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Research Article

Existence Solutions of Vector Equilibrium Problems and Fixed Point of Multivalued Mappings

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Let K be a nonempty compact convex subset of a topological vector space. In this paper-sufficient conditions are given for the existence of $x \in K$ such that $F(T) \cap \mathrm{VEP}(F) \neq \emptyset$, where $F(T)$ is the set of all fixed points of the multivalued mapping T and $\mathrm{VEP}(F)$ is the set of all solutions for vector equilibrium problem of the vector-valued mapping F. This leads us to generalize and improve some existence results in the recent references.

1. Introduction

Let X be a Hausdorff topological vector space, a fixed point of such a multi-valued mapping $T : X \to 2^X$, where 2^X denotes, the family of subsets of X means a point p in X such that $p \in T(p)$ and the set of all fixed points of T is denoted by $F(T)$. The study of fixed point theorems of multi-valued mappings started from von Neumann [1] in case of continuous mappings to multi-valued mappings. Since then, various notions of the fixed point theorems for the multi-valued mappings have been studied in [2–4]. Recently, the fixed point theorems for multi-valued mapping were generalized and improved by many authors: see, for example, [5–10].

On the other hand, the equilibrium problems were introduced by Blum and Oettli [11] and by Noor and Oettli [12] in 1994 as generalizations of variational inequalities and optimization problems. The equilibrium problem theory provides a novel and united treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity, and optimization. This theory has had a great impact and influence in the development of several branches of pure and applied sciences. Classical examples of equilibrium problems are variational inequalities, optimization problems, and complementarity problems. Presently, many results on the existence of solutions for vector variational inequalities (in short, VVI) and vector equilibrium problems (in short, VEP) have been established (see, e.g., [13–20]). For the more generalized from of (VEP) and (VVI) as special case, we assume that X and Y are Hausdorff topological vector spaces, K is a nonempty convex subset of X, and C is a pointed closed convex cone in Y with $\operatorname{int} C \neq \emptyset$. Let $T : K \to 2^K$ be a set-valued mapping and for a given vector valued mapping $F : K \times K \to Y$ such that $F(x, x) = 0$ for each $x \in K$, the vector quasi-equilibrium problem (VQEP) and the vector quasi-variational inequality (VQVI), respectively: find $x \in K$ such that

$$\begin{aligned}
&(\mathrm{VQEP}) : x \in T(x), \qquad F(x, y) \notin -C \setminus \{0\} \quad \forall y \in T(x), \\
&(\mathrm{VQVI}) : x \in T(x), \qquad \langle Ax, y - x \rangle \notin -C \setminus \{0\} \quad \forall y \in T(x),
\end{aligned} \tag{1}$$

where $A : K \to L(X, Y)$, and $L(X, Y)$ is denoted by the space of all continuous linear operators for X to Y: see, for example, [21–26] and references therein.

From our mentions on the importance of the fixed point theorems and the equilibrium problems, we have an inspired question to find when the set of the solution of both problems will have a joint solution, not an empty set.

To answer the question, we assume that $T : K \to 2^K$ is a set-valued mapping and for a given vector-valued mapping

$F : K \times K \to Y$ such that $F(x, x) = 0$ for each $x \in K$, let us present the fixed point problem of multi-valued mapping together with the vector equilibrium problem; in particular, it is to find $x \in K$ such that

$$x \in T(x), \quad F(x, y) \notin -C \setminus \{0\} \quad \forall y \in K. \tag{2}$$

This problem shows the relationship in sense of intersection between fixed points of the multi-valued mapping and the vector equilibrium problem so the set of all solutions of the problem (2) is denoted by $F(T) \cap \mathrm{VEP}(F)$. This problem includes vector quasi-equilibrium problems (in short, VQEP) and vector quasi-variational inequalities (in short, VQVI) as special cases.

The main purpose of this paper, we provide sufficient conditions and prove the existence solutions of intersection between the set of all fixed points of the multi-valued mapping and the set of all solutions for vector equilibrium problem by using the generalization of the Fan-Browder fixed point theorem. We also study the existence solutions of intersection between the set of all fixed points of the multi-valued mapping and the set of all solutions for vector variational inequality. Consequently, our results extend the existence theorems of vector quasi-equilibrium problems and vector quasi-variational inequalities.

2. Preliminaries

Throughout this paper, unless otherwise specified, we assume that X, and Y are Hausdorff topological vector spaces, K is a nonempty convex subset of X and C is a pointed closed convex cone in Y with $\operatorname{int} C \neq \emptyset$. From problem (2), some special cases are as follows.

(I) If $T(x) \equiv K$ for all $x \in K$, then problem (2) reduces to the vector equilibrium problem (in short, VEP): find $x \in K$ such that

$$F(x, y) \notin -C \setminus \{0\} \quad \forall y \in K. \tag{3}$$

The set of all solutions for vector equilibrium problem is denoted by $\mathrm{VEP}(F)$.

Moreover, if we set $Y = \mathbb{R}$ and $C = [0, \infty)$, then VEP reduces to the equilibrium problem that the set of all solutions is denoted by $\mathrm{EP}(F)$: find $x \in K$ such that

$$F(x, y) \geq 0 \quad \forall y \in K. \tag{4}$$

(II) If $Y = \mathbb{R}$ and $C = [0, \infty)$, then problem (2) reduces to the problem: find $x \in K$ such that

$$x \in T(x), \quad F(x, y) \geq 0 \quad \forall y \in K. \tag{5}$$

Moreover, this problem includes the quasi-equilibrium problems (in short, QEPs) considered and studied by Lin and Park [27] which is to find $x \in K$ such that

$$x \in T(x), \quad F(x, y) \geq 0 \quad \forall y \in T(x). \tag{6}$$

(III) If $F(x, y) = \langle Ax, y - x \rangle$ for all $x, y \in K$, problem (3) reduces to the vector variational inequality (in short, VVI): find $x \in K$ such that

$$\langle Ax, y - x \rangle \notin -C \setminus \{0\} \quad \forall y \in K. \tag{7}$$

The solution sets of problem (7) is denoted by $\mathrm{VVI}(K, A)$, where $A : K \to L(X, Y)$, and $L(X, Y)$ is denoted by the space of all continuous linear operators for X to Y.

Furthermore, if $Y = \mathbb{R}$ and $C = [0, \infty)$, then VVI reduces to the variational inequality that the set of all solutions is denoted by $\mathrm{VI}(K, A)$: find $x \in K$ such that

$$\langle Ax, y - x \rangle \geq 0 \quad \forall y \in K. \tag{8}$$

(IV) If $F(x, y) = \langle Ax, y - x \rangle$ for all $x, y \in K$, then problem (2) reduces to the following problems that is to find $x \in K$ such that

$$\overline{x} \in T(\overline{x}), \quad \langle A\overline{x}, y - \overline{x} \rangle \notin -C \setminus \{0\} \quad \forall y \in K. \tag{9}$$

This problem shows the relationship in sense of intersection between fixed points of the multi-valued mapping and the vector variational inequality so the set of all solutions of the problem (9) is denoted by $F(T) \cap \mathrm{VVI}(K, A)$.

Let us recall some concepts and properties that are needed in this sequel. Given the multi-valued mappping $T : X \to 2^Y$, the inverse T^{-1} of T is the multi-valued map from $\mathcal{R}(T)$, the range of T, to X defined by

$$x \in T^{-1}(y) \Longleftrightarrow y \in T(x). \tag{10}$$

The mapping $f : X \to Y$ is continuous at some point $x \in X$ if and only if for any neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.

Definition 1 (see [28]). Let X and Y be topological vector spaces. Let K be a nonempty subset of X and C be a point closed convex cone in Y with $\operatorname{int} C \neq \emptyset$, where $\operatorname{int} C$ denotes the topological interior of C. A bifunction $F : K \times K \to Y$ is said to be *C-strongly pseudomonotone* if, for any $x, y \in K$,

$$F(x, y) \notin -C \setminus \{0\} \Longrightarrow F(y, x) \in -C. \tag{11}$$

A mapping $f : K \to Y$ is said to be C-convex if for all $x, y \in K$ and for all $\lambda \in [0, 1]$,

$$\lambda f(x) + (1 - \lambda) f(y) - f(\lambda x + (1 - \lambda) y) \in C. \tag{12}$$

And the mapping f is said to be hemicontinuous if, for all $x, y \in K$ and for all $\lambda \in [0, 1]$,

$$\lambda \longmapsto f(x + \lambda(y - x)) \quad \text{is upper semicontinuous at } 0^+. \tag{13}$$

Remark 2. If $Y = \mathbb{R}$ and $C = [0, \infty)$, then

(1) the C-strongly pseudomonotonicity of $F : K \times K \to Y$ reduces to the monotonicity of $F : K \times K \to \mathbb{R}$ (i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in K$). In fact, $F(x, y) \notin -C \setminus \{0\} \Leftrightarrow F(x, y) \geq 0$, this implies that $-F(y, x) \geq 0$ and it is equivalence to $F(y, x) \in -C$;

(2) the C-convexity of $f : K \to Y$ reduces to the convexity of $f : K \to \mathbb{R}$ (i.e., $f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y)$).

Definition 3 (see [29]). Let X be a topological space and let Y be a set. A map $T : X \to 2^Y$ is said to have the *local intersection property* if for each $x \in X$ with $T(x) \neq \emptyset$ there exists an open neighborhood $V(x)$ of x such that $\bigcap_{z \in V(x)} T(z) \neq \emptyset$.

The following lemma is useful in what follows and can be found in [30].

Lemma 4. *Let X be a topological space and let Y be a set. Let $T : X \to 2^Y$ be a map with nonempty values. Then, the following are equivalent.*

(i) *T has the local intersection property.*

(ii) *There exists a map $F : X \to 2^Y$ s.t. $F(x) \subset T(x)$ for each $x \in X$, $F^{-1}(y)$ is open for each $y \in Y$ and $X = \bigcup_{y \in Y} F^{-1}(y)$.*

Subsequently, Browder [2] obtained in 1986 the following fixed point theorem.

Theorem 5 (Fan-Browder fixed point theorem). *Let X be a nonempty compact convex subset of a Hausdorff topological vector space and $T : X \to 2^X$ be a map with nonempty convex values and open fibers (i.e., for $y \in Y$, $T^{-1}(y)$ is called the fiber of T on y). Then T has a fixed point.*

The generalization of the Fan-Browder fixed point theorem was obtained by Balaj and Muresan [31] in 2005 as follows.

Theorem 6. *Let X be a compact convex subset of a t.v.s and $T : X \to 2^X$ be a map with nonempty convex values having the local intersection property. Then T has a fixed point.*

3. Main Theorem

In this section, the existence solutions of the fixed point for multi-valued mappings and the vector equilibrium problems will be presented. To do this, the following lemma is necessary.

Lemma 7. *Let K be a nonempty and convex subset of X. Let $T : K \to 2^K$ be a set-valued mapping such that for any $x \in K$, $T(x)$ is nonempty convex subset of K. Assume that $F : K \times K \to Y$ be a hemicontinuous in the first argument, C-convex in the second argument, and C-strong pseudomonotone. Then the following statements are equivalent.*

(i) *Find $x \in K$ such that $x \in T(x)$ and $F(x, y) \notin -C \setminus \{0\}$ $\forall y \in K$.*

(ii) *Find $x \in K$ such that $x \in T(x)$ and $F(y, x) \in -C$ $\forall y \in K$.*

Proof. (i) → (ii) It is clear by the C-strong pseudomonotone.

(ii) → (i) Let $x \in K$ such that

$$x \in T(x), \quad F(y, x) \in -C \quad \forall y \in K. \tag{14}$$

For any $y \in K$ and $\alpha \in (0, 1)$, we set $z_\alpha = \alpha y + (1 - \alpha)x$ and so we have $z_\alpha \in K$ because K is convex. By the assumption, we conclude that

$$x \in T(x), \quad F(z_\alpha, x) \in -C. \tag{15}$$

Since F is C-convex in the second argument and by (15), we get

$$\begin{aligned} 0 &= F(z_\alpha, \alpha y + (1 - \alpha) x) \\ &\in \alpha F(z_\alpha, y) + (1 - \alpha) F(z_\alpha, x) - C \\ &\subseteq \alpha F(z_\alpha, y) + (-C) + (-C) \\ &\subseteq \alpha F(z_\alpha, y) - C. \end{aligned} \tag{16}$$

This implies that $\alpha F(z_\alpha, y) \in C$ and since C is a convex cone then we have $F(z_\alpha, y) \in C$. Since F is a hemicontinuous in the first argument and $z_\alpha \to x$ as $\alpha \to 0^+$, we have $F(x, y) \in C$ for all $y \in K$. Therefore we obtain that

$$x \in T(x), \quad F(x, y) \notin -C \setminus \{0\} \quad \forall y \in K. \tag{17}$$

This completes the proof. □

Theorem 8. *Let K be a nonempty compact convex subset of X and let $F : K \times K \to Y$ be a C-strong pseudomonotone, hemicontinuous in the first argument and C-convex, l.s.c. in the second argument such that $0 = F(x, x)$ for all $x \in K$. Let $T : K \to 2^K$ be a set-valued mapping such that for any $x \in K$, $T(x)$ is nonempty convex subset of K and for any $y \in K$, $T^{-1}(y)$ is open in K. Assume the set $P := \{x \in X \mid x \in T(x)\}$ is open in K and for any $x \in K$, $T(x) \cap \{y \in K \mid F(y, x) \notin -C\} \neq \emptyset$. Then $F(T) \cap$ VEP $(F) \neq \emptyset$.*

Proof. For any $x \in K$, we define the set-valued mapping $A, B : K \to 2^K$ by

$$\begin{aligned} A(x) &= \{y \in K \mid F(y, x) \notin -C\}, \\ B(x) &= \{y \in K \mid F(x, y) \in -C \setminus \{0\}\}. \end{aligned} \tag{18}$$

Also, we define the set-valued mapping $H : K \to 2^K$ by

$$H(x) = \begin{cases} B(x), & \text{if } x \in P, \\ T(x), & \text{if } x \in K \setminus P. \end{cases} \tag{19}$$

Then we have $H(x)$ is convex. Indeed, let $y_1, y_2 \in B(x)$ and $\alpha \in (0, 1)$. Since F is C-convex in the second argument, we have

$$\begin{aligned} F(x, \alpha y_1 + (1 - \alpha) y_2) &\in \alpha F(x, y_1) + (1 - \alpha) F(x, y_2) - C \\ &\subseteq (-C \setminus \{0\}) - C \\ &= -C \setminus \{0\}. \end{aligned} \tag{20}$$

Then $\alpha y_1 + (1 - \alpha) y_2 \in B(x)$ and hence $B(x)$ is convex. Since $T(x)$ is convex, then $H(x)$ is also convex.

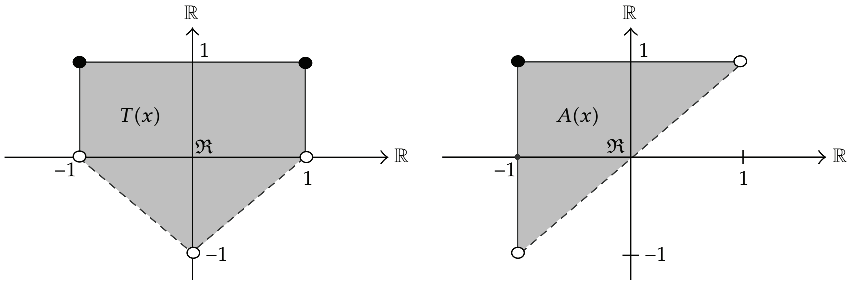

Figure 1

By the defining of H, we see that H has no fixed point Indeed, suppose that there is $x \in K$ such that $x \in H(x)$. It is impossible for $x \in K \setminus P$, then $x \in P$ and so $x \in B(x)$. Thus $F(x,x) \in -C \setminus \{0\}$, a contradiction with $0 = F(x,x)$. Using the contrapositive of Theorem 6, we obtain that H has no local intersection property. Define the set-valued mapping $G : K \to 2^K$ by

$$G(x) = \begin{cases} A(x), & \text{if } x \in P, \\ A(x) \cap T(x), & \text{if } x \in K \setminus P. \end{cases} \tag{21}$$

From the C-strong pseudmonotonicity of F, we have $G(x) \subseteq H(x)$ for any $x \in K$. Next, we will show that for each $y \in K, G^{-1}(y)$ is open in K. For any $y \in K$, we denote the complement of $A^{-1}(y)$ by $[A^{-1}(y)]^C = \{x \in K \mid F(y,x) \in -C\}$. Since C is closed and F is l.s.c. in the second argument, we have $[A^{-1}(y)]^C$ is closed in K and so $A^{-1}(y)$ is open in K. We note that

$$\begin{aligned} G^{-1}(y) &= \left(A^{-1}(y) \cap P\right) \cup \left(A^{-1}(y) \cap T^{-1}(y) \cap (K \setminus P)\right) \\ &= \left[A^{-1}(y) \cup \left(A^{-1}(y) \cap T^{-1}(y) \cap (K \setminus P)\right)\right] \\ &\quad \cap \left[P \cup \left(A^{-1}(y) \cap T^{-1}(y) \cap (K \setminus P)\right)\right] \\ &= \left\{A^{-1}(y) \cap \left[A^{-1}(y) \cup (K \setminus P)\right]\right\} \\ &\quad \cap \left\{\left[P \cup \left(A^{-1}(y) \cap T^{-1}(y)\right)\right] \cap K\right\} \\ &= A^{-1}(y) \cap \left[P \cup \left(A^{-1}(y) \cap T^{-1}(y)\right)\right]. \end{aligned} \tag{22}$$

Since for any $y \in K, T^{-1}(y), A^{-1}(y)$, and P are open, we have $G^{-1}(y)$ is open in K. Thus, by the contrapositive of Lemma 4, we have

$$K \not\subseteq \bigcup_{y \in K} G^{-1}(y). \tag{23}$$

Hence, there exists $\overline{x} \in K$ such that $\overline{x} \notin G^{-1}(y)$ for all $y \in K$. That is $G(\overline{x}) = \emptyset$. If $\overline{x} \in K \setminus P$ then $A(x) \cap T(x) = \emptyset$, which contradicts with the assumption. Therefore, $\overline{X} \in P$ and $A(x) = \emptyset$. This implies that $\overline{x} \in A(\overline{x})$ and $F(y,x) \in -C$ for all $y \in K$. This completes the proof by Lemma 7. □

The following example guarantees the assumption that the set $T(x) \cap A(x) \neq \emptyset$, where $A(x) = \{y \in KF(y,x) \notin -C\}$.

Example 9. Let $X, Y = \mathbb{R}$, $K = [-1,1]$, and $C = [0,\infty)$. For any $x, y \in K$, we define two mappings $F : K \times K \to 2^Y$ and $T : K \to 2^K$ by

$$\begin{aligned} F(x,y) &= x - y \quad \forall x, y \in [-1,1], \\ T(x) &= \begin{cases} (-1-x, 1], & \text{if } -1 \le x \le 0, \\ (x-1, 1], & \text{if } 0 \le x \le 1. \end{cases} \end{aligned} \tag{24}$$

Clearly, $T(x)$ is nonempty convex subset of K and $T^{-1}(y)$ is open in K. If $F(x,y) \notin -C \setminus \{0\}$ for all $x, y \in K$, then $x \ge y$ and it implies that for $x \ge y$, $F(x,y) \in -C$. This shows that F is C-strong pseudomonotone. Let $x, y_1, y_2 \in K$ and $\alpha \in [0,1]$ and since $0 \in K$, we obtain that

$$\begin{aligned} F(x, \alpha y_1 + (1-\alpha) y_2) &= x - (\alpha y_1 + (1-\alpha) y_2) \\ &= \alpha (x - y_1) + (1-\alpha) y_2 - 0 \\ &\in \alpha F(x, y_1) + (1-\alpha) F(x, y_2) - C. \end{aligned} \tag{25}$$

Then F is C-convex in the second argument and it is easy to see that F is a hemicontinuous in the first argument and l.s.c. in the second argument.

Note that

$$\begin{aligned} A(x) &= \{y \in K \mid F(y,x) \notin -C\} \\ &= \{y \in [-1,1] \mid y > x\} \\ &= (x, 1], \quad \text{where } x < 1. \end{aligned} \tag{26}$$

If $-1 \le x \le 0$, then $T(x) = (-1-x, 1]$ which includes $(0,1]$. Also $(0,1]$ is contained $A(x)$ for all $-1 \le x \le 0$. Otherwise, $(x,1] \subseteq (x-1,1]$ for any $0 \le x < 1$. This is to confirm the set $T(x) \cap A(x) \neq \emptyset$ for each $x \in K$ (see Figure 1). Moreover, this example asserts that the set $P = \{x \in X \mid x \in T(x)\}$ is open in K because it is equal to the set $(-0.5, 1]$ which is open in K.

Taking $T(x) \equiv K$ for all $x \in K$ in Theorem 8, we have the following results.

Corollary 10. *Let K be a nonempty compact convex subset of X and $F : K \times K \to Y$ be a C-strong pseudomonotone, hemicontinuous in the first argument and C-convex, l.s.c. in the second argument such that $0 = F(x,x)$ for all $x \in K$. Then,* VEP *has a solution.*

If we set the vector-valued mapping $F \equiv 0$, then Theorem 8 reduces to the following corollary introduced by Browder (see [2, Theorem 1]).

Corollary 11. *Let K be a nonempty compact convex subset of X. Let $T : K \to 2^K$ be a set-valued mapping such that for any $x \in K, T(x)$ is a nonempty convex subset of K and for any $y \in K, T^{-1}(y)$ is open in K. Then there exists $\overline{x}$ in K such that $\overline{x} \in T(\overline{x})$.*

If we set $Y = \mathbb{R}$ and $C = [0, \infty)$ in Theorem 8 together with Remark 2, we have the following result.

Corollary 12. *Let K be a nonempty compact convex subset of X and let $F : K \times K \to \mathbb{R}$ be a monotone, hemicontinuous in the first argument and convex, l.s.c. in the second argument such that $0 = F(x, x)$ for all $x \in K$. Let $T : K \to 2^K$ be a set-valued mapping such that for any $x \in K, T(x)$ is a nonempty convex subset of K and for any $y \in K$, $T^{-1}(y)$ is open in K. Assume the set $P := \{x \in X \mid x \in T(x)\}$ is open in K and for any $x \in K$, $T(x) \cap \{y \in K \mid F(y, x) > 0\} \neq \emptyset$. Then $F(T) \cap \mathrm{EP}(F) \neq \emptyset$.*

Let $L(X, Y)$ be a space of all linear continuous operators from X to Y. A mapping $A : K \to L(X, Y)$ is said to be C-*strong pseudomonotone* if it satisfies

$$\forall x, y \in K,$$
$$\langle A(x), y - x\rangle \notin -C \setminus \{0\} \Longrightarrow \langle A(y), x - y\rangle \in -C \qquad (27)$$

and it is called *hemicontinuous* if, for all $x, y \in K$ and for all $\lambda \in [0, 1]$, the mapping $\lambda \mapsto \langle T(x+\lambda(y-x), z)\rangle$ is continuous at 0^+.

As a direct consequence of Theorem 8, we obtain the following result.

Theorem 13. *Let K be a nonempty compact convex subset of X and let $A : K \to L(X, Y)$ be a C-strong pseudomonotone and hemicontinuous. Let $T : K \to 2^K$ be a set-valued mapping such that for any $x \in K$, $T(x)$ is nonempty convex subset of K and for any $y \in K, T^{-1}(y)$ is open in K. Assume the set $P := \{x \in X \mid x \in T(x)\}$ is open in K and for any $x \in K$, $T(x) \cap \{y \in K \mid \langle A(y), x - y\rangle \notin -C\} \neq \emptyset$. Then there exists $\overline{x} \in K$ such that*

$$\overline{x} \in T(\overline{x}), \quad \langle A(\overline{x}), y - \overline{x}\rangle \notin -C \setminus \{0\}, \quad \forall y \in K. \qquad (28)$$

Proof. We define the vector value mapping $F : K \times K \to Y$ by

$$F(x, y) = \langle Ax, y - x\rangle. \qquad (29)$$

We will show that F satisfies all conditions in Theorem 8. Clearly $F(x, x) = 0$ and by the assumptions of A, we have F is C-strong pseudomonotone and hemicontinuous in the first argument. Let $x \in K$ be fixed. For any $y, z \in K$ and $\theta \in [0, 1]$, we obtain that

$$\begin{aligned} F(x, \theta y + (1-\theta) z) &= \langle Ax, (\theta y + (1-\theta) z) - x\rangle \\ &= \theta \langle Ax, y - x\rangle + (1-\theta)\langle Ax, z - x\rangle \\ &\in \theta \langle Ax, y - x\rangle \\ &\quad + (1-\theta)\langle Ax, z - x\rangle - C \\ &= \theta F(x, y) + (1-\theta) F(x, z) - C. \end{aligned} \qquad (30)$$

Then F is C-convex in the second argument.

Next, we will show that F is l.s.c. in the second argument. Let $x \in K$ be fixed. Let $y_0 \in K$ and N be a neighborhood of $F(x, y_0)$. Since the linear operator Ax is continuous, there exists an open neighborhood M of y_0 such that for all $y \in M$, $\langle Ax, y-x\rangle \in N$ because N is a neighborhood of $\langle Ax, y_0-x\rangle$. Thus for all $y \in M$, $F(x, y) \in N$. Hence, F is continuous in the second argument and so it is l.s.c. in the second argument. Then all hypotheses of the Theorem 8 hold and hence, there exists $\overline{x} \in K$ such that

$$\overline{x} \in T(\overline{x}), \quad F(\overline{x}, y) = \langle A\overline{x}, y - x\rangle \notin -C \setminus \{0\}, \quad \forall y \in K. \qquad (31)$$

This completes the proof. □

If we take $T(x) \equiv K$ for all $x \in K$ in Theorem 13, we have the following corollary.

Corollary 14. *Let K be a nonempty compact convex subset of X and let $A : K \to L(X, Y)$ be a C-strong pseudomonotone and hemicontinuous. Then,* VVI *has a solution.*

If we set $Y = \mathbb{R}$ and $C = [0, \infty)$ in Theorem 13, we have the following result.

Corollary 15. *Let K be a nonempty compact convex subset of X and let $A : K \to L(X, \mathbb{R})$ be a monotone and hemicontinuous in the first argument. Let $T : K \to 2^K$ be a set-valued mapping such that for any $x \in K$, $T(x)$ is a nonempty convex subset of K and for any $y \in K$, $T^{-1}(y)$ is open in K. Assume the set $P := \{x \in X \mid x \in T(x)\}$ is open in K and for any $x \in K$, $T(x) \cap \{y \in K \mid \langle A(y), x - y\rangle > 0\} \neq \emptyset$. Then $F(T) \cap \mathrm{VI}(K, A) \neq \emptyset$.*

Remark 16. (1) Theorems 8 and 13 are the extensions of vector quasi-equilibrium problems and vector quasi-variational inequalities, respectively.

(2) If X is a real Banach space, then Corollary 10 comes to be Theorem 2.3 in [28].

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