

Research Article

Asymmetrical Impact of Allee Effect on a Discrete-Time Predator-Prey System

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Received 28 August 2012; Revised 2 March 2013; Accepted 8 March 2013

Academic Editor: Vu Phat

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A discrete-time predator-prey model is proposed with Leslie-type numerical response, and the asymmetrical influence of Allee effect on the proposed system is investigated. By mathematical analysis, locally stable conditions for the equilibrium points of the considered systems with or without Allee effect are obtained firstly. Furthermore, numerical simulation is used to verify the results and detect some new outcomes. The results show that Allee effect on predator leads the system to its stable state in much longer time. Conversely, the prey population with Allee effect makes it much faster. In particular, a large value of Allee effect on prey results in periodic dynamics of the system.

1. Introduction

In recent years, the Allee effect has drawn a considerable attention in almost every aspect of ecology and conservation. This effect, a reduction of individual fitness at low population density, may come about through a number of mechanisms, such as mate-search difficulty, increased predation risk due to failing flocking or schooling behavior, or reduced foraging efficiency in social predators [1–4]. Strong Allee effects lead to threshold population densities, below which the population growth is negative and the population is likely to go extinct [5, 6]. Such effects may be observed on different organisms including vertebrates, invertebrates, and plants. The effect usually saturates or vanishes as the population size gets larger. Although some work has shown that Allee effects might play a stabilizing and protective role [7–10], most studies have highlighted its negative impact on population persistence [4, 5, 11–14].

Many investigators have explored the consequences and the different ways of incorporating Allee effects into deterministic continuous-time population models [15–19]. For example, several recent papers have explored the role that Allee effects played in competition and predator-prey interactions. Wang et al. [15] highlighted that introducing an Allee

effect into a Lotka-Volterra competitor system would destabilize it. Courchamp et al. [16] have shown that cooperative species, because they are sensitive to an Allee effect, are more sensitive to interactions with natural enemies such as competitors or predators. Some studies have also shown that structured prey populations and stage-specific predation lead to Allee effects in the (top) predator population [17–19].

In this paper, we are interested in deterministic discrete-time population models incorporating Allee effects. Considering many researches and studies in mathematical biology, we can say that discrete-time models described by difference equations are more appropriate and realistic than the continuous-time models in the case of populations having distinct, nonoverlapping generations. In particular, discrete-time models can also provide more efficient computational models for numerical simulations and richer dynamics can be observed from them. In two previous papers [20, 21], the stability of a simple discrete-time predator-prey system with Holling's type I functional response is studied, while only the prey obeys the Allee effect. However, Allee effect can occur not only on prey but also on predator populations in real ecosystems. If the predator and prey have the Allee effect respectively, what will happen? For these issues, we propose

a discrete-time Leslie-type predator-prey model and explore the asymmetrical impact of Allee effect on that system.

This paper is organized as follows. In Section 2, we briefly introduce the mathematical model which we will analyze and obtain the local stability conditions for the equilibrium points. After that, in Section 3, we study local behavior of the equilibrium points when the predator and prey population are subject to an Allee effect, respectively. In Section 4, some numerical simulations are given and the asymmetrical impact of Allee effect on the system is investigated. Finally, the last section of the paper is devoted to the discussion and remarks.

2. The Mathematical Model and Stability Analysis

We introduce the following discrete-time predator-prey system:

$$\begin{aligned} x_{n+1} &= x_n + rx_n \left(1 - \frac{x_n}{K}\right) - \alpha x_n y_n, \\ y_{n+1} &= y_n + \beta y_n \left(1 - \frac{y_n}{x_n}\right), \end{aligned} \quad (1)$$

where r , α , and β are positive constants, and K is the carrying capacity. The parameter r is the maximal growth rate of the prey, α is the predation parameter, and β is the coefficient of food utilization. The variables x_n , y_n represent the populations density of prey and its predator at the n th generation, respectively. Here, the term $x_n + rx_n(1 - x_n/K)$ stands for the increasing rate of the prey population in the absence of predator, while the term $\alpha x_n y_n$ represents the decreasing rate due to predation. Finally, the term $y_n + \beta y_n(1 - y_n/x_n)$ stands for the variation of predator density which depends on both the number of preys and predators. The predator population grows logistically with a carrying capacity, x_n , that is proportional to the number of prey. This was firstly introduced by Leslie [22]. Notice that, if the predator density disappears in this model, then the prey density satisfies the discrete logistic-type model.

Next we will consider the local stability of the equilibrium points of system (1). To get this, firstly observe that corresponding equilibrium points of system (1) are $(K, 0)$ and (x_0^*, y_0^*) , where

$$x_0^* = y_0^* = \frac{rK}{r + \alpha K}. \quad (2)$$

Clearly, (x_0^*, y_0^*) is the unique positive equilibrium point of system (1).

The Jacobian matrix of system (1) at the equilibrium point $(K, 0)$ is

$$J_0^1 = \begin{pmatrix} 1-r & -\alpha \\ 0 & 1+\beta \end{pmatrix}. \quad (3)$$

Hence, $(K, 0)$ is not asymptotically stable. Usually, such a point is called a nonhyperbolic equilibrium point.

For the positive equilibrium point (x_0^*, y_0^*) , the Jacobian matrix is given by

$$J_0^2 = \begin{pmatrix} 1 - \frac{r^2}{r + \alpha K} & -\frac{\alpha r K}{r + \alpha K} \\ \beta & 1 - \beta \end{pmatrix}. \quad (4)$$

Then we have the corresponding characteristic equation to the matrix J_0^2 as follows:

$$\Phi_0(\lambda) = \lambda^2 - \left(2 - \beta - \frac{r^2}{r + \alpha K}\right)\lambda + 1 - \beta - \frac{r^2}{r + \alpha K} + \beta r. \quad (5)$$

It follows from the well-known *Jury conditions* (see in [23]) that the modulus of all roots of (5) is less than 1 if and only if

$$\Phi_0(1) > 0, \quad \Phi_0(-1) > 0, \quad \det J_0^2 < 1. \quad (6)$$

Clearly, $\Phi_0(1) = \beta r > 0$ for any $\beta, r > 0$. On the other hand, $\Phi_0(-1) > 0$ holds if and only if $2(2 - \beta - r^2/(r + \alpha K)) + \beta r > 0$, which implies

$$\frac{r}{2} - \frac{r^2}{\beta(r + \alpha K)} > 1 - \frac{2}{\beta}. \quad (7)$$

Again, the final inequality $\det J_0^2 < 1$ holds if and only if $1 - \beta - r^2/(r + \alpha K) + \beta r < 1$, which is equivalent to

$$r - \frac{r^2}{\beta(r + \alpha K)} < 1. \quad (8)$$

Now combining (7) and (8), we can get the following conclusion.

Theorem 1. *The positive equilibrium point (x_0^*, y_0^*) of system (1) is locally asymptotically stable if*

$$1 + \frac{r}{2} - \frac{2}{\beta} < r - \frac{r^2}{\beta(r + \alpha K)} < 1. \quad (9)$$

The following result is an immediate consequence of Theorem 1.

Corollary 2. *The positive equilibrium point (x_0^*, y_0^*) of system (1) is unstable if and only if*

$$1 - \frac{2}{\beta} > \frac{r}{2} - \frac{r^2}{\beta(r + \alpha K)} \quad \text{or} \quad r - \frac{r^2}{\beta(r + \alpha K)} > 1. \quad (10)$$

3. Asymmetrical Allee Effect

3.1. Allee Effect on Predator Population. Many predators may be easier to experience an Allee effect because a low number of viable hunters will suffer ineffectiveness in predation [24]. In order to show the influence of Allee effect on the dynamics of system (1), we consider the system as subjecting to an Allee effect on predator population and obtain the following model:

$$\begin{aligned} x_{n+1} &= x_n + rx_n \left(1 - \frac{x_n}{K}\right) - \alpha x_n y_n, \\ y_{n+1} &= y_n + \beta y_n \left(\frac{y_n}{y_n + u_1} - \frac{y_n}{x_n}\right), \end{aligned} \quad (11)$$

where $y_n/(y_n + u_1)$ denotes the predator with the “weak” Allee effect, and u_1 the Allee constant satisfying the assumption

$$0 < u_1 < K. \tag{12}$$

All the equilibrium points of system (11) are $(K, 0)$ and $(x_{u_1}^*, y_{u_1}^*)$, where

$$x_{u_1}^* = \frac{K(r + \alpha u_1)}{r + \alpha K}, \quad y_{u_1}^* = \frac{r(K - u_1)}{r + \alpha K}. \tag{13}$$

Under the above assumption, the predator-prey system (11) has unique positive equilibrium point $(x_{u_1}^*, y_{u_1}^*)$. It is clear that $x_{u_1}^* > x_0^*$ and $y_{u_1}^* < y_0^*$.

We can easily see that $(K, 0)$ is also a nonhyperbolic equilibrium point of system (11). Next, we will focus on the positive equilibrium point $(x_{u_1}^*, y_{u_1}^*)$. The Jacobian matrix is

$$J_{u_1} = \begin{pmatrix} 1 - \frac{r}{K}x_{u_1}^* & -\alpha x_{u_1}^* \\ \beta \left(1 - \frac{u_1}{x_{u_1}^*}\right)^2 & 1 - \beta \left(1 - \frac{u_1}{x_{u_1}^*}\right)^2 \end{pmatrix}. \tag{14}$$

Thus the matrix J_{u_1} yields the characteristic equation

$$\Phi_{u_1}(\lambda) = \lambda^2 - (\text{tr } J_{u_1})\lambda + \det J_{u_1} = 0, \tag{15}$$

where

$$\begin{aligned} \text{tr } J_{u_1} &= 2 - \frac{r}{K}x_{u_1}^* - \beta \left(1 - \frac{u_1}{x_{u_1}^*}\right)^2, \\ \det J_{u_1} &= 1 - \frac{r}{K}x_{u_1}^* - \beta \left(1 - \frac{u_1}{x_{u_1}^*}\right)^2 \\ &\quad + \frac{\beta(r + \alpha K)}{K}x_{u_1}^* \left(1 - \frac{u_1}{x_{u_1}^*}\right)^2. \end{aligned} \tag{16}$$

According to the Jury conditions we obtain that the modulus of all roots of (15) is less than 1 which implies the equilibrium point $(x_{u_1}^*, y_{u_1}^*)$ is asymptotically stable if $\Phi_{u_1}(1) > 0$, $\Phi_{u_1}(-1) > 0$, and $\det J_{u_1} < 1$.

Firstly, $\Phi_{u_1}(1) > 0$ holds for any positive parament of system (11), since

$$\Phi_{u_1}(1) = \frac{\beta(r + \alpha K)}{K}x_{u_1}^* \left(1 - \frac{u_1}{x_{u_1}^*}\right)^2. \tag{17}$$

Again, under the assumption $0 < u_1 < K$, $\Phi_{u_1}(-1) > 0$ holds if

$$\begin{aligned} &2 \left[2 - \frac{r}{K}x_{u_1}^* - \beta \left(1 - \frac{u_1}{x_{u_1}^*}\right)^2 \right] \\ &+ \frac{\beta(r + \alpha K)}{K}x_{u_1}^* \left(1 - \frac{u_1}{x_{u_1}^*}\right)^2 > 0, \end{aligned} \tag{18}$$

that is

$$1 - \frac{2}{r + \alpha u_1} > \left[\frac{r(r + \alpha u_1)}{r + \alpha K} - 2 \right] \frac{2K^2(r + \alpha u_1)}{\beta r^2(K - u_1)^2}. \tag{19}$$

Finally, $\det J_{u_1} < 1$ holds if

$$1 - \frac{r}{K}x_{u_1}^* - \beta \left(1 - \frac{u_1}{x_{u_1}^*}\right)^2 + \frac{\beta(r + \alpha K)}{K}x_{u_1}^* \left(1 - \frac{u_1}{x_{u_1}^*}\right)^2 < 1, \tag{20}$$

that is

$$1 - \frac{1}{r + \alpha u_1} < \frac{K^2(r + \alpha u_1)^2}{\beta r(r + \alpha K)(K - u_1)^2}. \tag{21}$$

Now we obtain the following conclusion.

Theorem 3. *Assuming that $0 < u_1 < K$, the positive equilibrium point $(x_{u_1}^*, y_{u_1}^*)$ of system (11) is locally asymptotically stable if the conditions (19) and (21) are satisfied.*

The next result is an immediate consequence of Theorem 3.

Corollary 4. *The positive equilibrium point $(x_{u_1}^*, y_{u_1}^*)$ of system (11) is unstable if and only if*

$$1 - \frac{2}{r + \alpha u_1} < \left[\frac{r(r + \alpha u_1)}{r + \alpha K} - 2 \right] \frac{2K^2(r + \alpha u_1)}{\beta r^2(K - u_1)^2} \tag{22}$$

or

$$1 - \frac{1}{r + \alpha u_1} > \frac{K^2(r + \alpha u_1)^2}{\beta r(r + \alpha K)(K - u_1)^2}. \tag{23}$$

When the predator is with Allee effect, the predator will decrease and the prey will increase. According to Theorem 3 and Corollary 4, the predator must improve rates of their predation and food utilization to avoid extinction (see Figure 1(a)). The predator and prey would coexist when all parameters of the system satisfy Theorem 3.

3.2. Allee Effect on Prey Population. Because of difficulties in finding mates, social dysfunctions, inbreeding depression, and dispersal cost, the prey is usually suffered from Allee effect [5, 25]. In this section, we will consider the predator-prey system (11) as subject to an Allee effect on prey population and analyze the following system:

$$x_{n+1} = x_n + rx_n \left(1 - \frac{x_n}{K}\right) \frac{x_n}{x_n + u_2} - \alpha x_n y_n, \tag{24}$$

$$y_{n+1} = y_n + \beta y_n \left(1 - \frac{y_n}{x_n}\right),$$

where we take $x_n/(x_n + u_2)$ as the Allee effect function and u_2 as the Allee constant satisfying the assumption

$$0 < u_2 < \frac{r}{\alpha}. \tag{25}$$

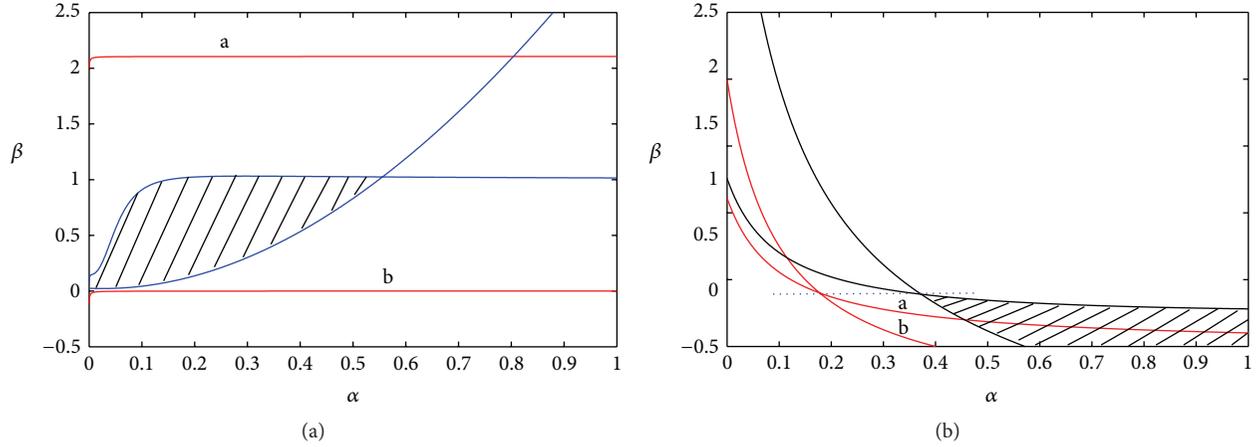


FIGURE 1: Compare the parameter regions of coexistence when the predator and prey are with Allee effect (shaded area) with those when they are without Allee effect, respectively. The area between lines a and b is the regions of coexistence when the predator-prey system is without Allee effect. (a) The situation of the predator with Allee effect. Parameter values are $r = 10$, $K = 100$, and $u_1 = 6$; (b) the situation of the prey with Allee effect. Parameter values are $r = 0.1$, $K = 100$, and $u_2 = 6$.

Then we have two equilibrium points of system (24) as $(K, 0)$ and $(x_{u_2}^*, y_{u_2}^*)$, where

$$x_{u_2}^* = y_{u_2}^* = \frac{K(r - \alpha u_2)}{r + \alpha K}. \tag{26}$$

Clearly, the equilibrium point $(K, 0)$ is a nonhyperbolic equilibrium point of system (24).

Under the assumption (25), the predator-prey system (24) has unique positive equilibrium point $(x_{u_2}^*, y_{u_2}^*)$. It is clear that $x_{u_2}^* < x_0^*$ and $y_{u_2}^* < y_0^*$. After some simple calculations, the Jacobian matrix of (24) turns out to be

$$J_{u_2} = \begin{pmatrix} 1 - \omega x_{u_2}^* & -\alpha x_{u_2}^* \\ \beta & 1 - \beta \end{pmatrix}, \tag{27}$$

where

$$\omega = \frac{r - \alpha u_2}{u_2 + K} - \frac{\alpha u_2 (r + \alpha K)}{r(u_2 + K)}. \tag{28}$$

Then the characteristic equation of matrix J_{u_2} is

$$\Phi_{u_2}(\lambda) = \lambda^2 - (\text{tr } J_{u_2})\lambda + \det J_{u_2} = 0, \tag{29}$$

where

$$\begin{aligned} \text{tr } J_{u_2} &= 2 - \beta - \omega x_{u_2}^*, \\ \det J_{u_2} &= 1 - \beta - \omega x_{u_2}^* + \beta(\alpha + \omega)x_{u_2}^*. \end{aligned} \tag{30}$$

Again, by using the Jury conditions we obtain that the equilibrium point $(x_{u_2}^*, y_{u_2}^*)$ is asymptotically stable if and only if $\Phi_{u_2}(1) > 0$, $\Phi_{u_2}(-1) > 0$ and $\det J_{u_2} < 1$.

We first obtain that $\Phi_{u_1} > 0$ if and only if $\beta(\alpha + \omega)x_{u_2}^* > 0$, and this implies $\alpha + \omega > 0$. Now assume the following function:

$$\varphi(u_2) = \alpha + \omega = \frac{(r + \alpha K)(r - \alpha u_2)}{r(K + u_2)}, \quad u_2 \in \left[0, \frac{r}{\alpha}\right]. \tag{31}$$

Since

$$\varphi'(u_2) = -\frac{(r + \alpha K)^2}{r(K + u_2)^2} < 0, \tag{32}$$

φ is a strictly decreasing function on $[0, r/\alpha]$. Thus, φ attains its minimum value at $u_2 = r/\alpha$. Since $\varphi(r/\alpha) = 0$, we conclude that

$$\alpha + \omega > 0 \quad \forall u_2 \in \left(0, \frac{r}{\alpha}\right). \tag{33}$$

So $\Phi_{u_2}(1) > 0$ if and only if the condition (25) is satisfied. Again,

$$\Phi_{u_2}(-1) > 0 \iff 2(2 - \beta - \omega x_{u_2}^*) + \beta(\alpha + \omega)x_{u_2}^* > 0$$

$$\begin{aligned} &\iff \frac{\alpha K(r - \alpha u_2)}{\beta(r + \alpha K)} + \left(\frac{1}{\beta} - \frac{1}{2}\right) \\ &\quad \times \frac{K(r - \alpha u_2)^2}{r(K + u_2)} > 1 - \frac{2}{\beta}, \end{aligned}$$

$$\det J_{u_2} < 1 \iff 1 - \beta - \omega x_{u_2}^* + \beta(\alpha + \omega)x_{u_2}^*$$

$$\iff \frac{\alpha K(r - \alpha u_2)}{\beta(r + \alpha K)} + \left(\frac{1}{\beta} - 1\right) \frac{K(r - \alpha u_2)^2}{r(K + u_2)} < 1. \tag{34}$$

Now considering (34), we can get the following results.

Theorem 5. Assuming that $0 < u_2 < r/\alpha$, the positive equilibrium point $(x_{u_2}^*, y_{u_2}^*)$ of system (24) is locally asymptotically stable if

$$\begin{aligned} &1 - \frac{2}{\beta} + \frac{K(r - \alpha u_2)^2}{2r(K + u_2)} \\ &< \frac{\alpha K(r - \alpha u_2)}{\beta(r + \alpha K)} + \left(\frac{1}{\beta} - 1\right) \frac{K(r - \alpha u_2)^2}{r(K + u_2)} < 1. \end{aligned} \tag{35}$$

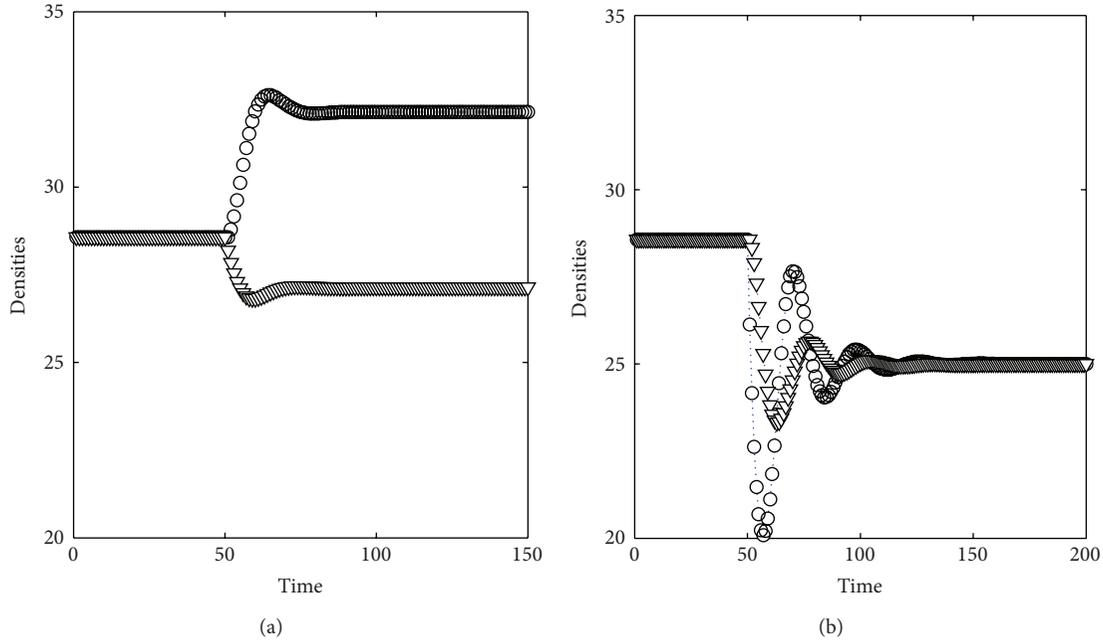


FIGURE 2: Typical examples of predator (∇) and prey (\circ) population dynamics. Allee effect of predator (or prey) starts to occur at $t = 50$. (a) Predator with Allee effect, and (b) prey with Allee effect. Before the Allee effect ($t < 50$), the system stays in steady state. Parameter values: $r = 0.8, \alpha = 0.02, \beta = 0.09, K = 100$, and $u_1 = u_2 = 5$.

The next result is an immediate consequence of Theorem 5.

Corollary 6. *The positive equilibrium point $(x_{u_2}^*, y_{u_2}^*)$ of system (11) is unstable if and only if*

$$1 - \frac{2}{\beta} > \frac{\alpha K (r - \alpha u_2)}{\beta (r + \alpha K)} + \left(\frac{1}{\beta} - \frac{1}{2}\right) \frac{K(r - \alpha u_2)^2}{r(K + u_2)} \quad (36)$$

or

$$\frac{\alpha K (r - \alpha u_2)}{\beta (r + \alpha K)} + \left(\frac{1}{\beta} - 1\right) \frac{K(r - \alpha u_2)^2}{r(K + u_2)} > 1. \quad (37)$$

When the prey is with Allee effect, both densities of prey and predator will decrease. According to Theorem 5 and Corollary 6, decrease of the rates of prey predation and food utilization is the way of the predator and prey to keep coexistence (see Figure 1(b)).

4. Numerical Simulations and Analysis

In this section, some numerical simulations are performed to verify our theoretical results showed in the previous sections and further analyze the asymmetrical influence of Allee effect on the predator-prey system. We used the MATLAB for these computations. Mainly, we show the asymmetrical impact of Allee effect on the predator-prey system in the following three ways: (i) both species without Allee effect (model (1)), (ii) only predator with Allee effect (model (11)), and (iii) only prey with Allee effect (model (24)).

We assume that $K = 100$ throughout this section.

Typical population dynamics are shown in Figure 2, where (a) and (b) denote the cases that the predator species and prey start to be with Allee effect at $t = 50$, respectively. Before the Allee effect, the system stays in steady state. Figure 2 exhibits the asymmetrical influence of Allee effect between prey and predator. It is found from Figure 2(a) that the density of prey species increases and predator decreases in the final equilibrium by the Allee effect in predator (model (11), $u_1 = 5$). In contrast, when the prey species obeys the Allee effect (model (24), $u_2 = 5$), then both species finally decrease their population size (see Figure 2(b)).

In Figure 3, we illustrate the trajectories of predator and prey densities in systems (1), (11), and (24). Here (a1) and (b1) show the trajectories of predator and prey densities in models (1), (a2), and (b2) correspond to model (24) that the predator is subject to the Allee effect; however, (a3) and (b3) correspond to model (11) that the prey is subject to the Allee effect. We use $r = 0.7, \alpha = 0.05, \beta = 0.6, u_1 = u_2 = 5$, and the initial conditions $x_0 = 25, y_0 = 15$ in Figure 3(a1–a3). We see from Figure 3(a1–a3), that, when the predator population is subject to an Allee effect, the local stability of the equilibrium point decreases and it takes a long time to approximate to the corresponding equilibrium point, while the prey population obeys an Allee effect, the local stability of the equilibrium point increases and trajectory of the solution approximates to the corresponding equilibrium point much faster. Furthermore, Figure 3(b1–b3) presents peculiar trajectories of predator and prey population by taking $r = 3, \alpha = 0.04, \beta = 0.04, u_1 = u_2 = 8$, and the initial conditions $x_0 = 30, y_0 = 20$. we find

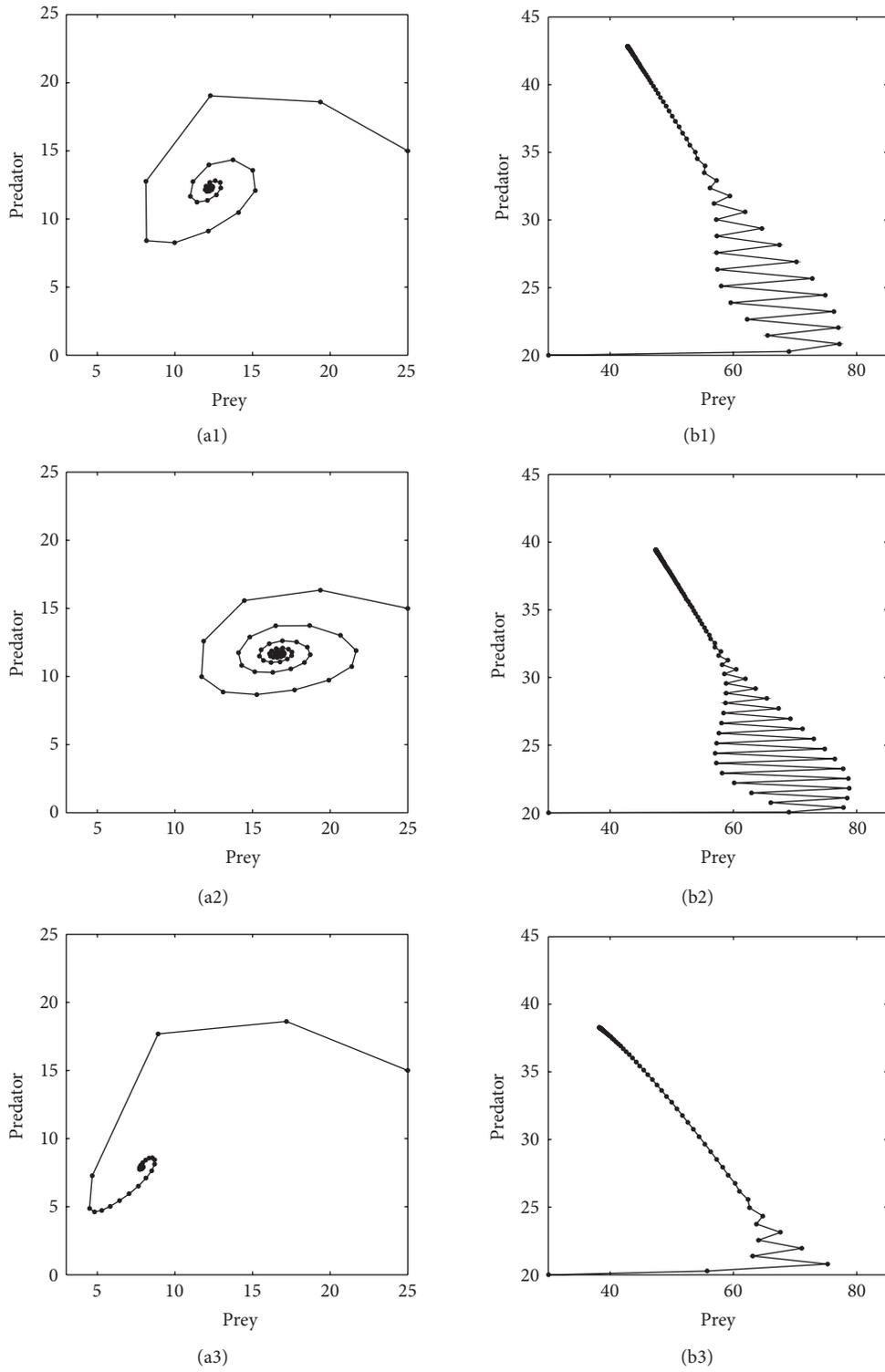


FIGURE 3: The trajectories of the prey and predator population described by model (1) ((a1) and (b1)), model (11) ((a2) and (b2)), and model (24) ((a3) and (b3)). Parameter values: (a1–a3): $r = 0.7$, $\alpha = 0.05$, $\beta = 0.6$, $K = 100$; $u_1 = u_2 = 5$, $x_0 = 25$, and $y_0 = 15$; (b1–b3): $r = 3$, $\alpha = 0.04$, $\beta = 0.04$, $K = 100$, $u_1 = u_2 = 8$, $x_0 = 30$, and $y_0 = 20$.

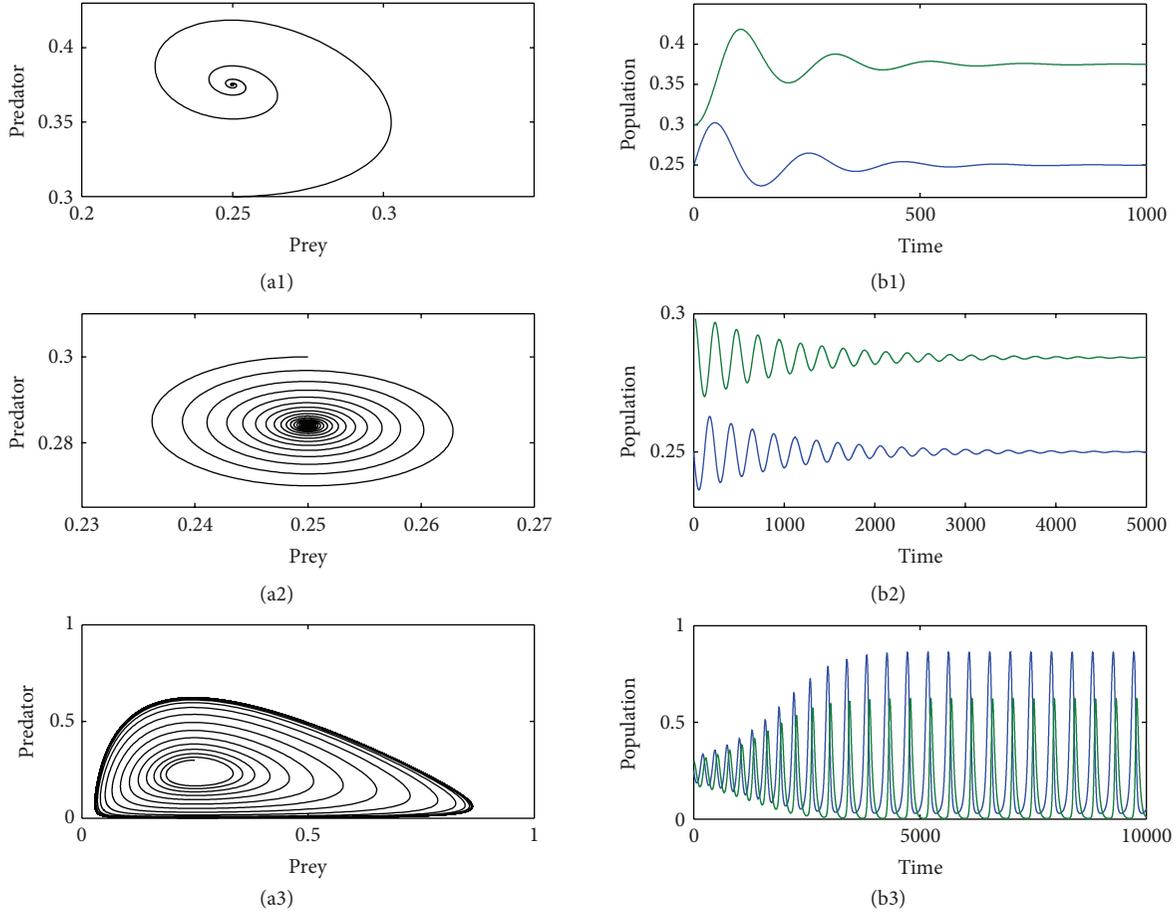


FIGURE 4: The population dynamics of the prey-predator system with and without Allee effect. ((a1) and (b1)) Without Allee effect, ((a2) and (b2)) predator with Allee effect, and ((a3) and (b3)) prey with Allee effect. All parameter values are the same as Figure 2.

that the population dynamics of system (1) become more oscillated and complicated when the predator is with Allee effect. However, when the prey is subject to Allee effect, the system becomes more stable.

The corresponding population dynamics (time versus population) of systems (1), (11), and (24) are shown in Figure 4. The parameters are the same as Figure 3. We can clearly conclude that the predator with Allee effect retards the time of system to reach the corresponding equilibrium point, while the prey with Allee effect makes it much faster.

However, when the prey is with too large value of Allee effect ($u_2 = 18$ in Figure 5(a) and $u_2 = 23$ in Figure 5(b)), there exists the limited cycle in system (24). In order to obtain a more systematic investigation of system (24), we proceeded to construct bifurcation diagrams of the maximum densities of predator and prey versus u_2 and r , respectively. To do this we first integrated the system numerically with $r = 3$, $\alpha = 0.04$, and $\beta = 0.04$. According to Theorem 1, the equilibrium point (x_0^*, y_0^*) of system (1) is asymptotically stable. Letting the system (24) approach the attractor for each value of u_2 , we then plotted successive maxima of both species as a function of u_2 . The resulting bifurcation diagram is plotted in Figure 5. It provides evidence for a qualitative change in the steady state

to periodic dynamics and finally to chaotic dynamics as u_2 is increased.

To construct bifurcation diagrams of the maximum value of prey and predator versus r , we do almost the same work as done to obtain those of both species versus u_2 (Figure 6), except successive maximum densities of predator and prey are plotted of a function of r for $\alpha = 0.03$, $\beta = 0.1$, and $u_2 = 18$. Then the periodic dynamics when $0.7 < r < 1.8$ are showed in Figure 7.

5. Discussion

For many years, Allee concept was mentioned as a minor topic in most ecology textbooks but never given very much importance. Over the last two decades, however, Allee concepts began to surface from obscurity with emerging interest in conservation biology and biological invasions. Ecologists started to ponder the dynamics of low-density populations and found that Allee effects can create thresholds below which populations decline toward extinction [5, 6]. It became apparent that Allee effects are prevalent in low-density populations, arising from a multitude of causes such as mate-location failure, lack of predator satiation, and inability to

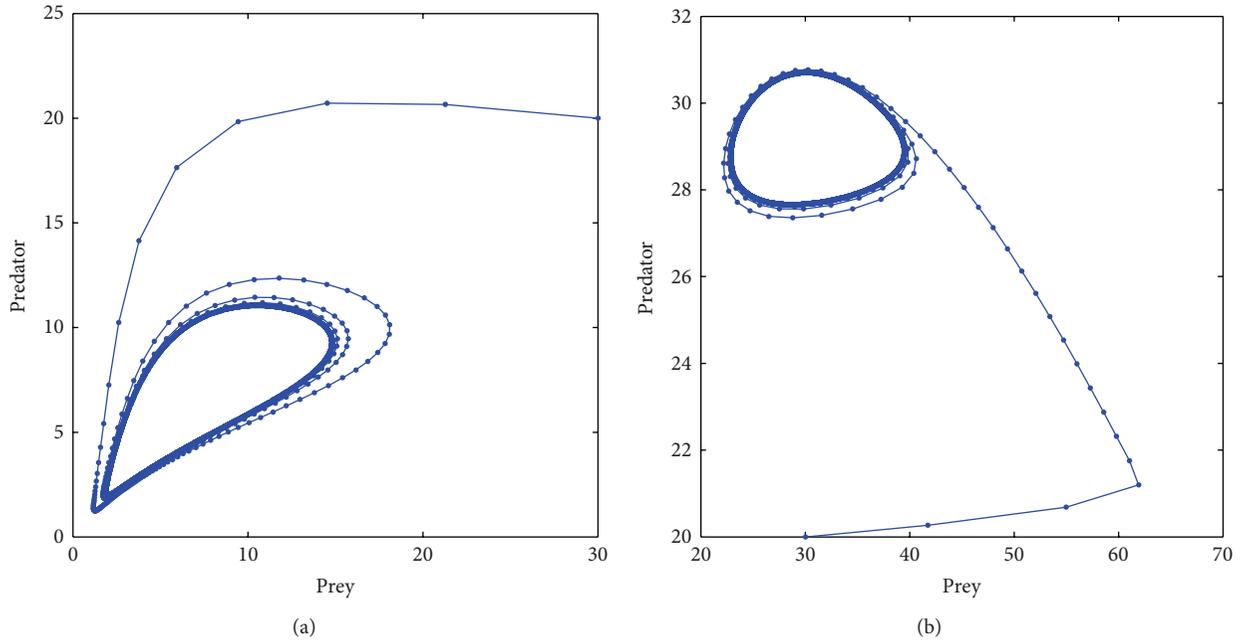


FIGURE 5: The phase portrait of system (24) by using the initial conditions $x_0 = 30$, $y_0 = 20$. There is a periodic solution when the prey obeys the Allee effect which gets to a large value. Parameter values: (a) $r = 0.7$, $\alpha = 0.03$, $\beta = 0.1$, $K = 100$, and $u_2 = 18$; (b) $r = 3$, $\alpha = 0.04$, $\beta = 0.04$, $K = 100$, and $u_2 = 23$.

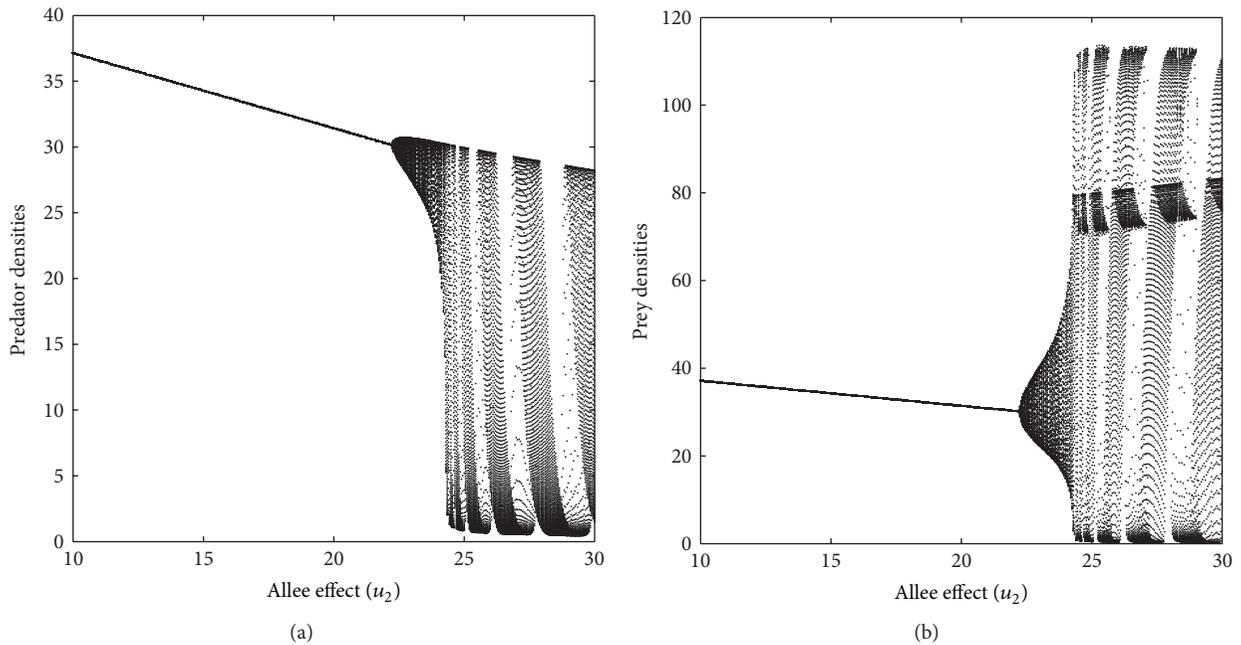


FIGURE 6: Bifurcation diagrams of predator and prey densities in model (24) with the initial conditions $x_0 = 30$, $y_0 = 20$ and the parameter values $r = 3$, $\alpha = 0.04$, $\beta = 0.04$, and $K = 100$.

engage in group feeding. We now know that Allee effects are critical to understanding the dynamics and persistence of both endangered and invading populations [5, 26, 27].

Previous studies demonstrated that Allee effects play an important role in the stability analysis of equilibrium points

of a population dynamics model. An Allee effect may have a stabilizing or a destabilizing effect on population dynamics [5, 7–12, 14]. Even if the system is stable at an equilibrium point, the system subject to an Allee effect may reach its stable state in much longer time.

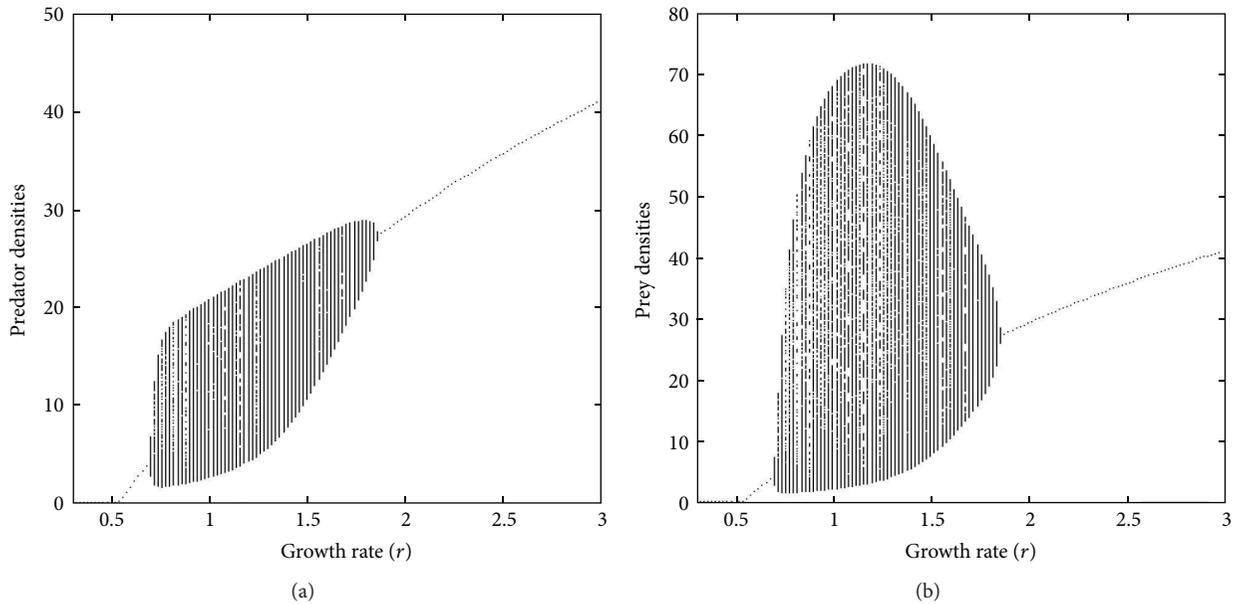


FIGURE 7: Bifurcation diagrams of predator and prey densities in model (24) with the initial conditions $x_0 = 30$, $y_0 = 20$ and the parameter values $\alpha = 0.03$, $\beta = 0.1$, $K = 100$, and $u_2 = 18$.

This paper focused on the asymmetrical influence of Allee effect on two interacting species, each with nonoverlapping generations. By combining mathematical analysis and numerical simulation, we attempt to clarify the overall characteristics of such systems with and without Allee effect. When the predator obeys Allee effect, the predator-prey system will take longer time to approach the steady state and become more complicated, which is in agreement with that of Zhou et al. [14]. However, when the prey population is with Allee effect, the system may reach the equilibrium point much faster. Here, an increase in stability refers to the case that the system with Allee effect approximates to the equilibrium point much faster than that without it. Clearly, in our system, Allee effects not only stabilize but also destabilize the population dynamics, which is in disagreement with previous work [10, 14, 18–21]. The impact of the Allee effect on the stability of population models shows different dynamics when the corresponding model is difference. We study the predator and prey subject to Allee effect in the same system, respectively, and conclude that the Allee effect has a stabilizing or a destabilizing effect also depending on which species of the interacting system obeys it.

The asymmetrical Allee effects have an important influence on interacting species. Compared the dynamics when the Allee effect on predator with that on prey, we obtain successive new conclusions as follow.

- (1) When the predator is with Allee effect, the density of prey species increases and predator decreases in the final equilibrium. The system reaches the stable state in much longer time and the dynamics of it become oscillated and complicated.
- (2) When the prey is with Allee effect, both species finally decrease their population densities. The system

approaches the stable state much faster. However, a large value of Allee effect on prey results in periodic dynamics which is in disagreement with the results obtained by previous researchers [5, 11, 14].

In this paper, we investigated the Allee effect which occurs on the predator and prey population, respectively. However, it may be a very complicated structure when both populations are subject to an Allee effect in our system. Thus, in the future studies, it would be very interesting to improve such structures.

Acknowledgments

The project is supported by the NSF of China (nos. 11226257 and 31170430) and the Fundamental Research Funds for the Central Universities (no. zyz2012074).

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