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Research Article

A Reproducing Kernel Hilbert Space Method for Solving Systems of Fractional Integrodifferential Equations

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We present a new version of the reproducing kernel Hilbert space method (RKHSM) for the solution of systems of fractional integrodifferential equations. In this approach, the solution is obtained as a convergent series with easily computable components. Several illustrative examples are given to demonstrate the effectiveness of the present method. The method described in this paper is expected to be further employed to solve similar nonlinear problems in fractional calculus.

1. Introduction

In this paper, we consider the following system of fractional integrodifferential equations:

$$\begin{aligned} D^{\alpha_i} x_i(t) &= F_i\Big(t, x_1(t), \dots, x_1^{(k)}(t), \dots, x_{i-1}(t), \dots, x_{i-1}^{(k)}(t), \\ &\qquad x_{i+1}(t), \dots, x_{i+1}^{(k)}(t), \dots, x_n(t), \dots, x_n^{(k)}(t)\Big) \\ &\quad + \int_0^t G_i\Big(t, \tau, x_1(\tau), \dots, x_1^{(k)}(\tau), \dots, x_n(\tau), \\ &\qquad \dots, x_n^{(k)}(\tau)\Big) d\tau, \end{aligned} \tag{1}$$

where $i = 1, \dots, n$, $k = 0, 1, \dots, m$, $0 \le t \le 1$, and D^{α_i} is derivative of order α_i in the sense of Caputo and $m-1 < \alpha_i \le m$, subject to the initial conditions:

$$x_i^{(j)}(a) = a_{ji}, \quad j = 0, 1, \dots, m-1, \; i = 1, 2, \dots, n, \; a \ge 0. \tag{2}$$

In the last two decades, fractional calculus has found diverse applications in various scientific and technological fields [1, 2], such as thermal engineering, acoustics, electromagnetism, control, robotics, viscoelasticity, diffusion, edge detection, turbulence, signal processing, and many other physical and biological processes. Fractional differential equations have also been applied in modeling many physical and engineering problems. Most systems of fractional integrodifferential equations do not have exact solutions, so numerical techniques are used to solve such systems. The homotopy perturbation method, the Adomian decomposition method, and other methods are used to give an approximate solution to linear and nonlinear problems; see [3–13] and the references therein.

In our previous work [14], we proposed a reproducing kernel Hilbert space method for solving integrodifferential equations of fractional order based on the reproducing kernel theory [14, 15]. In this paper, we will generalize the idea of the RKHSM to provide a numerical solution for systems of fractional integrodifferential equations (1). To demonstrate the effectiveness of the RKHSM algorithm, several numerical experiments of linear and nonlinear systems of fractional equations (1) will be presented.

This paper is organized as follows. An introduction of the algorithm for solving systems of fractional integrodifferential equations is given in Section 2. In Section 3, we introduce several examples to show the efficiency of the method. Finally, a conclusion is given in Section 4.

2. The Algorithm

After homogenizing the initial conditions (2), we apply the operator I^{α_i}, the Riemann-Liouville fractional integral of order α_i [2, 16–20], to both sides of (1) to have

$$x_i(t) = M_i(t), \quad i = 1, 2, \ldots, n, \tag{3}$$

where

$$\begin{aligned} M_i(t) &= I^{\alpha_i}\Bigg(F_i\big(t, x_1(t), \ldots, x_1^{(k)}(t), \ldots, x_{i-1}(t), \ldots, x_{i-1}^{(k)}(t), \\ &\quad x_{i+1}(t), \ldots, x_{i+1}^{(k)}(t), \ldots, x_n(t), \ldots, x_n^{(k)}(t)\big) \\ &\quad + \int_0^t G_i\big(t, \tau, x_1(\tau), \ldots, x_1^{(k)}(\tau), \ldots, x_n(\tau), \\ &\quad \ldots, x_n^{(k)}(\tau)\big)\, d\tau \Bigg), \\ &\quad i = 1, 2, \ldots, n, \quad k = 0, 1, \ldots, m. \end{aligned} \tag{4}$$

It is clear that (3) is equivalent to (1), so every solution of the integral equation (3) is also a solution of our original problem (1) and vice versa.

To solve (3) by means of the reproducing kernel Hilbert space method, first, we need to construct a reproducing kernel of certain spaces $W_2^{m+1}[a,b] := \{u \mid u^{(j)}$ is absolutely continuous, $j = 1, 2, \ldots, m-1$, and $u^{(m)} \in L^2[a,b]\}$ in which every function satisfies the homogenous initial conditions of (1).

(i) The inner product of the space $W_2^1[0,1] = \{u \mid u$ is absolutely continuous real value function, $u' \in L^2[0,1]\}$ is given by

$$\langle u, v\rangle_{W_2^1} := \int_0^1 \big(u(t)v(t) + u'(t)v'(t)\big)\, dt \tag{5}$$

and norm $\|u\|_{W_2^1} = \sqrt{\langle u, u\rangle_{W_2^1}}$.

In [21], Li and Cui proved that $W_2^1[0,1]$ is a reproducing kernel Hilbert space and its reproducing kernel is given by

$$R(x,y) := \frac{1}{2\sinh 1} \times \left[\cosh(x+y-1) + \cosh|x-y| - 1\right]. \tag{6}$$

(ii) The inner product of the space $W_2^2[0,1] = \{u \mid u, u'$ are absolutely continuous real value functions, $u'' \in L^2[0,1], u(0) = 0\}$ is given by

$$\langle u, v\rangle_{W_2^2} := u(0)v(0) + u'(0)v'(0) + \int_0^1 u''(t)v''(t)\, dt \tag{7}$$

and norm $\|u\|_{W_2^2} = \sqrt{\langle u, u\rangle_{W_2^2}}$. $W_2^2[0,1]$ is a reproducing kernel Hilbert space and its reproducing kernel is given by

$$S(x,y) := \begin{cases} \dfrac{1}{6} y\left(-y^2 + 3x(2+y)\right), & y \le x \\ \dfrac{1}{6} x\left(-x^2 + 3y(2+x)\right), & y > x. \end{cases} \tag{8}$$

(iii) The inner product of the space $W_2^3[0,1] = \{u \mid u, u', u''$ are absolutely continuous real value functions, $u''' \in L^2[0,1], u(0) = u'(0) = 0\}$ is given by

$$\langle u, v\rangle_{W_2^3} := \sum_{i=0}^{2} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u'''(t)v'''(t)\, dt \tag{9}$$

and norm $\|u\|_{W_2^3} = \sqrt{\langle u, u\rangle_{W_2^3}}$. $W_2^3[0,1]$ is a reproducing kernel Hilbert space and its reproducing kernel is given by

$$M(x,y) := \begin{cases} f(x,y), & y \le x \\ f(y,x), & y > x, \end{cases} \tag{10}$$

where $f(x,y) = (1/120)y^2(-x^2(-126 + 10x - 5x^2 + x^3) + 5(-1+x)xy^2 - (-1+x^2)y^3)$.

The method of obtaining the reproducing kernel can be found in [15].

Let $L_i : W_2^{m+1}[0,1] \to W_2^1[0,1]$ such that $L_i x_i(t) = x_i(t)$. Then L_i, $i = 1, 2, \ldots, n$, are bounded linear operators.

Let $\{t_j\}_{j=1}^{\infty}$ be a countable dense set in $[0,1]$. Let $\varphi_j^i(t) = R(t_j, t)$ and $\psi_j^i(t) = L_i^* \varphi_j^i(t)$, where L_i^* is the adjoint operator of L_i.

By Gram-Schmidt process we can construct an orthonormal system $\{\overline{\psi}_j^i(t)\}_{j=1}^{\infty}$ of $W_2^{m+1}[0,1]$, where

$$\overline{\psi}_j^i(t) = \sum_{k=1}^{j} \beta_{jk}^i \psi_k^i(t), \quad \beta_{jj}^i > 0, \qquad \forall j = 1, 2, \ldots, \quad i = 1, 2, \ldots, n. \tag{11}$$

Theorem 1. *Let $\{t_j\}_{j=1}^{\infty}$ be a dense set in $[0,1]$. Then $\{\psi_j^i(t)\}_{j=1}^{\infty}$ is a complete system of $W_2^{m+1}[0,1]$.*

For the proof, see [14].

Theorem 2. *Let $\{t_j\}_{j=1}^{\infty}$ be a dense set in $[0,1]$ and the solution of (3) is unique on $W_2^{m+1}[0,1]$. Then the solution of (3) is given by $x_i(t) = \sum_{j=1}^{\infty} A_j \overline{\psi}_j^i(t)$, where $A_j = \sum_{k=1}^{j} \beta_{jk}^i M_i(t_k)$.*

For the proof, see [14].

One can get an approximate solution $x_{in}(t)$ by taking finitely many terms in the series representation of $x_i(t)$ and $x_{in}(t) = \sum_{j=1}^{n} A_j \overline{\psi}_j^i(t)$.

Since $W_2^{m+1}[0,1]$ is a Hilbert space, then $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \beta_{jk}^i M_i(t_k) < \infty$.

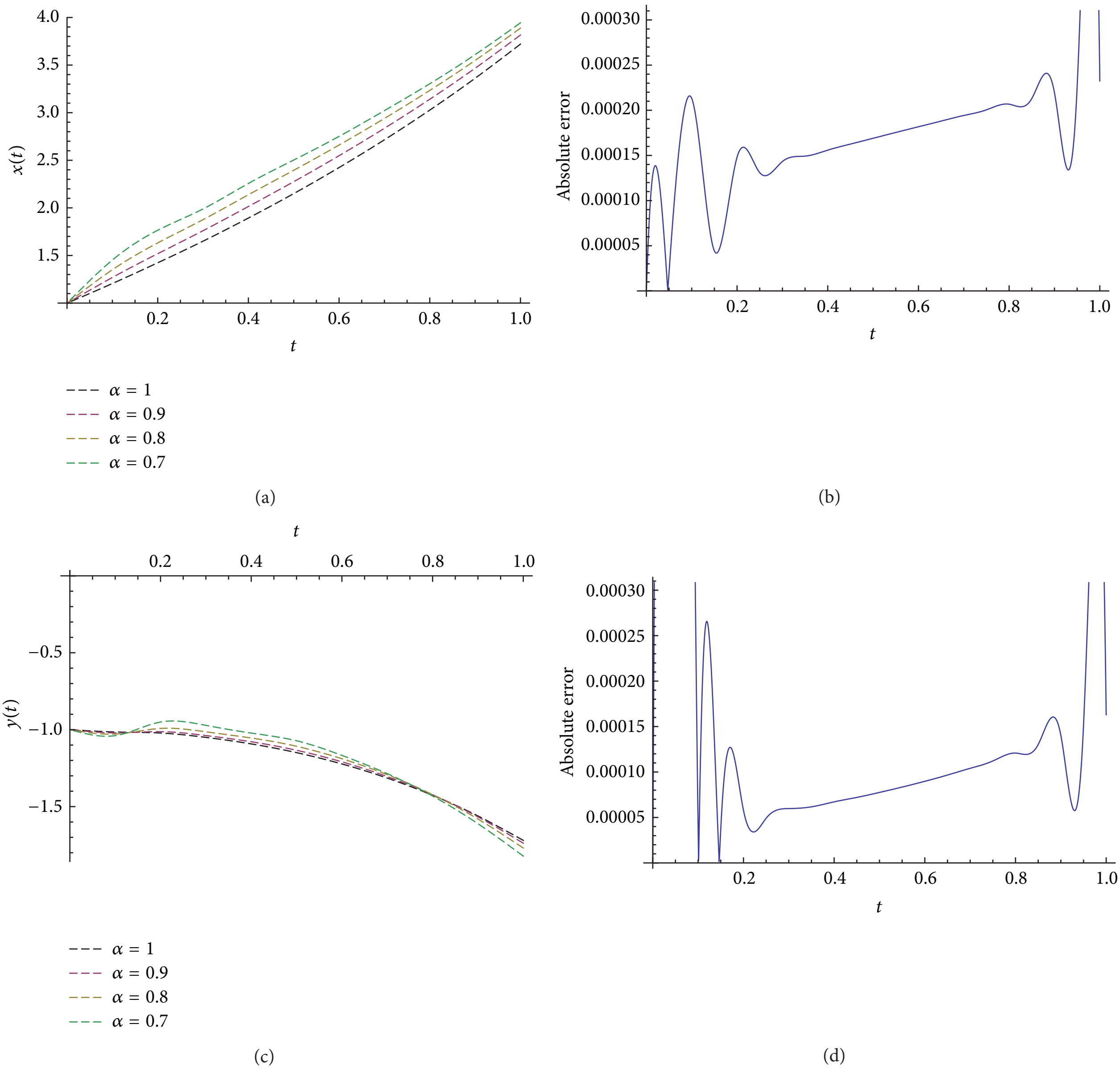

Figure 1: Graphical results for Example 1 when $\alpha_1 = \alpha_2 = \alpha = 1, 0.9, 0.8$, and 0.7.

Theorem 3. *The approximate solution $x_{in}(t)$ and its derivatives $x_{in}^{(j)}$ are uniformly convergent to $x_i^{(j)}(t)$, $i = 1, 2, \ldots, n$, $j = 0, 1, \ldots$.*

Proof. By the reproducing kernel property of $K(x, y)$ and Schwarz inequality, we can obtain

$$\begin{aligned} \left|x_{in}(t) - x_i(t)\right| &= \left|\left\langle x_{in}(t) - x_i(t), K(x, y)\right\rangle_{W_2^{m+1}}\right| \\ &\leq \left\|K(x, y)\right\|_{W_2^{m+1}} \left\|x_{in}(t) - x_i(t)\right\|_{W_2^{m+1}} \\ &\leq c_0 \left\|x_{in}(t) - x_i(t)\right\|_{W_2^{m+1}}, \end{aligned} \tag{12}$$

where c_0 is a constant.

By the representation of $K(x, y)$ we can obtain

$$\begin{aligned} &\left|x_{in}^{(j)}(t) - x_i^{(j)}(t)\right| \\ &\quad = \left|\left\langle x_{in}^{(j)}(t) - x_i^{(j)}(t), K^{(j)}(x, y)\right\rangle_{W_2^{m+1}}\right| \\ &\quad \leq \left\|K^{(j)}(x, y)\right\|_{W_2^{m+1}} \left\|x_{in}^{(j)}(t) - x_i^{(j)}(t)\right\|_{W_2^{m+1}}. \end{aligned} \tag{13}$$

Since $K^{(j)}(x, y)$, $j = 1, 2, \ldots$, is uniformly bounded about x and y, we have

$$\left\|K^{(j)}(x, y)\right\|_{W_2^{m+1}} \leq c_j, \quad j = 1, 2, \ldots, \tag{14}$$

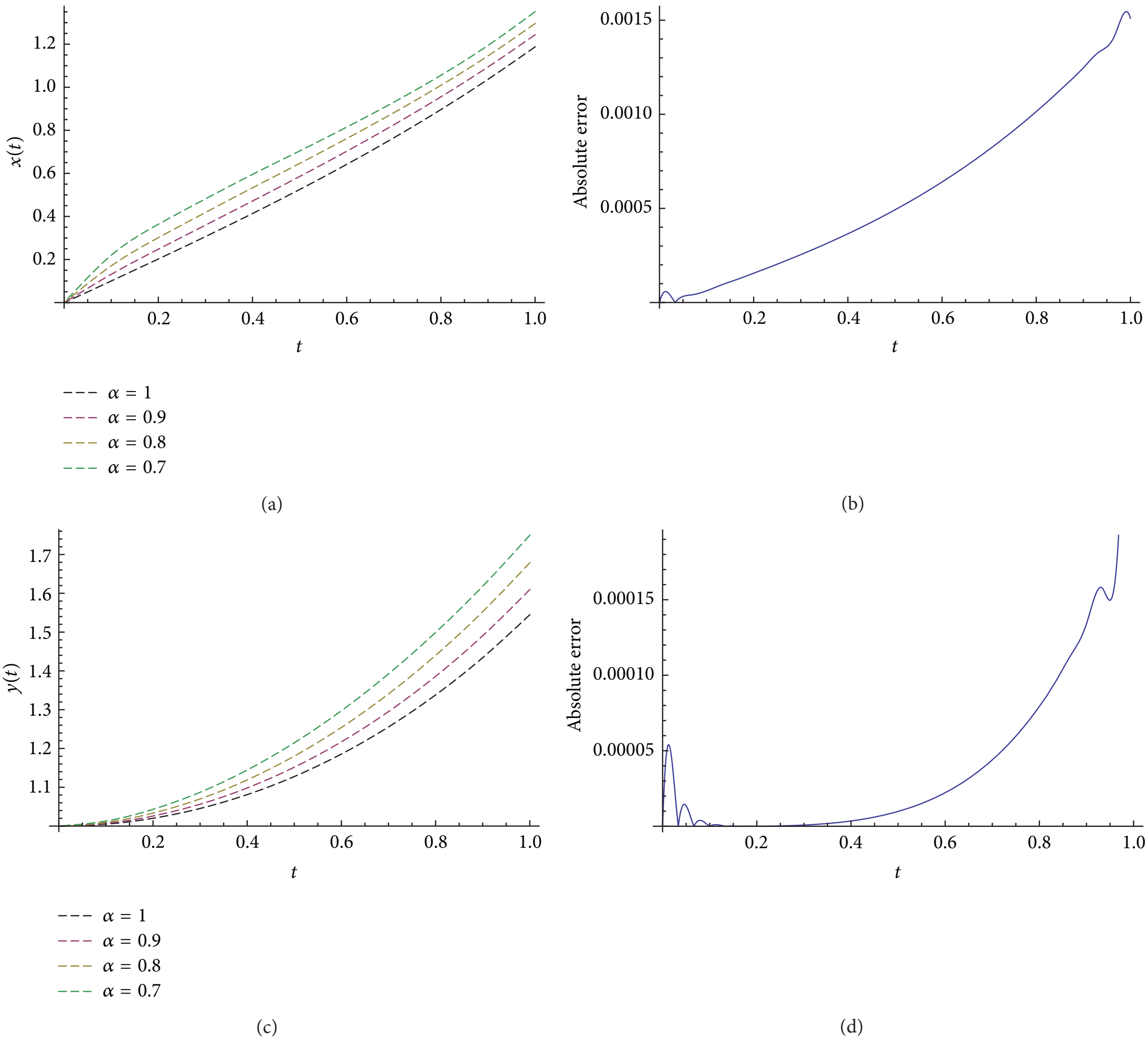

Figure 2: Graphical results for Example 2 when $\alpha_1 = \alpha_2 = \alpha = 1, 0.9, 0.8$, and 0.7.

and so

$$\left|x_{in}^{(j)}(t) - x_i^{(j)}(t)\right| \le c_j \left\|x_{in}^{(j)}(t) - x_i^{(j)}(t)\right\|_{W_2^{m+1}}, \quad j = 1, 2, \ldots. \tag{15}$$

Thus $x_i(t)$ and its derivatives $x_{in}^{(j)}(t)$ are uniformly convergent to $x_i^{(j)}(t)$, $j = 1, 2, \ldots$. □

3. Numerical Results

In this paper, three numerical examples are given to show the accuracy of this method. The computations are performed by Mathematica 8.0. We compare the results by this method with the exact solution of each example.

Example 1. Consider the following linear system of fractional integrodifferential equations:

$$\begin{gathered} D^{\alpha_1} x(t) = 1 + t + t^2 - y(t) - \int_0^t \left(x(\tau) + y(\tau)\right) d\tau, \\ D^{\alpha_2} x(t) = -1 - t + x(\tau) - \int_0^t \left(x(\tau) - y(\tau)\right) d\tau, \\ x(0) = 1, \quad y(0) = -1, \quad 0 < \alpha_1, \quad \alpha_2 \le 1. \end{gathered} \tag{16}$$

The exact solution for $\alpha_1 = \alpha_2 = 1$ is $x(t) = t + e^t$, $y(t) = t - e^t$.

After homogenizing the initial conditions and using this method, taking $t_i = i/n$, $i = 1, 2, \ldots, n$, and $n = 20$, the graphs of the approximate solutions for different values of α_1 and α_2 are plotted in Figure 1. From Figure 1, it is clear that the approximate solutions are in good agreement with the exact solutions when $\alpha_1 = \alpha_2 = 1$, and the solution continuously depends on the fractional derivative.

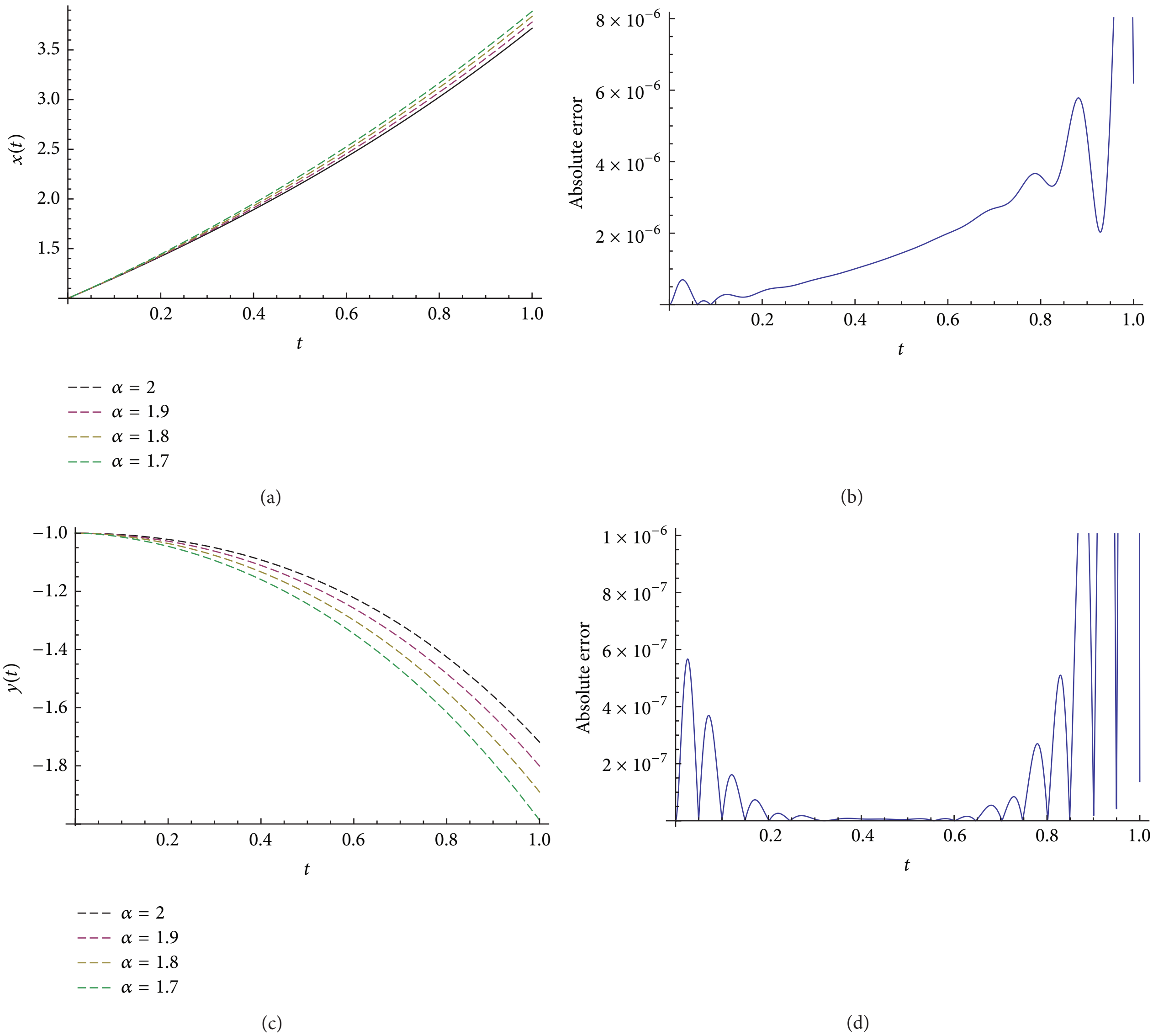

Figure 3: Graphical results for Example 3 when $\alpha_1 = \alpha_2 = \alpha = 2, 1.9, 1.8$, and 1.7.

Example 2. Consider the following nonlinear system of fractional integrodifferential equations:

$$D^{\alpha_1} x(t) = 1 - \frac{1}{2} y'^2(t) + \int_0^t \left[(t - \tau) y(\tau) + x(\tau) y(\tau)\right] d\tau,$$
$$D^{\alpha_2} x(t) = 2t + \int_0^t \left[(t - \tau) x(\tau) - y^2(\tau) + x^2(\tau)\right] d\tau,$$
$$x(0) = 0, \quad y(0) = 1, \quad 0 < \alpha_1, \quad \alpha_2 \leq 1. \tag{17}$$

The exact solution for $\alpha_1 = \alpha_2 = 1$ is $x(t) = \sinh t$, $y(t) = \cosh t$.

After homogenizing the initial conditions and using this method, taking $t_i = i/n, i = 1, 2, \ldots, n$, and $n = 30$, the graphs of the approximate solutions for different values of α_1 and α_2 are plotted in Figure 2.

Example 3. Consider the following nonlinear system of fractional integrodifferential equations:

$$D^{\alpha_1} x(t) = 1 - \frac{t^3}{3} - \frac{1}{2} y'^2(t) + \frac{1}{2} \int_0^t \left(x^2(\tau) + y^2(\tau)\right) d\tau,$$
$$D^{\alpha_2} x(t) = -1 + t^2 - t x(t) + \frac{1}{4} \int_0^t \left(x^2(\tau) - y^2(\tau)\right) d\tau,$$
$$x(0) = 1, \quad x'(0) = 2, \quad y(0) = -1,$$
$$y'(0) = 0, \quad 1 < \alpha_1, \quad \alpha_2 \leq 2. \tag{18}$$

The exact solution for $\alpha_1 = \alpha_2 = 2$ is $x(t) = t + e^3$, $y(t) = t - e^t$.

After homogenizing the initial conditions and using this method, taking $t_i = i/n, i = 1, 2, \ldots, n$, and $n = 20$, the graphs of the approximate solutions for different values of α_1 and α_2 are plotted in Figure 3.

4. Conclusion

In this paper, we introduce a new algorithm for solving systems of fractional integrodifferential equations. The approximate solution obtained by this method and its derivative are both uniformly convergent. The obtained results demonstrate the reliability of the algorithm and its wider applicability to linear and nonlinear systems of fractional differential equations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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