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Research Article

Base Axioms of Modular Supermatroids

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This paper studies axiom systems of supermatroids. Barnabei et al.'s base axioms concerning poset matroids (i.e., distributive supermatroids) are generalized to modular supermatroids, and a mistake in the proof of base axioms of poset matroids is pointed out.

1. Introduction

Matroids, as an important combinatorial structure, have been generalized by many authors, such as polymatroids (Edmonds [1]), supermatroids (Dunstan et al. [2]), greedoids (Korte et al. [3]), and fuzzy matroids (Goetschel and Voxman [4]).

One of the most natural extensions may be supermatroids, which generalize the underlying sets of matroids to arbitrary finite partially ordered sets. Supermatroids connect closely with other extensions; for example, integral polymatroids are essentially supermatroids on a special class of finite distributive sublattices of $\mathbb{R}^n$ and greedoids are strong supermatroids on graded posets. As supermatroids, Faigle's ordered geometries [5] combine the exchange properties with finite posets, meanwhile, ordered geometries generalize a particular interest case of supermatroids and distributive supermatroids (i.e., supermatroids on distributive lattices). Tardos [6] showed an intersection theorem for distributive supermatroids. Moreover, Barnabei et al. [7] studied distributive supermatroids in more detail (in the name of poset matroids). Another approach to generalizing the concept of distributive supermatroids was proposed by Fujishige et al. [8]; they studied supermatroids on lattices of closed sets of convex geometries (cg-matroids). For a related general framework, see [3, 9].

To study axiom systems of generalized matroidal structures are fundamental problems, no matter finite or infinite extensions of matroids (see, e.g., [8, 10–12]). Barnabei et al. [7] gave many equivalent axiomatizations of distributive supermatroids. The main missing axiom in [7], that is, flat lattice axioms of distributive supermatroids, was completed by Wild [9]. The present work studies mainly bases axioms of supermatroids on modular lattice (named modular supermatroids). We give two equivalent characterizations of modular supermatroids in terms of bases and the importance of the condition of modular lattices reflected in the proofs. We note here that, without further mention, all sets and structures in this paper are finite.

2. Preliminaries

In this section, we introduce some basic concepts and results on matroids, lattices, and supermatroids. For a more detailed exposition of these topics, refer to [3, 13, 14], respectively.

Definition 1 (Welsh [13]). A matroid M is an ordered pair $(E, \mathcal{I})$ consisting of a finite set E and a nonempty collection $\mathcal{I}$ of subsets of E having the following two properties.

(I1) If $I_1 \in \mathcal{I}$, $I_2 \subseteq I_1$, then $I_2 \in \mathcal{I}$.

(I2*) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then there is an element $e \in I_2 - I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

Let $M = (E, \mathcal{I})$ be a matroid; then the members of $\mathcal{I}$ are the independent sets of M, and we call a maximal independent set in M a base of M. Definition 1 defines a matroid via independent sets; we now present base axioms of matroids.

Theorem 2 (Welsh [13]). *Let $\mathscr{B}$ be a set of subsets of a finite set E. Then $\mathscr{B}$ is a collection of bases of a matroid on E if and only if it has the following properties.*

(B1) *$\mathscr{B}$ is nonempty.*

(B2*) *If $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 - B_2$, then there is an element y of $B_2 - B_1$ such that $(B_1 - \{x\}) \cup \{y\} \in \mathscr{B}$ (base exchange axiom).*

Axiom (B2) can be replaced by the following middle base axiom.*

Theorem 3 (White [15]). *Let $\mathscr{B}$ be a set of subsets of a finite set E. Then $\mathscr{B}$ is a collection of bases of a matroid on E if and only if $\mathscr{B}$ satisfies (B1) and the following conditions.*

(B2) *For $A \subseteq B \subseteq E$, if $\mathscr{B}$ has members B_1, B_2 such that $A \subseteq B_1, B_2 \subseteq B$, then $\mathscr{B}$ has a member B_3 such that $A \subseteq B_3 \subseteq B$ (middle base axiom).*

(B3) *If $B_1, B_2 \in \mathscr{B}$ and $B_1 \subseteq B_2$, then $B_1 = B_2$.*

Remark 4. Matroids have several equivalent descriptions. Extensions of matroids are used to generalize different axiom systems of matroids. For example, Goetschel and Voxman fuzzy matroids generalize (I2*) to fuzzy sets. It can be verified that (I2) in Definition 1 can be replaced by the following condition.

(I2) If $X \subseteq E$, I_1, I_2 are maximal members of $\{I \mid I \in \mathscr{I} \text{ and } I \subseteq X\}$, then $|I_1| = |I_2|$. Supermatroids use an extension of (I2) in posets. As mentioned above, poset matroids are equivalent to distributive supermatroids, but the original definition of poset matroids extends middle base axiom to posets.

Now we summarize some facts of posets and lattices needed in this paper.

Let X be a poset, $x, y \in X$. If $x < y$ and $x \le z \le y$ implies $z = x$ or $z = y$, we say that y covers x (denoted by $x \prec y$ or $y \succ x$). An ideal of a poset is a subset $A \subseteq X$ such that $y \in A$, $x \le y$ implies $x \in A$. Down(X) denotes the set of all ideals of X. Dually, a filter A is a subset such that $y \in A$, $x \ge y$ implies $x \in A$. Let $\downarrow x = \{y \in X \mid y \le x\}$); this ideal is called the principal ideal generated by x, and principal filters are defined analogously. The intersection of a principal ideal $\downarrow y$ and a principal filter $\uparrow x$ is an interval $[x, y]$; that is, $[x, y] = \{z \in X \mid x \le z \le y\}$. A subset $C \subseteq X$ such that any two elements of C are comparable (incomparable) is a (an) chain (antichain). The length of the chain is $|C| - 1$. The height $h(x)$ of an element x is the length of the longest chain from 0 to x.

An element u is the meet of x and y (denoted by $x \wedge y$) if for any z we have that $z \le u$ if and only if $z \le x$ and $z \le y$. An element w is the join of x and y (denoted by $x \vee y$) if for any z we have that $z \ge w$ if and only if $z \ge x$ and $z \ge y$. A poset X is a lattice if any two elements $x, y \in X$ have $x \vee y$ and $x \wedge y$. A lattice X is said to satisfy the upper covering condition if $a \prec b$ implies $a \vee c \prec b \vee c$ or $a \vee c = b \vee c$ for all $a, b, c \in X$. The lower covering condition is the dual. A lattice X is a modular lattice if X satisfies both the lower covering condition and the upper covering condition. The following theorem gives some equivalent forms of modular lattice.

Theorem 5 (Roman [14]). *For a lattice X, the following conditions are equivalent.*

(1) *X is modular.*

(2) *X does not contain N_5 (also called a pentagon).*

(3) *For all $x, y \in X$, $h(x) + h(y) = h(x \vee y) + h(x \wedge y)$.*

In 1972, Dunstan et al. [2] introduced a generalization of matroids to partial sets as follows.

Definition 6 (Dunstan et al. [2]). Let X be a poset with a minimal element 0 and height function h. $\varsigma \subseteq X$ is a supermatroid if ς satisfies the following three conditions.

(LI0) $0 \in \varsigma$.

(LI1) If $x \in \varsigma$, $y \le x$, then $y \in \varsigma$.

(LI2) For any $x \in X$, all maximal elements of $\downarrow x \cap \varsigma$ have the same height.

The elements of ς are called independent elements of the supermatroids; otherwise an element is dependent. A base of ς is a maximal independent element.

We now introduce another matroidal structure on poset sets proposed by Barnabei et al. in 1998.

Definition 7 (Barnabei et al. [7]). A poset matroid on the partial set X is a family $\mathscr{B}$ of filters of X, called bases, and satisfies the following axioms.

(DLB1) $\mathscr{B} \ne \emptyset$.

(DLB2) For $A \subseteq B \subseteq E$, if $\mathscr{B}$ has members B_1, B_2 such that $A \subseteq B_1, B_2 \subseteq B$, then $\mathscr{B}$ has a member B_3 such that $A \subseteq B_3 \subseteq B$.

(DLB3) If $B_1, B_2 \in \mathscr{B}$ and $B_1 \subseteq B_2$, then $B_1 = B_2$.

Remark 8. Poset matroids deal with filters of posets, which seems different from supermatroids that concern mainly elements of partial sets. However, by a fundamental theorem of Birkhoff (every finite distributive lattice is isomorphic to the lattice of all filters of a finite partially ordered set. Conversely, every finite partially ordered set is isomorphic to the partially ordered set of the meet-irreducible elements of a distributive lattice), poset matroids are just supermatroids on distributive lattices. Though supermatroids were introduced before poset matroids, unfortunately, Barnabei et al.'s paper [7] never mentioned the name "supermatroids."

3. Base Axioms of Modular Supermatroids

In this section, we will characterize modular supermatroids in terms of bases. First, we will prove some properties of bases of modular supermatroids.

Theorem 9. *Let (X, ς) be a supermatroid on a lattice X and let β be the set of bases of (X, ς); then β satisfies the following condition.*

(LB2) *Suppose $b_1, b_2 \in \beta$; then for every pair $x, y \in X$ satisfying $y \le b_1, b_2 \le x$, and $y < x$, there exists $z \in \beta$ such that $y \le z \le x$.*

Proof. (LB2) obviously holds if $y = b_1$ or $b_2 = x$. In the following, we assume $y < b_1$ and $b_2 < x$. It follows from (LI2) that $h(b_1) = h(b_2)$; thus, $h(y) < h(b_2)$. Since maximal elements of $\downarrow (y \vee b_2) \wedge \varsigma$ have the same height, there is $z_1 \in \varsigma$ such that $y < z_1 \le y \vee b_2 \le x$.

If $h(z_1) < h(b_2)$, similarly, it follows from (LI2) that there exists $z_2 \in \varsigma$ satisfying $z_1 < z_2 \le z_1 \vee b_2 \le x$.

If $h(z_2) < h(b_2)$, then there exists $z_3 \in \varsigma$ satisfying $z_2 < z_3 \le z_2 \vee b_2 \le x$.

If $h(z_3) < h(b_2)$, we continue the above process. After finite steps, we have $z_n \in \varsigma$, $y < z_1 < \cdots < z_n \le z_{n-1} \vee b_2 \le x$, and $h(z_n) = h(b_2)$. Note that maximal elements of ς have the same height; thus, $z_n \in \beta$; this concludes the proof.

□

Remark 10. (B2) characterizes matroids (Theorem 3), and this fact can be extended to distributive supermatroids (poset matroids are actually defined by generalized form of (B2), and then independence axioms of poset matroids, i.e., generalized forms of (I2*), and equivalent descriptions of independence axioms of poset matroids were proved in [7]). However, an antichain with (LB2) cannot define a supermatroid on a lattice, as the following example shows.

Example 11. Consider the lattice X in Figure 1(a). ς is the set of black-filled points. Let $\beta = \operatorname{Max} \varsigma$; then β satisfies (LB2), but (X, ς) is not a supermatroid. Note that the two elements in β have different heights, and the following result tells us that this cannot happen when X is a modular lattice.

Lemma 12. *Suppose that X is a modular lattice; $\varsigma \in Down(X)$. Let $\beta = \operatorname{Max} \varsigma$; if β satisfies (LB2), then all elements in β have the same height.*

Proof. Suppose $b_1, b_2 \in \beta$; we apply induction on $h[b_1 \wedge b_2, b_1 \vee b_2]$. Obviously, $h[b_1 \wedge b_2, b_1 \vee b_2] \ge 2$ and for $h[b_1 \wedge b_2, b_1 \vee b_2] = 2$ the assertion is true. Assume that b_1 and b_2 have the same height when $h[b_1 \wedge b_2, b_1 \vee b_2] \le n$. Let $h[b_1 \wedge b_2, b_1 \vee b_2] = n+1$. Choose $x \in [b_1 \wedge b_2, b_2]$ and $x \prec b_2$; then $x \ge b_1 \wedge b_2$. Consider the following two cases.

Case 1 ($x = b_1 \wedge b_2$ (thus $b_1 \wedge b_2 \prec b_2$)). By the upper covering condition, $b_1 = b_1 \vee (b_1 \wedge b_2) \prec b_1 \vee b_2$ ($b_1 = b_1 \vee b_2$ contradicts the fact that b_1, b_2 are maximal elements of ς). Choose $y \in [b_1 \wedge b_2, b_1]$ and $y \prec b_1$. If $y = b_1 \wedge b_2$, then $h(b_2) = h(b_1 \wedge b_2) + 1 = h(b_1)$. We now consider the case $y > b_1 \wedge b_2$.

Since $y \prec b_1$, it follows from the upper covering condition that $y \vee b_2 \prec b_1 \vee b_2$ or $y \vee b_2 = b_1 \vee b_2$. If $y \vee b_2 = b_1 \vee b_2$, it is easy to verify that the sublattice $\{b_1 \wedge b_2, b_1 \vee b_2, y, b_1, b_2\}$ of X is just a pentagon, contrary to the fact that X is a modular lattice. Thus, $y \vee b_2 \prec b_1 \vee b_2$.

Since $y \le b_1, b_2 \le y \vee b_2$, and $y \le y \vee b_2$, it follows from (LB2) that there exists $z \in \beta$ such that $y \le z \le y \vee b_2$. Because $[z \wedge b_2, z \vee b_2] \subseteq [y \wedge b_2, y \vee b_2]$, we have

$$\begin{aligned} h[z \wedge b_2, z \vee b_2] &\le h[y \wedge b_2, y \vee b_2] \\ &= h[b_1 \wedge b_2, b_1 \vee b_2] - 1 = n. \end{aligned} \tag{1}$$

By the induction hypothesis, we have $h(z) = h(b_2)$. $z \wedge b_1 = y$ and $z \vee b_1 \le b_1 \vee b_2$ imply $[z \wedge b_1, z \vee b_1] \subseteq [y, b_1 \vee b_2]$. Note $y > b_1 \wedge b_2$, so

$$\begin{aligned} h[z \wedge b_1, z \vee b_1] &\le h[y, b_1 \vee b_2] \\ &\le h[b_1 \wedge b_2, b_1 \vee b_2] - 1 = n. \end{aligned} \tag{2}$$

Thus $h(z) = h(b_1)$ holds by the induction hypothesis. As a result, we get $h(b_1) = h(b_2)$ (cf. Figure 1(b)).

Case 2 ($x > b_1 \wedge b_2$). Since $x \prec b_2$, it holds by the upper covering condition that $x \vee b_1 \prec b_1 \vee b_2$ or $x \vee b_1 = b_1 \vee b_2$. If $x \vee b_1 = b_1 \vee b_2$, then $\{b_1 \wedge b_2, b_1 \vee b_2, x, b_1, b_2\}$ forms a sublattice N_5 of X, contradicting the fact that X is a modular lattice. Thus, $x \vee b_1 \prec b_1 \vee b_2$. Consider that $x \le b_2, b_1 \le x \vee b_1$, and $x \le x \vee b_1$; it follows from (LB2) that there exists $z \in \beta$ satisfying $x \le z \le x \vee b_1$. The subsequent proof is similar to Case 1; we can verify $h(b_1) = h(z) = h(b_2)$, as desired (cf. Figure 1(c)). □

Now we prove the first base axioms of modular supermatroids.

Theorem 13 (base axioms of modular supermatroids). *Let X be a modular lattice and $\beta \subseteq X$. Then β is the set of bases of supermatroids on X if and only if it has the following property.*

(LB2) *Suppose $b_1, b_2 \in \beta$; then for every pair $x, y \in X$ satisfying $y \le b_1$, $b_2 \le x$, and $y < x$, there exists $z \in \beta$ such that $y \le z \le x$.*

(LB3) *If $b_1, b_2 \in \beta$ and $b_1 \le b_2$, then $b_1 = b_2$.*

Proof. We have already proved that the collection β of bases of a supermatroid satisfies (LB2) (i.e., Theorem 9) and β obviously satisfies (LB3). To show the converse, let $\varsigma = \bigcup_{x \in \beta} \downarrow x$. It follows from (LB3) that $\beta = \operatorname{Max} \varsigma$. Assume $x, y \in \varsigma$ and $h(y) < h(x)$. We will prove the following.

(LI2*) If $x, y \in \varsigma$, $h(y) < h(x)$, then there exists $z \in \varsigma$ such that $y \le z \le x \vee y$ holds. For (LI2*) ⇒ (LI2) is trivial, we conclude the proof.

The definition of β implies that there exists $b_1, b_2 \in \beta$ such that $y \le b_1$ and $x \le b_2$. Since $y \le b_1, b_2 \le y \vee b_2$, and $y < y \vee b_2$, it holds from (LB2) that there exists $b_3 \in \beta$ satisfying $y \le b_3 \le y \vee b_2$. If $x = b_2$, then (LI2*) holds. We now consider the case $x < b_2$. By $y \le b_3 \le y \vee b_2$, we have $b_3 \vee b_2 = y \vee b_2$. From Lemma 12, $h(b_3) = h(b_2) = h(b_1) > h(x) > h(y)$ holds; thus, $y < b_3$. Clearly, $y \wedge x \le b_3 \wedge x$. To prove $y \wedge x < b_3 \wedge x$, we assume $y \wedge x = b_3 \wedge x$ and derive a contradiction. X is a modular lattice; thus,

$$h(x) + h(y) = h(x \vee y) + h(x \wedge y). \tag{3}$$

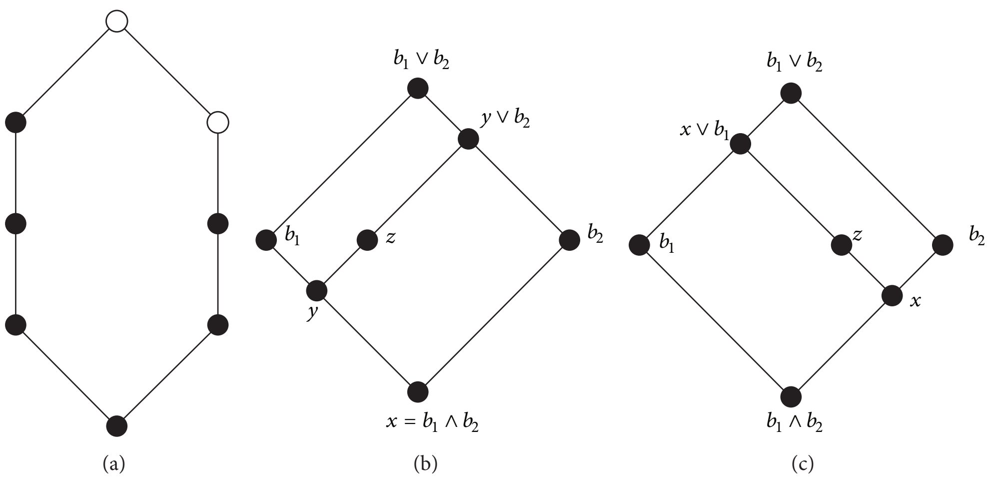


Figure 1

By $h(b_3) = h(b_2) > h(x) > h(y)$, so

$$h(b_3) - h(y) > h(b_2) - h(x). \quad (4)$$

It follows from (3) and (4) that

$$h(b_3) + h(x) > h(b_2) - h(x) + h(x \vee y) + h(x \wedge y). \quad (5)$$

And since

$$h(x) + h(b_3) = h(x \vee b_3) + h(x \wedge b_3), \quad (6)$$

with (5), we have

$$h(x \vee b_3) + h(x \wedge b_3) > h(b_2) - h(x) + h(x \vee y) + h(x \wedge y). \quad (7)$$

Note the assumption $y \wedge x = b_3 \wedge x$; (7) implies

$$h(x \vee b_3) - h(x \vee y) > h(b_2) - h(x). \quad (8)$$

It holds from $x \vee b_3 \leq b_2 \vee b_3$ and (8) that

$$h(b_2 \vee b_3) - h(x \vee y) \geq h(x \vee b_3) - h(x \vee y) > h(b_2) - h(x). \quad (9)$$

Let $x \prec a_1 \prec a_2 \prec \cdots \prec a_m \prec b_2$. By the upper covering condition, $\{a \vee y \mid a \in \{a_1, a_2, \ldots, a_m\}\}$ forms a maximal chain from $x \vee y$ to $b_2 \vee y$. Therefore, $h(b_2) - h(x) \geq h(b_2 \vee y) - h(x \vee y)$, contrary to (9) (note $b_3 \vee b_2 = y \vee b_2$). Thus, $y \wedge x < b_3 \wedge x$ is proved.

Choose $z \in [y \wedge x, b_3 \wedge x]$ and $y \wedge x \prec z$. It is trivial that $y \wedge x = y \wedge z$. Since

$$h(z) + h(y) = h(z \vee y) + h(z \wedge y), \quad h(y \wedge x) + 1 = h(z), \quad (10)$$

we have $h(y \vee z) = h(y) + 1$; thus, $y \vee z > y$. $y \vee z \leq b_3$ implies $y \vee z \in \varsigma$, and obviously $y \vee z \leq x \vee y$, so (LI2*) holds and this completes the proof. □

Remark 14. Barnabei et al. defined poset matroids in terms of base axioms. They proved the equivalence of base axioms and independence axioms for distributive supermatroids (see Theorem 5.1 in [7]). Our theorem (Theorem 13) actually generalizes their results about base axioms to modular supermatroids.

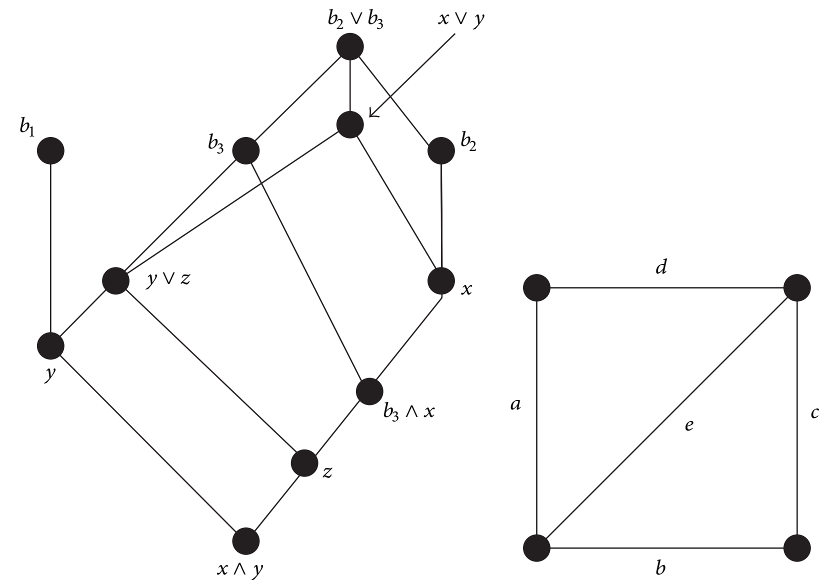
Remark 15. In the proof of Theorem 5.1 in [7], (LB2*) ⇒ (LI2*) was verified by induction ((LB2*) is equivalent to (LB2) for distributive supermatroids; see Theorem 4.3 of [7]. We will prove the equivalence of (LB2*) and (LB2) for modular supermatroids in the following). Unfortunately, there is something wrong with the induction. We take a matroid, that is, a special case of poset matroids, as an example to verify our assertion. Consider the cycle matroid $M(G)$, where G is shown in Figure 2(b). $\{a, b\}$ is an independent set and $\{c, d, e\}$ is a base. Though for every $x \in \{c, d, e\}$, obviously, $\{a, b, x\}$ is an independent set (i.e., we can augment $\{a, b\}$ by an element x in $\{c, d, e\} - \{a, b\}$), it is easy to check that we cannot augment $\{a, b\}$ to a larger independent set by the base exchange axiom. This example shows that the assertion "if $n = 0$, the thesis follows immediately by the exchange property and by Theorem 4.1" in the proof of [7, Theorem 5.1] is not right. We give a direct proof of a more generalized result of [7, Theorem 5.1] in Theorem 13 instead of the induction approach and our proof presents the important role of modular properties.

We now give another base axiom of modular supermatroids.

Theorem 16. *Let X be a modular lattice, $\varsigma \in Down(X)$, and $\beta = \mathrm{Max}\,\varsigma$. Then the following are equivalent.*

(LB2) *Suppose $b_1, b_2 \in \beta$; then for every pair $x, y \in X$ satisfying $y \leq b_1$, $b_2 \leq x$, and $y < x$, there exists $z \in \beta$ such that $y \leq z \leq x$.*

(LB2*) *Suppose $b_1, b_2 \in \beta$; then for every $x \prec b_1$, there exists $y \leq b_2$ such that $x \vee y \in \beta$.*



(a) (b)

FIGURE 2

Proof. (LB2) $\Rightarrow$ (LB2*). Let $b_1, b_2 \in \beta$, $x \in X$, satisfying $x \prec b_1$. Since $x \le b_1$, $b_2 \le b_2 \vee x$, and $x \le b_2 \vee x$, it holds from (LB2) that there exists $b_3 \in \beta$ such that $x \le b_3 \le b_2 \vee x$. By Lemma 12, $h(b_1) = h(b_3) = h(x) + 1$; thus $x \prec b_3$. Then the lower covering condition implies

$$x \wedge b_2 \prec b_3 \wedge b_2 \quad \text{or} \quad x \wedge b_2 = b_3 \wedge b_2. \tag{11}$$

If $x \wedge b_2 = b_3 \wedge b_2$, observe that $b_3 \le b_2 \vee x$ implies $b_3 \vee b_2 = b_2 \vee x$; then $\{x, b_2, b_3, b_2 \vee x, b_2 \wedge x\}$ is the sublattice N_5 of X, contradicting the fact that X is a modular lattice. Hence, $x \wedge b_2 \prec b_3 \wedge b_2$.

It is easy to check $x \wedge b_2 = x \wedge (b_3 \wedge b_2)$. Since b_2 and b_3 are incomparable, we have

$$h(b_3) = h(x) + 1 > h(b_3 \wedge b_2) = h(x \wedge b_2) + 1. \tag{12}$$

Thus, $x > x \wedge b_2$. Then it follows from $x \wedge b_2 \prec b_3 \wedge b_2$ that x and $b_3 \wedge b_2$. Because $x < x \vee (b_3 \wedge b_2) \le b_3$ and $x \prec b_3$, $x \vee (b_3 \wedge b_2) = b_3$, as desired.

(LB2*) $\Rightarrow$ (LB2). Let $b_1, b_2 \in \beta$ and $x, y \in X$, satisfying $y \le b_1$, $b_2 \le x$, and $y < x$. If $y = b_1$ or $b_2 = x$, then (LB2) holds. We now consider the case $y < b_1$ and $b_2 < x$. In the subset of β that all elements are equal to or greater than y, we choose a b_3 such that the height of $b_3 \wedge b_2$ is maximal (i.e., $b_3 \in \{a \in \beta \mid a \ge y\}$ and for all $a_0 \in \{a \in \beta \mid a \ge y\}$, $h(b_3 \wedge b_2) \ge h(a_0 \wedge b_2)$). Note that $b_3 < x$ or they are incomparable. We assert $b_3 < x$; then this finishes our proof. To prove our assertion, we assume that b_3 and x are incomparable and derive a contradiction.

Since $b_3 \wedge x > b_3$, choose $y_0 \in [b_3 \wedge x, b_3]$, satisfying $y_0 \prec b_3$. By (LB2), there exists $y_1 \le b_2$ such that $y_0 \vee y_1 \in \beta$. Let $b_4 = y_0 \vee y_1$; then $b_4 > y_0 \ge b_3 \wedge x \ge y \wedge x = y$. Obviously, $b_3 \vee b_2 \ge (y_0 \vee y_1) \vee b_2 = b_4 \vee b_2$. We consider the following two cases.

Case 1 *($b_3 \vee b_2 = b_4 \vee b_2$).* Note $b_4 \vee b_2 = (y_0 \vee y_1) \vee b_2 = y_0 \vee b_2 = b_3 \vee b_2$, and it is easy to verify $y_0 \wedge b_2 = b_3 \wedge b_2$; hence, $\{b_2, b_3, y_0, b_3 \wedge b_2, b_3 \vee b_2\}$ forms a sublattice N_5 of X, contradicting the fact that X is a modular lattice.

Case 2 *($b_3 \vee b_2 > b_4 \vee b_2$).* In this case, $h(b_3 \vee b_2) > h(b_4 \vee b_2)$ holds. Since X is a modular lattice, we have

$$\begin{aligned} h(b_2) + h(b_3) &= h(b_3 \vee b_2) + h(b_3 \wedge b_2), \\ h(b_2) + h(b_4) &= h(b_4 \vee b_2) + h(b_4 \wedge b_2). \end{aligned} \tag{13}$$

(13) implies

$$\begin{aligned} h(b_4 \wedge b_2) - h(b_3 \wedge b_2) = h(b_4) - h(b_3) \\ + h(b_3 \vee b_2) - h(b_4 \vee b_2). \end{aligned} \tag{14}$$

It is easy to check $h(b_4) - h(b_3) \ge 0$ ((LB2*) implies (LB2); then it holds from Lemma 12 that $h(b_4) = h(b_3)$. We certainly can prove directly that (LB2*) implies that all elements in β have the same height (but not needed here), though the proof is not trivial); thus it follows from (14) that $h(b_4 \wedge b_2) - h(b_3 \wedge b_2) > 0$, contrary to the definition of b_3. □

4. Conclusions

This paper establishes the middle base axiom and base exchange axiom of modular supermatroids. For independence axioms, we can prove that (I2*) and its other equivalent characterizations can also be generalized to modular supermatroids, and most of these extensions do not need the restriction of modular lattices. However, we cannot extend directly circuit axioms of distributive supermatroids (i.e., Theorem 9.1 of [7]) to modular supermatroids. Note that there are many equivalent descriptions of circuits for matroids; thus, choosing one proper definition and constructing circuit axioms of modular supermatroids are our future work. Apart from axiom systems, the second reason for us to consider properties of circuits is to study the connectedness of supermatroids. Connectedness is an important topic in matroid theory, while few papers concerning extensions of matroids refer to connectedness. Our work of this paper, meanwhile, proposes preliminaries for subsequent study about circuits.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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