d^2\boldsymbol{\tau}\\
&\quad + \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\tau_1} g\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{f}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right) e^{-\mathbf{j}\omega_2\tau_2}\, d^2\boldsymbol{\tau}\\
&\quad + \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\tau_1} g\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right) e^{-\mathbf{j}\omega_2\tau_2}\, d^2\boldsymbol{\tau}\\
&= \mathscr{W}^D_f(\mathbf{t},\boldsymbol{\omega}) + \mathscr{W}^D_{f,g}(\mathbf{t},\boldsymbol{\omega}) + \mathscr{W}^D_{g,f}(\mathbf{t},\boldsymbol{\omega}) + \mathscr{W}^D_g(\mathbf{t},\boldsymbol{\omega}).
\end{aligned} \tag{49}$$

The proof is complete. □

This shows that the two-sided QWVD of the sum of two quaternion signals is not simply the sum of the two-sided QWVD of the signals.

Proposition 14 (modulation). *Let $f, g \in L^2(\mathbb{R}^2;\mathbb{H})$ be quaternion signals. If we spectrum-shift the signal f by $\boldsymbol{\omega}_0$, then we obtain*

$$\begin{aligned}
\mathscr{W}^D_{M_{\boldsymbol{\omega}_0}f,g}(\mathbf{t},\boldsymbol{\omega}) &= e^{\mathbf{i}u_0t_1}\mathscr{W}^D_{f,g_0}\left(\mathbf{t},\boldsymbol{\omega}-\frac{\boldsymbol{\omega}_0}{2}\right)e^{\mathbf{j}v_0t_2}\\
&\quad - e^{\mathbf{i}u_0t_1}\mathscr{W}^D_{f,g_1}\left(\mathbf{t},\omega_1-\frac{u_0}{2},-\omega_2-\frac{v_0}{2}\right)e^{\mathbf{j}v_0t_2}\mathbf{i}\\
&\quad - e^{\mathbf{i}u_0t_1}\mathscr{W}^D_{f,g_2}\left(\mathbf{t},\boldsymbol{\omega}-\frac{\boldsymbol{\omega}_0}{2}\right)e^{\mathbf{j}v_0t_2}\mathbf{j}\\
&\quad - e^{\mathbf{i}u_0t_1}\mathscr{W}^D_{f,g_3}\left(\mathbf{t},\omega_1-\frac{u_0}{2},-\omega_2-\frac{v_0}{2}\right)e^{\mathbf{j}v_0t_2}\mathbf{k}.
\end{aligned} \tag{50}$$

Proof. In this proof we will use the decomposition of quaternion functions. We first remember the fact that $e^{\mathbf{j}v_0(t_2+(\tau_2/2))}\mathbf{i}g_1(\mathbf{t}-(\boldsymbol{\tau}/2)) = \mathbf{i}g_1(\mathbf{t}-(\boldsymbol{\tau}/2))e^{-\mathbf{j}v_0(t_2+(\tau_2/2))}$ and so on. This further leads to

$$\begin{aligned}
&\mathscr{W}^D_{M_{\boldsymbol{\omega}_0}f,g}(\mathbf{t},\boldsymbol{\omega})\\
&= \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\tau_1}\left(e^{\mathbf{i}u_0(t_1+(\tau_1/2))}\right.\\
&\qquad \left.\times f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)e^{\mathbf{j}v_0(t_2+(\tau_2/2))}\overline{g}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\right)e^{-\mathbf{j}\omega_2\tau_2}\, d^2\boldsymbol{\tau}\\
&= \int_{\mathbb{R}^2} e^{\mathbf{i}u_0t_1}\left(e^{-\mathbf{i}(\omega_1-(u_0/2))\tau_1} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)e^{\mathbf{j}v_0(t_2+(\tau_2/2))}\right.\\
&\qquad \times\left(\overline{g_0}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)-\mathbf{i}\overline{g_1}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)-\mathbf{j}\overline{g_2}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\right.\\
&\qquad \left.\left. -\mathbf{k}\overline{g_3}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\right)\right)e^{-\mathbf{j}\omega_2\tau_2}\, d^2\boldsymbol{\tau}\\
&= e^{\mathbf{i}u_0t_1}\left(\int_{\mathbb{R}^2} e^{-\mathbf{i}(\omega_1-(u_0/2))\tau_1} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\right.\\
&\qquad \times\overline{g_0}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)e^{-\mathbf{j}(\omega_2-(v_0/2))\tau_2}\, d^2\boldsymbol{\tau}\\
&\qquad - \int_{\mathbb{R}^2} e^{-\mathbf{i}(\omega_1-(u_0/2))\tau_1} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g_2}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\\
&\qquad \left.\times e^{-\mathbf{j}(\omega_2-(v_0/2))\tau_2}\mathbf{j}\, d^2\boldsymbol{\tau}\right)e^{\mathbf{j}v_0t_2}\\
&\quad - e^{\mathbf{i}u_0t_1}\left(\int_{\mathbb{R}^2} e^{-\mathbf{i}(\omega_1-(u_0/2))\tau_1} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g_1}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\right.\\
&\qquad \left.\times e^{\mathbf{j}(\omega_2+(v_0/2))\tau_2}\, d^2\boldsymbol{\tau}\right)e^{\mathbf{j}v_0t_2}\mathbf{i}\\
&\quad - e^{\mathbf{i}u_0t_1}\left(\int_{\mathbb{R}^2} e^{-\mathbf{i}(\omega_1-(u_0/2))\tau_1} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g_3}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\right.\\
&\qquad \left.\times e^{\mathbf{j}(\omega_2+(v_0/2))\tau_2}\, d^2\boldsymbol{\tau}\right)e^{\mathbf{j}v_0t_2}\mathbf{k}\\
&= e^{\mathbf{i}u_0t_1}\int_{\mathbb{R}^2} e^{-\mathbf{i}(\omega_1-(u_0/2))\tau_1} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g_0}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\\
&\qquad \times e^{-\mathbf{j}(\omega_2-(v_0/2))\tau_2}\, d^2\boldsymbol{\tau}\; e^{\mathbf{j}v_0t_2}\\
&\quad - e^{\mathbf{i}u_0t_1}\int_{\mathbb{R}^2} e^{-\mathbf{i}(\omega_1-(u_0/2))\tau_1} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g_1}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\\
&\qquad \times e^{\mathbf{j}(\omega_2+(v_0/2))\tau_2}\, d^2\boldsymbol{\tau}\; e^{\mathbf{j}v_0t_2}\mathbf{i}\\
&\quad - e^{\mathbf{i}u_0t_1}\int_{\mathbb{R}^2} e^{-\mathbf{i}(\omega_1-(u_0/2))\tau_1} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g_2}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\\
&\qquad \times e^{-\mathbf{j}(\omega_2-(v_0/2))\tau_2}\, d^2\boldsymbol{\tau}\; e^{\mathbf{j}v_0t_2}\mathbf{j}\\
&\quad - e^{\mathbf{i}u_0t_1}\int_{\mathbb{R}^2} e^{-\mathbf{i}(\omega_1-(u_0/2))\tau_1} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g_3}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\\
&\qquad \times e^{\mathbf{j}(\omega_2+(v_0/2))\tau_2}\, d^2\boldsymbol{\tau}\; e^{\mathbf{j}v_0t_2}\mathbf{k}\\
&= e^{\mathbf{i}u_0t_1}\mathscr{W}^D_{f,g_0}\left(\mathbf{t},\boldsymbol{\omega}-\frac{\boldsymbol{\omega}_0}{2}\right)e^{\mathbf{j}v_0t_2}\\
&\quad - e^{\mathbf{i}u_0t_1}\mathscr{W}^D_{f,g_1}\left(\mathbf{t},\omega_1-\frac{u_0}{2},-\omega_2-\frac{v_0}{2}\right)e^{\mathbf{j}v_0t_2}\mathbf{i}\\
&\quad - e^{\mathbf{i}u_0t_1}\mathscr{W}^D_{f,g_2}\left(\mathbf{t},\boldsymbol{\omega}-\frac{\boldsymbol{\omega}_0}{2}\right)e^{\mathbf{j}v_0t_2}\mathbf{j}\\
&\quad - e^{\mathbf{i}u_0t_1}\mathscr{W}^D_{f,g_3}\left(\mathbf{t},\omega_1-\frac{u_0}{2},-\omega_2-\frac{v_0}{2}\right)e^{\mathbf{j}v_0t_2}\mathbf{k}.
\end{aligned} \tag{51}$$

which completes the proof. □

Proposition 15 (dilation). *Let $f, g \in L^2(\mathbb{R}^2;\mathbb{H})$ be quaternion signals. Then*

$$\mathscr{W}^D_{\mathscr{D}_{cf},\mathscr{D}_{cg}}(\mathbf{t},\boldsymbol{\omega}) = \mathscr{W}^D_{f,g}\left(\frac{\mathbf{t}}{c}, \mathbf{c}\boldsymbol{\omega}\right). \quad (52)$$

Proof. Let $\boldsymbol{\tau}/c = \mathbf{x}$, then a direct computation yields

$$\begin{aligned}&\mathscr{W}^D_{\mathscr{D}_{cf},\mathscr{D}_{cg}}(\mathbf{t},\boldsymbol{\omega}) \\ &= \frac{1}{c^2}\int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\tau_1} f\left(\frac{\mathbf{t}}{c}+\frac{\boldsymbol{\tau}}{2c}\right)\overline{g}\left(\frac{\mathbf{t}}{c}-\frac{\boldsymbol{\tau}}{2c}\right)e^{-\mathbf{j}\omega_2\tau_2}\,d^2\boldsymbol{\tau} \\ &= \int_{\mathbb{R}^2} e^{-\mathbf{i}c\omega_1 x_1} f\left(\frac{\mathbf{t}}{c}+\frac{\mathbf{x}}{2}\right)\overline{g}\left(\frac{\mathbf{t}}{c}-\frac{\mathbf{x}}{2}\right)e^{-\mathbf{j}c\omega_2 x_2}\,d^2\mathbf{x} \\ &= \mathscr{W}^D_{f,g}\left(\frac{\mathbf{t}}{c}, c\boldsymbol{\omega}\right),\end{aligned} \quad (53)$$

which was to be proved. □

3.3. Main Properties of Two-Sided QWVD. In this subsection we investigate two main properties of the two-sided QWVD. Based on the properties of the two-sided QFT we can only establish the specific Moyal formula for the QWVD as follows.

Theorem 16 (Moyal's formula for the two-sided QWVD). *Let $f, g \in L^2(\mathbb{R}^2;\mathbb{H})$ be two quaternion signals. Then the following equation holds:*

$$\iint_{\mathbb{R}^2}\left|\mathscr{W}^D_{f,g}(\mathbf{t},\boldsymbol{\omega})\right|^2 d^2\boldsymbol{\omega}\,d^2\mathbf{t} = (4\pi)^2\|f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}\|g\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}. \quad (54)$$

Proof. We first notice that, for fixed $\mathbf{t}$,

$$\mathscr{W}^D_{f,g}(\mathbf{t},\boldsymbol{\omega}) = \mathscr{F}^D_q\{R_t(\boldsymbol{\tau})\}(\boldsymbol{\omega}). \quad (55)$$

Applying (55) and Parseval's formula (14) we get

$$\begin{aligned}&\int_{\mathbb{R}^2}\left|\mathscr{W}^D_{f,g}(\mathbf{t},\boldsymbol{\omega})\right|^2 d^2\boldsymbol{\omega} \\ &= \left\|\mathscr{F}^D_q\{R_t(\boldsymbol{\tau})\}\right\|^2_{L^2(\mathbb{R}^2;\mathbb{H})} \\ &= (2\pi)^2\|R_t(\boldsymbol{\tau})\|^2_{L^2(\mathbb{R}^2;\mathbb{H})} \\ &= (2\pi)^2\int_{\mathbb{R}^2} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\overline{f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)}d^2\boldsymbol{\tau} \\ &= (2\pi)^2\int_{\mathbb{R}^2} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)g\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\overline{f}\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)d^2\boldsymbol{\tau}.\end{aligned} \quad (56)$$

Integrating (56) with respect to $d^2\mathbf{t}$ we further obtain

$$\begin{aligned}&\iint_{\mathbb{R}^2}\left|\mathscr{W}^D_{f,g}(\mathbf{t},\boldsymbol{\omega})\right|^2 d^2\boldsymbol{\omega}\,d^2\mathbf{t} \\ &\quad= (2\pi)^2\iint_{\mathbb{R}^2} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right) \\ &\qquad\times g\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\overline{f}\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)d^2\boldsymbol{\tau}\,d^2\mathbf{t}.\end{aligned} \quad (57)$$

First use $\mathbf{t} + (\boldsymbol{\tau}/2) = \mathbf{x}$, then change the order of integration and use $\mathbf{t} - (\boldsymbol{\tau}/2) = \mathbf{y}$ we immediately get

$$\begin{aligned}&\iint_{\mathbb{R}^2}\left|\mathscr{W}^D_{f,g}(\mathbf{t},\boldsymbol{\omega})\right|^2 d^2\boldsymbol{\omega}\,d^2\mathbf{t} \\ &\quad= (2\pi)^2 4\iint_{\mathbb{R}^2} f(\mathbf{x})\overline{g}(\mathbf{y})g(\mathbf{y})\overline{f}(\mathbf{x})\,d^2\mathbf{y}\,d^2\mathbf{x} \\ &\quad= (4\pi)^2\int_{\mathbb{R}^2} f(\mathbf{x})\overline{f}(\mathbf{x})\,d^2\mathbf{x}\int_{\mathbb{R}^2}|g(\mathbf{y})|^2 d^2\mathbf{y} \\ &\quad= (4\pi)^2\int_{\mathbb{R}^2}|f(\mathbf{x})|^2 d^2\mathbf{x}\int_{\mathbb{R}^2}|g(\mathbf{y})|^2 d^2\mathbf{y}.\end{aligned} \quad (58)$$

This is the desired result. □

Due to the non-commutativity of the quaternion exponential products and quaternion multiplications, we only can establish a special condition of the convolution theorem for the two-sided QWVD (compared to [20]). This is described in the following theorem.

Theorem 17 (convolution for the two-sided QWVD). *Let $f, g \in L^2(\mathbb{R}^2;\mathbb{H})$ be two real-valued signals. If we assume that $\mathscr{W}^D_{f,g}$ is a real-valued function, then the following result holds:*

$$\mathscr{W}^D_{f\star g}(\mathbf{t},\boldsymbol{\omega}) = \int_{\mathbb{R}^2}\mathscr{W}^D_{f,g}(\mathbf{u},\boldsymbol{\omega})\,\mathscr{W}^D_{g,f}(\mathbf{t}-\mathbf{u},\boldsymbol{\omega})\,d^2\mathbf{u}, \quad (59)$$

where $\star$ is the quaternion convolution operator.

Proof. Applying the definition of the two-sided QWVD (20) and elementary properties of the quaternion convolution gives

$$\begin{aligned}&\mathscr{W}^D_{f\star g}(\mathbf{t},\boldsymbol{\omega}) \\ &= \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\tau_1}(f\star g)\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{(f\star g)}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)e^{-\mathbf{j}\omega_2\tau_2}d^2\boldsymbol{\tau} \\ &= \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\tau_1}(f\star g)\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\left(\overline{g}\star\overline{f}\right)\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)e^{-\mathbf{j}\omega_2\tau_2}d^2\boldsymbol{\tau} \\ &= \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\tau_1}\left[\int_{\mathbb{R}^2} f(\mathbf{x})\,g\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}-\mathbf{x}\right)d^2\mathbf{x}\right. \\ &\qquad\left.\times\int_{\mathbb{R}^2}\overline{g}(\mathbf{y})\,\overline{f}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}-\mathbf{y}\right)d^2\mathbf{y}\right]e^{-\mathbf{j}\omega_2\tau_2}d^2\boldsymbol{\tau}.\end{aligned} \quad (60)$$

By the change of variables $\mathbf{x} = \mathbf{u} + (\mathbf{p}/2)$, $\mathbf{y} = \mathbf{u} - (\mathbf{p}/2)$ and $\boldsymbol{\tau} = \mathbf{p} + \mathbf{q}$, the above identity can be written as

$$\begin{aligned}&\mathscr{W}^D_{f\star g}(\mathbf{t},\boldsymbol{\omega}) \\ &= \int_{\mathbb{R}^2} e^{-\mathbf{i}(p_1+q_1)\omega_1} \\ &\quad\times\left[\int_{\mathbb{R}^2} f\left(\mathbf{u}+\frac{\mathbf{p}}{2}\right)g\left(\mathbf{t}-\mathbf{u}+\frac{\mathbf{q}}{2}\right)d^2\mathbf{x}\right. \\ &\qquad\left.\times\int_{\mathbb{R}^2}\overline{g}\left(\mathbf{u}-\frac{\mathbf{p}}{2}\right)\overline{f}\left(\mathbf{t}-\mathbf{u}-\frac{\mathbf{q}}{2}\right)d^2\mathbf{y}\right]e^{-\mathbf{j}(p_2+q_2)\omega_2}d^2\boldsymbol{\tau}.\end{aligned} \quad (61)$$

Because f, g are real-valued signals, we may interchange the order of $g(\mathbf{t}-\mathbf{u}+(\mathbf{q}/2))$ and $\overline{g}(\mathbf{u}-(\mathbf{p}/2))$ to get

$$\begin{aligned}
&\mathscr{W}^D_{f\star g}(\mathbf{t},\boldsymbol{\omega}) \\
&= \int_{\mathbb{R}^2}\left[\int_{\mathbb{R}^2} e^{-\mathbf{i}q_1\omega_1}e^{-\mathbf{i}p_1\omega_1} f\left(\mathbf{u}+\frac{\mathbf{p}}{2}\right)\overline{g}\left(\mathbf{u}-\frac{\mathbf{p}}{2}\right)e^{-\mathbf{j}p_2\omega_2}d^2\mathbf{p}\right. \\
&\qquad \left.\times\int_{\mathbb{R}^2} g\left(\mathbf{t}-\mathbf{u}+\frac{\mathbf{q}}{2}\right)\overline{f}\left(\mathbf{t}-\mathbf{u}-\frac{\mathbf{q}}{2}\right)e^{-\mathbf{j}q_2\omega_2}\,d^2\mathbf{q}\right]d^2\mathbf{u} \\
&= \int_{\mathbb{R}^2}\left[e^{-\mathbf{i}q_1\omega_1}\mathscr{W}^D_{f,g}(\mathbf{u},\boldsymbol{\omega})\int_{\mathbb{R}^2} g\left(\mathbf{t}-\mathbf{u}+\frac{\mathbf{q}}{2}\right)\right. \\
&\qquad \left.\times\overline{f}\left(\mathbf{t}-\mathbf{u}-\frac{\mathbf{q}}{2}\right)\,e^{-\mathbf{j}q_2\omega_2}\,d^2\mathbf{q}\right]d^2\mathbf{u} \\
&= \int_{\mathbb{R}^2}\left[\mathscr{W}^D_{f,g}(\mathbf{u},\boldsymbol{\omega})\,e^{-\mathbf{i}q_1\omega_1}\right. \\
&\qquad \times\int_{\mathbb{R}^2} g\left(\mathbf{t}-\mathbf{u}+\frac{\mathbf{q}}{2}\right) \\
&\qquad\qquad \left.\times\overline{f}\left(\mathbf{t}-\mathbf{u}-\frac{\mathbf{q}}{2}\right)e^{-\mathbf{j}q_2\omega_2}\,d^2\mathbf{q}\right]d^2\mathbf{u} \\
&= \int_{\mathbb{R}^2}\mathscr{W}^D_{f,g}(\mathbf{u},\boldsymbol{\omega})\,\mathscr{W}^D_{g,f}(\mathbf{t}-\mathbf{u},\boldsymbol{\omega})\,d^2\mathbf{u},
\end{aligned} \tag{62}$$

where in the third line we used the assumption to interchange the order of the two-sided QWVD and the kernel of the two-sided QFT. This finishes the proof of the theorem. □

4. Right-Sided QWVD

Based on the properties of quaternions and the kernel of the right-sided QFT we may construct the right-sided QWVD. We will see that some properties of the right-sided QWVD are quite different from the two-sided QWVD.

Definition 18. The cross right-sided quaternion Wigner-Ville distribution of the 2D signals $f, g \in L^2(\mathbb{R}^2;\mathbb{H})$ is given by

$$\mathscr{W}^R_{f,g}(\mathbf{t},\boldsymbol{\omega}) = \int_{\mathbb{R}^2} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)e^{-\mathbf{i}\omega_1\tau_1}e^{-\mathbf{j}\omega_2\tau_2}d^2\boldsymbol{\tau}, \tag{63}$$

provided the integral exists.

Similar to inverse transform of the two-sided QWVD we get the following fundamental result.

Theorem 19. *The inverse right-sided QWVD of the signal $f \in L^1(\mathbb{R}^2;\mathbb{H})$ is given by*

$$f(t) = \frac{1}{(2\pi)^2\overline{g}(\mathbf{0})}\int_{\mathbb{R}^2}\mathscr{W}^R_{f,g}\left(\frac{\mathbf{t}}{2},\boldsymbol{\omega}\right)e^{\mathbf{j}\omega_2 t_2}e^{\mathbf{i}\omega_1 t_1}d^2\boldsymbol{\omega}, \tag{64}$$

provided $\overline{g}(\mathbf{0}) \neq \mathbf{0}$.

By using the change of variables $\mathbf{t}+(\boldsymbol{\tau}/2)=\mathbf{x}$, (63) can be expressed in the following form:

$$\begin{aligned}
&\mathscr{W}^R_{f,g}(\mathbf{t},\boldsymbol{\omega}) \\
&\quad = 4\int_{\mathbb{R}^2} f(\mathbf{x})\,\overline{g}(2\mathbf{t}-\mathbf{x})\,e^{-\mathbf{i}\omega_1(2x_1-2t_1)}e^{-\mathbf{j}\omega_2(2x_2-2t_2)}d^2\mathbf{x} \\
&\quad = 4\int_{\mathbb{R}^2} f(\mathbf{x})\,\overline{g}(2\mathbf{t}-\mathbf{x})\,e^{\mathbf{i}2\omega_1 t_1}e^{-\mathbf{i}2\omega_1 x_1}e^{-\mathbf{j}2\omega_2 x_2}d^2\mathbf{x}e^{\mathbf{j}2\omega_2 t_2}.
\end{aligned} \tag{65}$$

Observe first that if f and g are real signals, the above expression gives an equivalent definition of $\mathscr{W}_{f,g}(\mathbf{t},\boldsymbol{\omega})$ in the form

$$\mathscr{W}^R_{f,g}(\mathbf{t},\boldsymbol{\omega}) = 4e^{\mathbf{i}2\omega_1 t_1}Q_{Ig}f(2\mathbf{t},2\boldsymbol{\omega})\,e^{\mathbf{j}2\omega_2 t_2}, \tag{66}$$

where $Ig(\mathbf{x}) = g(-\mathbf{x})$. Here $Q_{Ig}f(2\mathbf{t},2\boldsymbol{\omega})$ is the windowed quaternionic Fourier transform which was introduced by Bahri et al. [17, 18].

Lemma 20 (time energy density). *If $f \in L^2(\mathbb{R}^2;\mathbb{H})$, then the right-sided QWVD satisfies the time energy density as*

$$\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}\mathscr{W}^R_f(\mathbf{t},\boldsymbol{\omega})\,d^2\mathbf{t}\,d^2\boldsymbol{\omega} = \|f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{67}$$

Proof. Indeed, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2}\mathscr{W}^R_{f,g}(\mathbf{t},\boldsymbol{\omega})\,d^2\boldsymbol{\omega} \\
&\quad = \iint_{\mathbb{R}^2} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)e^{-\mathbf{i}\omega_1\tau_1}e^{-\mathbf{j}\omega_2\tau_2}d^2\boldsymbol{\tau}\,d^2\boldsymbol{\omega} \\
&\quad = (2\pi)^2\int_{\mathbb{R}^2} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\delta(-\boldsymbol{\tau})\,d^2\boldsymbol{\tau} \\
&\quad = (2\pi)^2 f(\mathbf{t})\,\overline{g}(\mathbf{t})\,.
\end{aligned} \tag{68}$$

Integrating both sides of the above expression with respect to $\mathbf{t}$, we obtain

$$\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}\mathscr{W}^R_{f,g}(\mathbf{t},\boldsymbol{\omega})\,d^2\mathbf{t}\,d^2\boldsymbol{\omega} = \int_{\mathbb{R}^2} f(\mathbf{t})\,\overline{g}(\mathbf{t})\,d^2\mathbf{t}. \tag{69}$$

In particular, $f = g$, we get

$$\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}\mathscr{W}^R_f(\mathbf{t},\boldsymbol{\omega})\,d^2\boldsymbol{\omega}\,d^2\mathbf{t} = \int_{\mathbb{R}^2}|f(\mathbf{t})|^2 d^2\mathbf{t}. \tag{70}$$

This gives the desired result. □

4.1. Useful Properties. Proceeding as in the proof of propositions listed in Section 3.2 we obtain elementary properties of the right-sided QWVD.

Proposition 21. *For $f, g \in L^2(\mathbb{R}^2;\mathbb{H})$ the right-sided QWVD has the following useful properties.*

(i) Shift. One has

$$\mathscr{W}^R_{T_{\mathbf{a}}f,T_{\mathbf{a}}g}(\mathbf{t},\boldsymbol{\omega}) = \mathscr{W}^R_{f,g}(\mathbf{t}-\mathbf{a},\boldsymbol{\omega}). \quad (71)$$

(ii) Nonlinearity. One has

$$\begin{aligned}\mathscr{W}^R_{f+g}(\mathbf{t},\boldsymbol{\omega}) &= \mathscr{W}^R_{f}(\mathbf{t},\boldsymbol{\omega}) + \mathscr{W}_{f,g}(\mathbf{t},\boldsymbol{\omega}) \\ &\quad + \mathscr{W}^R_{g,f}(\mathbf{t},\boldsymbol{\omega}) + \mathscr{W}_{g}(\mathbf{t},\boldsymbol{\omega}).\end{aligned} \quad (72)$$

(iii) Dilation. One has

$$\mathscr{W}^R_{\mathscr{D}_{cf},\mathscr{D}_{cg}}(\mathbf{t},\boldsymbol{\omega}) = \mathscr{W}^R_{f,g}\left(\frac{\mathbf{t}}{c}, c\boldsymbol{\omega}\right). \quad (73)$$

Remark 22. It seems that modulation property is not valid for the right-sided QWVD. This shows that some properties of the right-sided QWVD follow the properties the right-sided QFT.

In the following we establish general Moyal's formula of the right-sided QWVD. We see that Moyal's formula of the two-sided QWVD is a special case of Moyal's formula of the right-sided QWVD.

Theorem 23 (Moyal's formula for right-sided the QWVD). *Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^2;\mathbb{H})$ be quaternion-valued signals. Then the following equation holds:*

$$\begin{aligned}&\iint_{\mathbb{R}^2} \mathscr{W}^R_{f_1,g_1}(\mathbf{t},\boldsymbol{\omega}) \overline{\mathscr{W}^R_{f_2,g_2}(\mathbf{t},\boldsymbol{\omega})} d^2\mathbf{t}\, d^2\boldsymbol{\omega} \\ &\quad = (4\pi)^2 \left(f_1 (\overline{g_1}, \overline{g_2})_{L^2(\mathbb{R}^2;\mathbb{H})}, f_2\right)_{L^2(\mathbb{R}^2;\mathbb{H})}.\end{aligned} \quad (74)$$

Proof. From the definition of the right-sided QWVD (63), we easily obtain

$$\begin{aligned}&\iint_{\mathbb{R}^2} \mathscr{W}^R_{f_1,g_1}(\mathbf{t},\boldsymbol{\omega}) \overline{\mathscr{W}^R_{f_2,g_2}(\mathbf{t},\boldsymbol{\omega})} d^2\mathbf{t}\, d^2\boldsymbol{\omega} \\ &= \iiint\int_{\mathbb{R}^2} f_1\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g_1}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right) e^{\mathbf{i}\omega_1(\tau_1'-\tau_1)} e^{-\mathbf{j}\omega_2(\tau_2'-\tau_2)} \\ &\qquad \times g_2\left(\mathbf{t}-\frac{\boldsymbol{\tau}'}{2}\right) \\ &\qquad \times \overline{f_2}\left(\mathbf{t}+\frac{\boldsymbol{\tau}'}{2}\right) d^2\boldsymbol{\tau}'\, d^2\boldsymbol{\tau}\, d^2\mathbf{t}\, d^2\boldsymbol{\omega} \\ &= (2\pi)^2 \iiint_{\mathbb{R}^2} f_1\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g_1}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right) \\ &\qquad \times \delta(\tau_1'-\tau_1)\,\delta(\tau_2'-\tau_2) \\ &\qquad \times g_2\left(\mathbf{t}-\frac{\boldsymbol{\tau}'}{2}\right)\overline{f_2}\left(\mathbf{t}+\frac{\boldsymbol{\tau}'}{2}\right) d^2\boldsymbol{\tau}'\, d^2\boldsymbol{\tau}\, d^2\mathbf{t} \\ &= (2\pi)^2 \iint_{\mathbb{R}^2} f_1\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g_1}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right) \\ &\qquad \times g_2\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\overline{f_2}\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right) d^2\boldsymbol{\tau}\, d^2\mathbf{t}.\end{aligned} \quad (75)$$

Letting $\mathbf{t} + (\boldsymbol{\tau}/2) = \mathbf{x}$, $d\boldsymbol{\tau} = 4d\mathbf{x}$ and putting $\mathbf{t} - (\boldsymbol{\tau}/2) = \mathbf{y}$, $d\mathbf{t} = d\mathbf{y}$

$$\begin{aligned}&\iint_{\mathbb{R}^2} \mathscr{W}^R_{f_1,g_1}(\mathbf{t},\boldsymbol{\omega}) \overline{\mathscr{W}^R_{f_2,g_2}(\mathbf{t},\boldsymbol{\omega})} d^2\mathbf{t}\, d^2\boldsymbol{\omega} \\ &\quad = (2\pi)^2 4 \iint_{\mathbb{R}^2} f_1(\mathbf{x})\,\overline{g_1}(\mathbf{y})\, g_2(\mathbf{y})\,\overline{f_2}(\mathbf{x})\, d^2\mathbf{x}\, d^2\mathbf{y}.\end{aligned} \quad (76)$$

Interchanging the order of integration gives

$$\begin{aligned}&\iint_{\mathbb{R}^2} \mathscr{W}^R_{f_1,g_1}(\mathbf{t},\boldsymbol{\omega}) \overline{\mathscr{W}^R_{f_2,g_2}(\mathbf{t},\boldsymbol{\omega})} d^2\mathbf{t}\, d^2\boldsymbol{\omega} \\ &\quad = (4\pi)^2 \int_{\mathbb{R}^2} f_1(\mathbf{x}) \int_{\mathbb{R}^2} \overline{g_1}(\mathbf{y})\, g_2(\mathbf{y})\, d^2\mathbf{y}\, \overline{f_2}(\mathbf{x})\, d^2\mathbf{x} \\ &\quad = (4\pi)^2 \left(f_1 (\overline{g_1}, \overline{g_2})_{L^2(\mathbb{R}^2;\mathbb{H})}, f_2\right)_{L^2(\mathbb{R}^2;\mathbb{H})}.\end{aligned} \quad (77)$$

This completes the proof of the theorem. □

Based on the above theorem, we may conclude the following important consequences.

(i) If $g_1 = g_2$, then

$$\begin{aligned}&\iint_{\mathbb{R}^2} \mathscr{W}^R_{f_1,g_1}(\mathbf{t},\boldsymbol{\omega}) \overline{\mathscr{W}^R_{f_2,g_1}(\mathbf{t},\boldsymbol{\omega})} d^2\mathbf{t}\, d^2\boldsymbol{\omega} \\ &\quad = (4\pi)^2 \|g_1\|_{L^2(\mathbb{R}^2;\mathbb{H})} (f_1, f_2)_{L^2(\mathbb{R}^2;\mathbb{H})}.\end{aligned} \quad (78)$$

This formula is quite similar to Moyal's formula for the classical WVD, for example; see [1, 3]. However, we must remember that (78) is a quaternion-valued function.

(ii) If $f_1 = f_2$, then

$$\begin{aligned}&\iint_{\mathbb{R}^2} \mathscr{W}^R_{f_1,g_1}(\mathbf{t},\boldsymbol{\omega}) \overline{\mathscr{W}^R_{f_1,g_2}(\mathbf{t},\boldsymbol{\omega})} d^2\mathbf{t}\, d^2\boldsymbol{\omega} \\ &\quad = (4\pi)^2 \left(f_1 (\overline{g_1}, \overline{g_2})_{L^2(\mathbb{R}^2;\mathbb{H})}, f_1\right)_{L^2(\mathbb{R}^2;\mathbb{H})}.\end{aligned} \quad (79)$$

(iii) If $f_1 = f_2$ and $g_1 = g_2$, then

$$\begin{aligned}&\iint_{\mathbb{R}^2} \left|\mathscr{W}^R_{f_1,g_1}(\mathbf{t},\boldsymbol{\omega})\right|^2 d^2\mathbf{t}\, d^2\boldsymbol{\omega} \\ &\quad = (4\pi)^2 \|f_1\|^2_{L^2(\mathbb{R}^2;\mathbb{H})} \|g_1\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}.\end{aligned} \quad (80)$$

This formula has the same form as the specific Moyal formula for the two-sided QWVD (54) and also the classical WVD.

5. Quaternion Ambiguity Function (QAF)

The classical ambiguity function (AF) is firstly introduced by Woodward in 1953 for mathematical analysis of sonar and radar signals [1]. This section will generalize the classical AF in the quaternion algebra setting.

Definition 24. The cross two-sided quaternionic ambiguity function (QAF) of the two-dimensional functions (or signals) $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$ is denoted by $\mathscr{A}_{f,g}(\mathbf{t}, \boldsymbol{\omega})$ and is defined by

$$\mathscr{A}_{f,g}^{D}(\mathbf{t}, \boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\tau_1} f\left(\boldsymbol{\tau} + \frac{\mathbf{t}}{2}\right) \overline{g}\left(\boldsymbol{\tau} - \frac{\mathbf{t}}{2}\right) e^{-\mathbf{j}\omega_2\tau_2}\, d^2\boldsymbol{\tau}, \tag{81}$$

provided the integral exists.

The following lemma describes the relationship between the two-sided QWVD and the two-sided QAF mentioned above.

Lemma 25. *The two-sided QWVD of the signal $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$ can be seen as the two-sided QAF of the signals $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$ by formula*

$$\mathscr{A}_{f,g}^{D}(2\mathbf{t}, 2\boldsymbol{\omega}) = \frac{1}{4}\mathscr{W}_{f,g}^{D}(\mathbf{t}, \omega). \tag{82}$$

Proof. Putting $\boldsymbol{\tau} + (\mathbf{t}/2) = \mathbf{x}$, we may write (81) in the form

$$\begin{aligned}
&\mathscr{A}_{f,g}^{D}(\mathbf{t}, \boldsymbol{\omega}) \\
&\quad = \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1(x_1-(t_1/2))} f(\mathbf{x})\, \overline{g}(\mathbf{x}-\mathbf{t})\, e^{-\mathbf{j}\omega_2(x_2-(t_2/2))} d^2\mathbf{x} \\
&\quad = e^{\mathbf{i}(\omega_1 t_1/2)} \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1 x_1} f(\mathbf{x})\, \overline{g}(\mathbf{x}-\mathbf{t})\, e^{-\mathbf{j}\omega_2 x_2} d^2\mathbf{x}\, e^{\mathbf{j}(\omega_2 t_2/2)}.
\end{aligned} \tag{83}$$

It means that we have

$$\begin{aligned}
&4\mathscr{A}_{f,g}^{D}(2\mathbf{t}, 2\boldsymbol{\omega}) \\
&\quad = 4e^{\mathbf{i}2\omega_1 t_1} \int_{\mathbb{R}^2} e^{-\mathbf{i}2\omega_1 x_1} f(\mathbf{x})\, \overline{g}(-(\mathbf{x}-2\mathbf{t}))\, e^{-\mathbf{j}2\omega_2 x_2} d^2\mathbf{x}\, e^{\mathbf{j}2\omega_2 t_2} \\
&\quad = \mathscr{W}_{f,g}^{D}(\mathbf{t}, \boldsymbol{\omega}).
\end{aligned} \tag{84}$$

We should remember that if we write $h_\tau(\mathbf{t}) = f(\boldsymbol{\tau}+(\mathbf{t}/2))\overline{g}(\boldsymbol{\tau}-(\mathbf{t}/2))$, then the two-sided QWVD defined in (81) is the two-sided QFT of the function $h_\tau(\mathbf{t})$ with respect to $\mathbf{t}$. That is,

$$\begin{aligned}
\mathscr{A}_{f,g}^{D}(\mathbf{t}, \boldsymbol{\omega}) &= \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\tau_1} h_\tau(\mathbf{t})\, e^{-\mathbf{j}\omega_2\tau_2} d^2\boldsymbol{\tau} \\
&= \mathscr{F}_q^{D}\{h_\tau(\mathbf{t})\}(\boldsymbol{\omega}),
\end{aligned} \tag{85}$$

which was to be proved. □

We also obtain the following results which correspond to classical WVD properties (compared to [1, 2]).

Lemma 26. *For $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$, the two-sided QFT of* (81) *with respect to $\boldsymbol{\omega}$ can be represented in the form*

$$\begin{aligned}
\mathscr{F}_q^{D}\left\{\mathscr{A}_{f,g}^{D}\right\}(\mathbf{t}, \boldsymbol{\sigma}) &= \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\sigma_1} \mathscr{A}_{f,g}^{D}(\mathbf{t}, \boldsymbol{\omega})\, e^{-\mathbf{j}\omega_2\sigma_2} d^2\boldsymbol{\omega} \\
&= (2\pi)^2 f\left(-\boldsymbol{\sigma} + \frac{\mathbf{t}}{2}\right) \overline{g}\left(-\boldsymbol{\sigma} - \frac{\mathbf{t}}{2}\right).
\end{aligned} \tag{86}$$

Proof. Direct calculations yield

$$\begin{aligned}
&\mathscr{F}_q^{D}\left\{\mathscr{A}_{f,g}^{D}\right\}(\mathbf{t}, \boldsymbol{\sigma}) \\
&\quad \overset{(11)}{=} \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\sigma_1} \mathscr{A}_{f,g}(\mathbf{t}, \boldsymbol{\omega})\, e^{-\mathbf{j}\omega_2\sigma_2} d^2\boldsymbol{\omega} \\
&\quad \overset{(85)}{=} \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\sigma_1} \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\tau_1} h_t(\boldsymbol{\tau})\, e^{-\mathbf{j}\omega_2\tau_2} d^2\boldsymbol{\tau} e^{-\mathbf{j}\omega_2\sigma_2} d^2\boldsymbol{\omega} \\
&\quad = \iint_{\mathbb{R}^2} e^{\mathbf{i}\omega_1(-\tau_1-\sigma_1)} h_t(\boldsymbol{\tau})\, e^{\mathbf{j}\omega_2(-\tau_2-\sigma_2)} d^2\boldsymbol{\omega}\, d^2\boldsymbol{\tau} \\
&\quad = (2\pi)^2 h_t(-\boldsymbol{\sigma}) = (2\pi)^2 f\left(-\boldsymbol{\sigma} + \frac{\mathbf{t}}{2}\right) \overline{g}\left(-\boldsymbol{\sigma} - \frac{\mathbf{t}}{2}\right),
\end{aligned} \tag{87}$$

which was to be proved. □

The following theorem shows that the quaternion signal can be recovered from the two-sided QAF up to a quaternion constant.

Theorem 27 (reconstruction formula for two-sided QAF). *The inverse transform of the cross two-sided QWVD of the signal $f \in L^2(\mathbb{R}^2; \mathbb{H})$ is given by*

$$f(t) = \frac{1}{(2\pi)^2 \overline{g}(\mathbf{0})} \int_{\mathbb{R}^2} e^{\mathbf{i}(\omega_1 t_1/2)} \mathscr{A}_{f,g}^{D}(\mathbf{t}, \boldsymbol{\omega})\, e^{\mathbf{j}(\omega_2 t_2/2)} d^2\boldsymbol{\omega}, \tag{88}$$

provided $\overline{g}(\mathbf{0}) \neq \mathbf{0}$.

Proof. We have from the inverse transform of the two-sided QFT (12)

$$f\left(\boldsymbol{\tau} + \frac{\mathbf{t}}{2}\right) \overline{g}\left(\boldsymbol{\tau} - \frac{\mathbf{t}}{2}\right) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{\mathbf{i}\omega_1\tau_1} \mathscr{A}_{f,g}^{D}(\mathbf{t}, \boldsymbol{\omega})\, e^{\mathbf{j}\omega_2\tau_2} d^2\boldsymbol{\omega}. \tag{89}$$

Taking the specific value, $\boldsymbol{\tau} = \mathbf{t}/2$, we have

$$f(\mathbf{t})\, \overline{g}(\mathbf{0}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{\mathbf{i}(\omega_1 t_1/2)} \mathscr{A}_{f,g}^{D}(\mathbf{t}, \boldsymbol{\omega})\, e^{\mathbf{j}(\omega_2 t_2/2)} d^2\boldsymbol{\omega}. \tag{90}$$

Or, equivalently,

$$f(t) = \frac{1}{(2\pi)^2 \overline{g}(\mathbf{0})} \int_{\mathbb{R}^2} e^{\mathbf{i}(\omega_1 t_1/2)} \mathscr{A}_{f,g}^{D}(\mathbf{t}, \boldsymbol{\omega})\, e^{\mathbf{j}(\omega_2 t_2/2)} d^2\boldsymbol{\omega}, \tag{91}$$

which completes the proof. □

5.1. Useful Properties of Two-Sided QAF. The properties of the two-sided QAF are summarized in the following proposition. It seem that they have a remarkable similarity with those of the two-sided QWVD.

Proposition 28. *For $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$ the two-sided QAF has the following important properties.*

(i) *Shift. One has*

$$\mathscr{A}_{T_\mathbf{a} f, T_\mathbf{a} g}^{D}(\mathbf{t}, \boldsymbol{\omega}) = e^{-\mathbf{i}\omega_1 a_1} \mathscr{A}_{f,g}^{D}(\mathbf{t}, \boldsymbol{\omega})\, e^{-\mathbf{j}\omega_2 a_2}. \tag{92}$$

(ii) *Nonlinearity. One has*

$$\mathscr{A}^D_{f+g}(\mathbf{t},\boldsymbol{\omega}) = \mathscr{A}^D_{f}(\mathbf{t},\boldsymbol{\omega}) + \mathscr{A}^D_{f,g}(\mathbf{t},\boldsymbol{\omega}) + \mathscr{A}^D_{g,f}(\mathbf{t},\boldsymbol{\omega}) + \mathscr{A}^D_{g}(\mathbf{t},\boldsymbol{\omega}). \tag{93}$$

(iii) *Modulation. One has*

$$\begin{aligned}\mathscr{A}^D_{M_{\omega_0}f,g}(\mathbf{t},\boldsymbol{\omega}) &= e^{\mathbf{i}u_0t_1}\mathscr{A}^D_{f,g_0}\left(\mathbf{t},\boldsymbol{\omega}-\frac{\boldsymbol{\omega}_0}{2}\right)e^{\mathbf{j}v_0t_2} \\ &\quad - e^{\mathbf{i}u_0t_1}\mathscr{A}_{f,g_1}\left(\mathbf{t},\omega_1-\frac{u_0}{2},\omega_2+\frac{v_0}{2}\right)e^{-\mathbf{j}v_0t_2}\mathbf{i} \\ &\quad - e^{\mathbf{i}u_0t_1}\mathscr{A}^D_{f,g_2}\left(\mathbf{t},\boldsymbol{\omega}-\frac{\boldsymbol{\omega}_0}{2}\right)e^{\mathbf{j}v_0t_2}\mathbf{j} \\ &\quad - e^{\mathbf{i}u_0t_1}\mathscr{A}_{f,g_3}\left(\mathbf{t},\omega_1-\frac{u_0}{2},\omega_2+\frac{v_0}{2}\right)e^{-\mathbf{j}v_0t_2}\mathbf{k}.\end{aligned} \tag{94}$$

(iv) *Dilation. One has*

$$\mathscr{A}^D_{\mathscr{D}_{cf},\mathscr{D}_{cg}}(\mathbf{t},\boldsymbol{\omega}) = \mathscr{A}^D_{f,g}\left(\frac{\mathbf{t}}{\mathbf{c}},\mathbf{c}\boldsymbol{\omega}\right). \tag{95}$$

Proof. For part (i), using the definition of the two-sided QAF (81), we obtain

$$\begin{aligned}&\mathscr{A}_{T_{\mathbf{a}}f,T_{\mathbf{a}}g}(\mathbf{t},\boldsymbol{\omega}) \\ &= \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\tau_1} f\left(\boldsymbol{\tau}-\mathbf{a}+\frac{\mathbf{t}}{2}\right)\overline{g}\left(\boldsymbol{\tau}-\mathbf{a}-\frac{\mathbf{t}}{2}\right)e^{-\mathbf{j}\omega_2\tau_2}d^2\boldsymbol{\tau} \\ &= \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1(\tau_1-a_1+a_1)} f\left(\boldsymbol{\tau}-\mathbf{a}+\frac{\mathbf{t}}{2}\right) \\ &\qquad\times\overline{g}\left(\boldsymbol{\tau}-\mathbf{a}-\frac{\mathbf{t}}{2}\right)e^{-\mathbf{j}\omega_2(\tau_2-a_2+a_2)}d^2\boldsymbol{\tau} \\ &= e^{-\mathbf{i}\omega_1a_1}\int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1(\tau_1-a_1)} f\left(\boldsymbol{\tau}-\mathbf{a}+\frac{\mathbf{t}}{2}\right) \\ &\qquad\times\overline{g}\left(\boldsymbol{\tau}-\mathbf{a}-\frac{\mathbf{t}}{2}\right)e^{-\mathbf{j}\omega_2(\tau_2-a_2)}d^2\boldsymbol{\tau}\, e^{-\mathbf{j}\omega_2a_2} \\ &= e^{-\mathbf{i}\omega_1a_1}\mathscr{A}^D_{f,g}(\mathbf{t},\boldsymbol{\omega})e^{-\mathbf{j}\omega_2a_2}.\end{aligned} \tag{96}$$

To derive part (iii), we apply (81) and the decomposition of quaternion function to get

$$\begin{aligned}&\mathscr{A}^D_{M_{\omega_0}f,g}(\mathbf{t},\boldsymbol{\omega}) \\ &= \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1\tau_1}\left(e^{\mathbf{i}u_0(\tau_1+(t_1/2))} f\left(\boldsymbol{\tau}+\frac{\mathbf{t}}{2}\right)e^{\mathbf{j}v_0(\tau_2+(t_2/2))}\right. \\ &\qquad\left.\times\overline{g}\left(\boldsymbol{\tau}-\frac{\mathbf{t}}{2}\right)\right)e^{-\mathbf{j}\omega_2\tau_2}d^2\boldsymbol{\tau} \\ &= \int_{\mathbb{R}^2} e^{\mathbf{i}(u_0t_1/2)}\left(e^{-\mathbf{i}(\omega_1-u_0)\tau_1} f\left(\boldsymbol{\tau}+\frac{\mathbf{t}}{2}\right)e^{\mathbf{j}v_0(\tau_2+(t_2/2))}\right. \\ &\qquad\times\left(\overline{g_0}\left(\boldsymbol{\tau}-\frac{\mathbf{t}}{2}\right)-\mathbf{i}\overline{g_1}\left(\boldsymbol{\tau}-\frac{\mathbf{t}}{2}\right)\right. \\ &\qquad-\mathbf{j}\overline{g_2}\left(\boldsymbol{\tau}-\frac{\mathbf{t}}{2}\right) \\ &\qquad\left.\left.-\mathbf{k}\overline{g_3}\left(\boldsymbol{\tau}-\frac{\mathbf{t}}{2}\right)\right)\right)e^{-\mathbf{j}\omega_2\tau_2}d^2\boldsymbol{\tau} \\ &= e^{\mathbf{i}(u_0t_1/2)}\left(\int_{\mathbb{R}^2} e^{-\mathbf{i}(\omega_1-(u_0/2))\tau_1} f\left(\boldsymbol{\tau}+\frac{\mathbf{t}}{2}\right)\right. \\ &\qquad\times\overline{g_0}\left(\boldsymbol{\tau}-\frac{\mathbf{t}}{2}\right)e^{-\mathbf{j}(\omega_2-(v_0/2))\tau_2}d^2\boldsymbol{\tau} \\ &\qquad-\int_{\mathbb{R}^2} e^{-\mathbf{i}(\omega_1-(u_0/2))\tau_1} f\left(\boldsymbol{\tau}+\frac{\mathbf{t}}{2}\right) \\ &\qquad\left.\times\overline{g_2}\left(\boldsymbol{\tau}-\frac{\mathbf{t}}{2}\right)e^{-\mathbf{j}(\omega_2-(v_0/2))\tau_2}d^2\boldsymbol{\tau}\right)e^{\mathbf{j}(v_0t_2/2)}\mathbf{j} \\ &\quad- e^{\mathbf{i}(u_0t_1/2)}\left(\int_{\mathbb{R}^2} e^{-\mathbf{i}(\omega_1-(u_0/2))\tau_1} f\left(\boldsymbol{\tau}+\frac{\mathbf{t}}{2}\right)\overline{g_1}\left(\boldsymbol{\tau}-\frac{\mathbf{t}}{2}\right)\right. \\ &\qquad\left.\times e^{\mathbf{j}(\omega_2+(v_0/2))\tau_2}d^2\boldsymbol{\tau}\right)e^{-\mathbf{j}(v_0t_2/2)}\mathbf{i} \\ &\quad- e^{\mathbf{i}(u_0t_1/2)}\left(\int_{\mathbb{R}^2} e^{-\mathbf{i}(\omega_1-(u_0/2))\tau_1} f\left(\boldsymbol{\tau}+\frac{\mathbf{t}}{2}\right)\overline{g_3}\left(\boldsymbol{\tau}-\frac{\mathbf{t}}{2}\right)\right. \\ &\qquad\left.\times e^{\mathbf{j}(\omega_2+(v_0/2))\tau_2}d^2\boldsymbol{\tau}\right)e^{-\mathbf{j}(v_0t_2/2)}\mathbf{k} \\ &= e^{\mathbf{i}(u_0t_1/2)}\int_{\mathbb{R}^2} e^{-\mathbf{i}(\omega_1-(u_0/2))\tau_1} f\left(\boldsymbol{\tau}+\frac{\mathbf{t}}{2}\right) \\ &\qquad\times\overline{g_0}\left(\boldsymbol{\tau}-\frac{\mathbf{t}}{2}\right)e^{-\mathbf{j}(\omega_2-(v_0/2))\tau_2}d^2\boldsymbol{\tau}e^{\mathbf{j}(v_0t_2/2)} \\ &\quad- e^{\mathbf{i}(u_0t_1/2)}\int_{\mathbb{R}^2} e^{-\mathbf{i}(\omega_1-(u_0/2))\tau_1} f\left(\boldsymbol{\tau}+\frac{\mathbf{t}}{2}\right)\overline{g_1}\left(\boldsymbol{\tau}-\frac{\mathbf{t}}{2}\right) \\ &\qquad\times e^{\mathbf{j}(\omega_2+(v_0/2))\tau_2}d^2\boldsymbol{\tau}\,e^{-\mathbf{j}(v_0t_2/2)}\mathbf{i} \\ &\quad- e^{\mathbf{i}(u_0t_1/2)}\int_{\mathbb{R}^2} e^{-\mathbf{i}(\omega_1-(u_0/2))\tau_1} f\left(\boldsymbol{\tau}+\frac{\mathbf{t}}{2}\right) \\ &\qquad\times\overline{g_2}\left(\boldsymbol{\tau}-\frac{\mathbf{t}}{2}\right)e^{-\mathbf{j}(\omega_2-(v_0/2))\tau_2}\,d^2\boldsymbol{\tau}\,e^{\mathbf{j}(v_0t_2/2)}\mathbf{j} \\ &\quad- e^{\mathbf{i}(u_0t_1/2)}\int_{\mathbb{R}^2} e^{-\mathbf{i}(\omega_1-(u_0/2))\tau_1} f\left(\boldsymbol{\tau}+\frac{\mathbf{t}}{2}\right) \\ &\qquad\times\overline{g_3}\left(\boldsymbol{\tau}-\frac{\mathbf{t}}{2}\right)e^{\mathbf{j}(\omega_2+(v_0/2))\tau_2}\,d^2\boldsymbol{\tau}e^{-\mathbf{j}(v_0t_2/2)}\mathbf{k} \\ &= e^{\mathbf{i}u_0t_1}\mathscr{A}^D_{f,g_0}\left(\mathbf{t},\boldsymbol{\omega}-\frac{\boldsymbol{\omega}_0}{2}\right)e^{\mathbf{j}v_0t_2} \\ &\quad- e^{\mathbf{i}u_0t_1}\mathscr{W}_{f,g_1}\left(\mathbf{t},\omega_1-\frac{u_0}{2},\omega_2+\frac{v_0}{2}\right)e^{-\mathbf{j}(v_0t_2/2)}\mathbf{i}\end{aligned}$$

$$- e^{\mathbf{i}(u_0 t_1/2)} \mathscr{A}^D_{f,g_2}\left(\mathbf{t}, \boldsymbol{\omega} - \frac{\boldsymbol{\omega}_0}{2}\right) e^{\mathbf{j}v_0 t_2}\mathbf{j}$$
$$- e^{\mathbf{i}(u_0 t_1/2)} \mathscr{W}_{f,g_3}\left(\mathbf{t}, \omega_1 - \frac{u_0}{2}, \omega_2 + \frac{v_0}{2}\right) e^{-\mathbf{j}(v_0 t_2/2)}\mathbf{k}. \quad (97)$$

For part (iv), let $\boldsymbol{\tau}/c = x$, we have, by definition,

$$\begin{aligned}
&\mathscr{A}^D_{\mathscr{D}_{cf},\mathscr{D}_{cg}}(\mathbf{t}, \boldsymbol{\omega}) \\
&= \frac{1}{c}\int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1 \tau_1} f\left(\frac{\mathbf{t}}{c} + \frac{\boldsymbol{\tau}}{2c}\right) \overline{g}\left(\frac{\mathbf{t}}{c} - \frac{\boldsymbol{\tau}}{2c}\right) e^{-\mathbf{j}\omega_2 \tau_2}\, d^2\boldsymbol{\tau} \\
&= \int_{\mathbb{R}^2} e^{-\mathbf{i}c\omega_1 x_1} f\left(\frac{\mathbf{t}}{c} + \frac{\mathbf{x}}{2}\right) \overline{g}\left(\frac{\mathbf{t}}{c} - \frac{\mathbf{x}}{2}\right) e^{-\mathbf{j}c\omega_2 x_2}\, d^2\mathbf{x} \\
&= \mathscr{A}^D_{f,g}\left(\frac{\mathbf{t}}{\mathbf{a}}, \mathbf{c}\boldsymbol{\omega}\right).
\end{aligned} \quad (98)$$

□

Theorem 29 (specific Moyal's formula for the two-sided QAF). *Suppose that $f \in L^2(\mathbb{R}^2;\mathbb{H})$ and $g \in L^2(\mathbb{R}^2;\mathbb{H})$ are two quaternion signals. Then the following equation holds:*

$$\iint_{\mathbb{R}^2} \left|\mathscr{A}^D_{f,g}(\mathbf{t}, \boldsymbol{\omega})\right|^2 d^2\boldsymbol{\omega}\, d^2\mathbf{t} = (4\pi)^2 \|f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})} \|g\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}. \quad (99)$$

Proof. Using Plancherel's formula for the right-sided QFT in (14) and then applying Lemma 26 we immediately obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left|\mathscr{A}^D_{f,g}(\mathbf{t}, \boldsymbol{\omega})\right|^2 d^2\boldsymbol{\omega} \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \left|\mathscr{F}^D_q\left\{\mathscr{A}^D_{f,g}\right\}(\mathbf{t}, \boldsymbol{\sigma})\right|^2 d^2\boldsymbol{\sigma} \\
&= (2\pi)^2 \int_{\mathbb{R}^2} f\left(-\boldsymbol{\sigma} + \frac{\mathbf{t}}{2}\right) \overline{g}\left(-\boldsymbol{\sigma} - \frac{\mathbf{t}}{2}\right) \\
&\qquad \times \overline{f\left(-\boldsymbol{\sigma} + \frac{\mathbf{t}}{2}\right) \overline{g}\left(-\boldsymbol{\sigma} - \frac{\mathbf{t}}{2}\right)} d^2\boldsymbol{\sigma} \\
&= (2\pi)^2 \int_{\mathbb{R}^2} f\left(-\boldsymbol{\sigma} + \frac{\mathbf{t}}{2}\right) \overline{g}\left(-\boldsymbol{\sigma} - \frac{\mathbf{t}}{2}\right) \\
&\qquad \times g\left(-\boldsymbol{\sigma} - \frac{\mathbf{t}}{2}\right) \overline{f}\left(-\boldsymbol{\sigma} + \frac{\mathbf{t}}{2}\right) d^2\boldsymbol{\sigma}.
\end{aligned} \quad (100)$$

Integrating (100) with respect to $d^2\mathbf{t}$ yields

$$\begin{aligned}
&\iint_{\mathbb{R}^2} \left|\mathscr{A}^D_{f,g}(\mathbf{t}, \boldsymbol{\omega})\right|^2 d^2\boldsymbol{\omega}\, d^2\mathbf{t} \\
&= (2\pi)^2 \iint_{\mathbb{R}^2} f\left(-\boldsymbol{\sigma} + \frac{\mathbf{t}}{2}\right) \overline{g}\left(-\boldsymbol{\sigma} - \frac{\mathbf{t}}{2}\right) g\left(-\boldsymbol{\sigma} - \frac{\mathbf{t}}{2}\right) \\
&\qquad \times \overline{f}\left(-\boldsymbol{\sigma} + \frac{\mathbf{t}}{2}\right) d^2\boldsymbol{\sigma}\, d^2\mathbf{t}.
\end{aligned} \quad (101)$$

Putting $-\boldsymbol{\sigma} + (\mathbf{t}/2) = \mathbf{x}$ and $-\boldsymbol{\sigma} - (\mathbf{t}/2) = \mathbf{y}$, then the right side of the above expression can be rewritten as

$$\begin{aligned}
&\iint_{\mathbb{R}^2} \left|\mathscr{A}^D_{f,g}(\mathbf{t}, \boldsymbol{\omega})\right|^2 d^2\boldsymbol{\omega}\, d^2\mathbf{t} \\
&= (4\pi)^2 \iint_{\mathbb{R}^2} f(\mathbf{x})\, \overline{g}(\mathbf{y})\, g(\mathbf{y})\, \overline{f}(\mathbf{x})\, d^2\mathbf{y}\, d^2\mathbf{x} \\
&= (4\pi)^2 \iint_{\mathbb{R}^2} f(\mathbf{x}) \left|g(\mathbf{y})\right|^2 \overline{f}(\mathbf{x})\, d^2\mathbf{y}\, d^2\mathbf{x}.
\end{aligned} \quad (102)$$

After switching the order of integration, we get

$$\iint_{\mathbb{R}^2} \left|\mathscr{A}^D_{f,g}(\mathbf{t}, \boldsymbol{\omega})\right|^2 d^2\boldsymbol{\omega}\, d^2\mathbf{t} = (4\pi)^2 \|f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})} \|g\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}. \quad (103)$$

This is the desired result. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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