Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2014, Article ID 138087, 6 pages
http://dx.doi.org/10.1155/2014/138087

Research Article

Modified Differential Transform Method for Two Singular Boundary Values Problems

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Received 6 September 2013; Accepted 17 November 2013; Published 5 January 2014

Academic Editor: Ying Hu

This paper deals with the two singular boundary values problems of second order. Two singular points are both boundary values points of the differential equation. The numerical solutions are developed by modified differential transform method (DTM) for expanded point. Linear and nonlinear models are solved by this method to get more reliable and efficient numerical results. It can also solve ordinary differential equations where the traditional one fails. Besides, we give the convergence of this new method.

1. Introduction

In the present paper, we consider the following two singular boundary value problems (BVPs) of second order:

$$\begin{gathered} u''(t) + \frac{f(t)}{t(t-1)}u'(t) + \frac{g(t)}{t(t-1)}N(u(t)) \\ = \frac{h(t)}{t(t-1)}, \quad 0 < t < 1,\ u(0) = p,\ u(1) = q, \end{gathered} \tag{1}$$

where $f(t)$, $g(t)$, and $h(t)$ are known and continuous functions, $t \in (0,1)$, and $N(u)$ is a nonlinear function of u. The equation is singular at these two boundary values $t = 0, 1$.

Scientists and engineers have been interested in the singular equation because of its importance in many applications such as physical and mathematical models. There are many research directions on these equations. Some studied their qualitative properties [1, 2]. For example, Bartolucci and Montefusco [1] studied the concentration-compactness problem and the mass quantization properties. Some others used theorems to establish the existence and uniqueness of solution [3, 4]. For example, Guo et al. [3] got the existence and uniqueness of solution using a fixed point theorem.

Recently great attention had been paid to numerical solutions [5–13]. For example, Duan and Rach [5] solved boundary value problems using a new modified Adomian decomposition method. Chowdhury and Hashim [7] used homotopy asymptotic method for finding the approximate solutions. Wazwaz [6, 8, 9] used Adomian decomposition method to get the numerical solutions.

Puhov in 1976 [14] proposed the concept of DTM. DTM is the extension of Taylor series method and had been applied to solve analytic solutions of ordinary [15, 16], partial [17–19], differential-algebraic equations [20, 21], differential-difference equations [22, 23], and integrodifferential equations [24, 25]. Numerical solutions are also obtained [26]. Furthermore, Alquran and Al-Khaled [27] applied DTM to solve some eigenvalue problems.

In the present paper, there are two singular points and these two singular points are just boundary values of the equation. Traditional DTM only can solve one-point singular BVP or two-point BVP (but not both the two points are just boundary values). We add some operation properties of DTM; then DTM can be used to calculate this type of problem. Furthermore, we have a convergent analysis of this method.

2. Modified Technique

Now, we have a brief description of standard DTM.

Let $u(t)$ be an analytic function in a domain D and let $t = a$ represent any point in D. The function $u(t)$ is then represented by a power series whose center is located at a. Then Taylor series expansion of $u(t)$ is expressed as

$$u(t) = \sum_{k=0}^{\infty} \frac{(t-a)^k}{k!} \left[\frac{d^k u(t)}{d^k} \right]_{t=a}, \quad t \in D. \tag{2}$$

The particular case of (2) when $a = 0$ is referred to as the Maclaurin series expressed as

$$u(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left[\frac{d^k u(t)}{d^k} \right]_{t=0}, \quad t \in D. \tag{3}$$

The differential transformed function $u(t)$ is defined as

$$U(k) \equiv \frac{H^k}{k!} \left[\frac{d^k(t)}{dt^k} \right]_{t=0}, \quad k = 0, 1, 2, \ldots, \tag{4}$$

where $U(k)$ represents the transformed function and $u(t)$ is the original function. The differential spectrum of $U(k)$ is confined in the interval $t \in [0, H]$; H is the given constant number.

The differential inverse transform of $U(k)$ is defined as

$$u(t) = \sum_{k=0}^{\infty} \left(\frac{t}{H} \right)^k U(k). \tag{5}$$

From the definition, the differential transformation technique is based upon the Taylor series expansion. In real applications, it is found that the number of arguments required to restore the unknown function precisely can be reduced by specifying an appropriate value of the constant H. In other words, the function $u(t)$ can be expressed in terms of a finite series as follows:

$$u(t) = \sum_{k=0}^{n} \left(\frac{t}{H} \right)^k U(k). \tag{6}$$

Next, we state some important properties of the Taylor differential transformation derived using the expressions in (4) and (5) which are needed in the sequel.

3. The Operation Properties of Differential Transformation

If $U(k)$ and $V(k)$ are the transformed functions corresponding to $u(t)$ and $v(t)$, then the fundamental mathematical operations of differential transformation are listed as follows. (For 9–12 is new, the proof of the others can see references.)

(1) If $z(t) = u(t) \pm v(t)$ then $Z(k) = U(k) \pm V(k)$.

(2) If $z(t) = au(t)$ then $Z(k) = aU(k)$.

(3) If $z(t) = d^m u(t)/dt^m$ then $Z(k) = (k+1)(k+2)\cdots(k+m)U(k+m)$.

(4) If $z(t) = u(t)v(t)$ then $Z(k) = \sum_{i=0}^{k} u(i)v(k-i)$.

(5) If $z(t) = t^m$ then

$$Z(k) = \delta(k-m) = \begin{cases} 1, & k = m, \\ 0, & k \neq m. \end{cases} \tag{7}$$

(6) If $z(t) = \exp(bt)$ then $Z(k) = b^k/k!$.

(7) If $z(t) = \sin(ct+a)$ then $Z(k) = (c^k/k!)\sin((\pi k/2)+a)$.

(8) If $z(t) = \cos(ct+a)$ then $Z(k) = (c^k/k!)\cos((\pi k/2)+a)$.

Next, the following lemmas are new, so we have to prove them.

(9) If $z(t) = t^n d(t)$, then $Z(k) = D(k-n)$.

Proof. For simplicity, letting $H = 1$

$$\begin{aligned} d(t) &= \sum_{k=0}^{\infty} t^k D(k) \\ &\Longrightarrow t^n d(t) = \sum_{k=0}^{\infty} t^{n+k} D(k) = \sum_{i=n}^{\infty} t^i D(i-n) \\ &= \sum_{k=n}^{\infty} t^k D(k-n). \end{aligned} \tag{8}$$

(10) Consider

$$z(t) = t^2 \frac{d^2 u(t)}{dt^2} \Longrightarrow Z(k) = k(k-1)U(k). \tag{9}$$

□

Proof. For $y(t) = d^2u(t)/dt^2 \Rightarrow Y(k) = (k+2)(k+1)U(k+2)$; for $z(t) = t^2 y(t) \Rightarrow Z(k) = Y(k-2) = k(k-1)U(k)$.

(11) One has

$$z(t) = t \frac{d^2 u(t)}{dt^2} \Longrightarrow Z(k) = (k+1)kU(k+1). \tag{10}$$

□

Proof. Similar method as (9).

(12) One has

$$z(t) = t \frac{du(t)}{dt} \Longrightarrow Z(k) = kU(k). \tag{11}$$

□

Proof. Similar method as (9). □

4. Convergence Analysis

Theorem 1. *Consider the following two singularly linear BVPs:*

$$t(1-t)u''(t) + (1-t)u'(t) + g(t)u(t) = f(t), \quad 0 < t < 1, \tag{12}$$

where $f(t) = f_0 + f_1 t + f_2 t^2 + \cdots$ and $g(t) = g_0 + g_1 t + g_2 t^2 + \cdots$.

If there exists a fixed n such that $n \geq m$, $|f_k| \leq Mr^k$ for some fixed M, $0 < r < 1$, all $k \geq n$, and $U(n) \leq Mr^n$, then the numerical solution using the present method absolutely converges.

Proof. For simplicity, let $g(t) = g_0$. The general case is similar to the special case.

By using (9) and (10), we have

$$t^2 u''(t) \Longrightarrow k(k-1)U(k)$$
$$tu''(t) \Longrightarrow (k+1)kU(k+1). \tag{13}$$

So the differential transformation of (12) is

$$(k+1)kU(k+1) - k(k-1)U(k) + (k+1)U(k+1) - kU(k) + g_0 U(k) = F(k), \quad k \geq m, \tag{14}$$

for fixed m. So we have

$$U(k+1) = \frac{1}{(k+1)^2}\left[F(k) + \left(k^2 + g_0\right)U(k)\right]. \tag{15}$$

Suppose $|U(k)| \leq Mr^k$ is true for all $k \geq n \geq m$. From (15), we have

$$|U(k+1)| \leq \frac{1}{(k+1)^2}\left[Mr^k + \left(k^2 + g_0\right)Mr^k\right] = Mr^k\left[\frac{1 + k^2 + |g_0|}{(k+1)^2}\right]. \tag{16}$$

Let $r = ((1 + k^2 + |g_0|)/(k+1)^2) < 1$ for k is large enough. By induction, the hypothesis is true. So we have

$$|u(t)| \leq \sum_{k=0}^{\infty} t^k |U(k)| \leq \sum_{k=0}^{\infty} |U(k)| \leq M\sum_{k=0}^{\infty} r^k = \frac{M}{1-r}. \tag{17}$$

Next we have the theorem for nonlinear BVP. □

Theorem 2. *Consider the following two singularly nonlinear BVPs*

$$t(1-t)u''(t) + (1-t)u'(t) + g(t)u(t) + u(t)^2 = f(t), \quad 0 < t < 1, \tag{18}$$

where $f(t) = f_0 + f_1 t + f_2 t^2 + \cdots$, $g(t) = g_0 + g_1 t + g_2 t^2 + \cdots$.

If there exists a fixed n such that $n \geq m$, $|f_k| \leq r^k$, $0 < r < 1$, all $k \geq n$, and $U(n) \leq r^n$, then the numerical solution using the present method absolutely converges.

Proof. For simplicity, let $g(t) = g_0$. The general case is similar to the special case.

So the differential transformation of (18) is

$$(k+1)kU(k+1) - k(k-1)U(k) + (k+1)U(k+1) - kU(k) + g_0 U(k) + B(k) = F(k), \quad k \geq m, \tag{19}$$

for fixed m, $B(k) = U(0)U(k) + U(1)U(k-1) + \cdots + U(k)U(0)$. So we have

$$U(k+1) = \frac{1}{(k+1)^2}\left[F(k) + \left(k^2 + g_0\right)U(k) - B(k)\right]. \tag{20}$$

Suppose $|U(k)| \leq r^k$ is true for all $k \geq n \geq m$. So we have

$$|U(i)U(k-i)| \leq r^i r^{k-i} = r^k \Longrightarrow |B(k)| \leq (k+1)r^k. \tag{21}$$

From(20), we have

$$|U(k+1)| \leq \frac{1}{(k+1)^2}\left[r^k + \left(k^2 + g_0\right)r^k + (k+1)r^k\right] = r^k\left[\frac{1 + k^2 + |g_0| + k + 1}{(k+1)^2}\right]. \tag{22}$$

Let $r = ((1 + k^2 + |g_0| + k + 1)/(k+1)^2) < 1$ for k is large enough. By induction, the hypothesis is true. So we have

$$|u(t)| \leq \sum_{k=0}^{\infty} t^k |U(k)| \leq \sum_{k=0}^{\infty} |U(k)| \leq \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}. \tag{23}$$

□

5. Numerical Examples

Differential transformation method (DTM) is used to solve the following examples. Some numerical results are also compared with RKHSM in [28]. The algorithm is performed by software with 16-digit precision.

Example 1. Consider the following two singularly linear BVPs:

$$t(1-t)u''(t) + (1-t)u'(t) + u(t) = f(t), \quad 0 < t < 1, \tag{24}$$

with boundary values $u(0) = 0$, $u(1) = 1$, and $f(t) = t(4-3t)$. The exact solution is $u(t) = t^2$.

By using (9) and (10), we have

$$t^2 u''(t) \Longrightarrow k(k-1)U(k),$$
$$tu''(t) \Longrightarrow (k+1)kU(k+1). \tag{25}$$

So the differential transformation of (24) is

$$(k+1)kU(k+1) - k(k-1)U(k) + (k+1)U(k+1) - kU(k) + U(k) = -3\delta(k-2) + 4\delta(k-1), \tag{26}$$

where

$$\delta(k-m) = \begin{cases} 1, & k = m, \\ 0, & k \neq m, \end{cases} \tag{27}$$

for $k \geq 2$. One has

$$\Longrightarrow U(k+1) = \frac{1}{(k+1)^2}\left[\left(k^2 - 1\right)U(k) - 3\delta(k-2) + 4\delta(k-1)\right]. \tag{28}$$

Table 1: Comparison of relative errors of the present method for Example 2.

x	True solution $u(x)$	Geng [28] $U_{5,20}$	Present DTM$_2$ method
0.08	0.0064	$1.91E-06$	0
0.16	0.0256	$1.54E-06$	0
0.24	0.0576	$1.58E-06$	0
0.32	0.1024	$1.59E-06$	0
0.48	0.2304	$1.20E-06$	0
0.64	0.4096	$3.96E-07$	0
0.80	0.6400	$6.07E-08$	0
0.96	0.9216	$8.13E-09$	0

For the coefficient of constant and t, because $u(0) = 0$, let $u(t) = U(1)t + U(2)t^2$; substituting in (24), we have

$$\left(t-t^2\right) 2U(2) + (1-t)\left[U(1) + 2U(2)t\right] + U(1)t. \quad (29)$$

The coefficient of constant is

$$U(1) = 0. \quad (30)$$

The coefficient of t that is $k = 1$ is

$$\begin{gathered} 2U(2) - U(1) + 2U(2) + U(1) = 4 \\ \Longrightarrow U(2) = 1. \end{gathered} \quad (31)$$

As $k = 2$ in (28),

$$U(3) = \frac{3U(2) - 3}{9} = 0. \quad (32)$$

For $k = 3, 4, 5, \ldots$,

$$U(k+1) = 0. \quad (33)$$

Then we have the numerical solution of the present method in Example 1:

$$u(t) = t^2. \quad (34)$$

It is also the exact solution and Table 1 presents the results.

Example 2. Consider the following two singularly linear BVPs in [28] Example 1:

$$\begin{gathered} t(1-t)u''(t) + (1-t)u'(t) + tu(t) \\ + t(1-t)u^2(t) = f(t), \quad 0 < t < 1, \end{gathered} \quad (35)$$

with boundary values $u(0) = 0$, $u(1) = 1$, and $f(t) = t(4 - 4t + t^2 + t^4 - t^5)$. The exact solution is $u(t) = t^2$.

The differential transformation of (35) is

$$\begin{aligned} & k(k+1)U(k) - k(k-1)U(k) + (k+1)U(k+1) \\ & \quad - kU(k) + U(k-1) + B(k-1) - B(k-2) \\ & = 4\delta(k-1) - 4\delta(k-2) + \delta(k-3) \\ & \quad + \delta(k-5) - \delta(k-6), \end{aligned} \quad (36)$$

where $B(k) = U(0)U(k) + U(1)U(k-1) + \cdots + U(k)U(0)$ for $k \geq 2$. Consider

$$\begin{aligned} & \Longrightarrow U(k+1) \\ & = \frac{1}{(k+1)^2}\left[k^2U(k) - U(k-1) + B(k-2)\right. \\ & \quad - B(k-1) + 4\delta(k-1) - 4\delta(k-2) \\ & \quad \left. + \delta(k-3) + \delta(k-5) - \delta(k-6)\right]. \end{aligned} \quad (37)$$

For the coefficient of constant and t, because $u(0) = 0$, let $u(t) = U(1)t + U(2)t^2$; substituting in (35), we have

$$\left(t-t^2\right) 2U(2) + (1-t)\left[U(1) + 2U(2)t\right] + tU(1)t. \quad (38)$$

The coefficient of constant is

$$U(1) = 0. \quad (39)$$

The coefficient of t that is $k = 1$ is

$$\begin{gathered} 2U(2) - U(1) + 2U(2) = 4 \\ \Longrightarrow U(2) = 1. \end{gathered} \quad (40)$$

As $k = 2$ in (37),

$$U(3) = \frac{4U(2) - U(1) + B(0) - B(1) - 4}{9} = 0. \quad (41)$$

As $k = 3$,

$$U(4) = \frac{9U(3) - U(2) + B(1) - B(2) + 1}{16} = 0. \quad (42)$$

For $k = 4, 5, \ldots$,

$$U(k+1) = 0. \quad (43)$$

Then we have the numerical solution of the present method in Example 2 as

$$u(t) = t^2. \quad (44)$$

It is also the exact solution.

Example 3. Consider the following two singularly linear BVPs:

$$\begin{gathered} t(1-t)u''(t) + (1-t)u'(t) + \left(1 + t^2 - t^3\right)u(t) = f(t), \\ 0 < t < 1, \end{gathered} \quad (45)$$

with boundary values $u(0) = 1$, $u(1) = e$, and $f(t) = (2 - t^3)e^t$. The exact solution is $u(t) = e^t$.

The differential transformation of (45) is

$$\begin{aligned} & k(k+1)U(k) - k(k-1)U(k) + (k+1)U(k+1) - kU(k) \\ & \quad + U(k) + U(k-2) - U(k-3) = \frac{2}{k!} - \frac{1}{(k-3)!}, \end{aligned} \quad (46)$$

Table 2: Comparison of relative errors of the present method for Example 3.

x	True solution $u(x)$	Present DTM_8 method
0.01	1.0101	0
0.08	1.0833	$1.90E-13$
0.16	1.1735	$7.50E-12$
0.32	1.3771	$1.00E-10$
0.48	1.6161	$3.90E-09$
0.64	1.8965	$5.30E-08$
0.8	2.2255	$4.00E-07$
0.96	2.6117	$2.10E-06$

Table 3: Comparison of relative errors of the present method for Example 4.

x	True solution $u(x)$	Geng [28] $U_{5,50}$	Present DTM_8 method
0.08	0.0064	over $1E-7$	0
0.16	0.0256	over $2E-7$	$1.9E-13$
0.24	0.0576	over $2E-6$	$7.3E-12$
0.32	0.1024	over $3E-6$	$9.7E-11$
0.48	0.2304	over $1E-5$	$3.7E-09$
0.64	0.4096	over $1E-6$	$4.9E-08$
0.80	0.6400	over $3E-6$	$3.7E-07$
0.96	0.9216	over $1E-7$	$7.0E-07$

for $k \geq 2$. Consider

$$\Longrightarrow U(k+1) = \frac{1}{(k+1)^2}\left[\left(k^2+1\right)U(k) - U(k-2) + U(k-3) + \frac{2}{k!} - \frac{1}{(k-3)!}\right]. \tag{47}$$

Because $u(0) = 1$, $U(1) = 1$. For $U(1) = 1$, $U(2) = 1/2$, we have Table 2.

Example 4. Consider the following two singularly linear BVPs in [28] Example 3:

$$t^3(1-t)^2u''(t) + 5u'(t) + (2+t)u(t) + (1-t)^3u^2(t) = f(t), \quad 0 < t < 1, \tag{48}$$

with boundary values $u(0) = 0$, $u(1) = 1 + \sin(1)$, and $f(t) = 5(1+\cos(t)) - (t-1)^2t^3\sin(t) + (t+2)(t+\sin(t)) - (t-1)^3(t+\sin(t))^2$. The exact solution is $u(t) = \sin(t) + t$.

Using the same method as the above examples, we have Table 3.

6. Conclusion

The modification proposed in this paper has demonstrated that the linear and nonlinear singular BVPs can be handled without difficulty. The computation can obtain more precise approximation. The results show a greater improvement over the He's HPM and RKHSM [28].

How to establish error analysis of this method? Maybe Adomian decomposition method with integrating factor has some beautiful results [29]. In the paper, we only have the convergence of the solution. The underlined theory for two singular points BVPs, however, still remains open and deserves further investigation.

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgment

This work was supported by the National Science Council of the Republic of China under Contract 101-2221-E-006-092-MY2.

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