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Research Article

A New Characteristic Nonconforming Mixed Finite Element Scheme for Convection-Dominated Diffusion Problem

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A characteristic nonconforming mixed finite element method (MFEM) is proposed for the convection-dominated diffusion problem based on a new mixed variational formulation. The optimal order error estimates for both the original variable u and the auxiliary variable σ with respect to the space are obtained by employing some typical characters of the interpolation operator instead of the mixed (or expanded mixed) elliptic projection which is an indispensable tool in the traditional MFEM analysis. At last, we give some numerical results to confirm the theoretical analysis.

1. Introduction

Consider the following convection-dominated diffusion problem:

$$\begin{gathered}
u_t + \mathbf{a}(x, y) \cdot \nabla u - \nabla \cdot (b(x, y) \nabla u) \\
= f(x, y, t), \quad (x, y, t) \in \Omega \times (0, T), \\
u(x, y, t) = 0, \quad (x, y, t) \in \partial\Omega \times (0, T), \\
u(x, y, 0) = u_0(x, y), \quad (x, y) \in \Omega,
\end{gathered} \tag{1}$$

where Ω is a bounded polygonal domain in $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial\Omega$, $J = (0, T], 0 < T < +\infty$. ∇ and $\nabla\cdot$ denote the gradient and the divergence operators, respectively.

Model (1) has been widely used to describe the conduction of heat in fluid, the diffusion of soluble minerals or pollutants in ground water, the incompressible miscible displacement in porous media, and so on. The parameters appearing in (1) satisfy the following assumptions [1, 2]:

(A1) u denotes, for example, the concentration or saturation of soluble substances;

(A2) $\mathbf{a}(x, y) = (a_1(x, y), a_2(x, y))$ represents Darcy velocity of mixed fluid, and f a source term;

(A3) $b(x, y)$ is sufficiently smooth and there exist constants b_1 and b_2, such that

$$0 < b_1 \le b(x, y) \le b_2 < +\infty, \quad \forall (x, y) \in \Omega. \tag{2}$$

It is well known that convection dominated-diffusion problem (1) often presents serious numerical difficulties. The standard numerical methods, such as finite difference method (FDM), FEM and MFEM, usually produce numerical diffusion along sharp fronts. In order to overcome this fatal defect, Douglas et al. [3] combined the method of characteristics with FE procedures so as to reduce the truncation error, and it allows us to use large time steps without lose of accuracy. Moreover, there have appeared many effective discretization schemes concentrating on the hyperbolic nature of the equation, for example, characteristic FD streamline diffusion method [4, 5], Eulerian-Lagrangian method [6, 7], characteristic-finite volume element method [2, 8, 9], characteristics-mixed covolume method [10, 11], the modified method of characteristic-Galerkin FE procedure [12], characteristic nonconforming FEM [13–15], characteristic MFEM [16–19] and expanded characteristic MFEM [1, 20], and so forth.

As for the characteristic MFEM or expanded characteristic MFEM, the convergence rates of u and σ in existing literature were suboptimal [11, 18, 21, 22] and the convergence analysis was valid only to the case of the lowest order MFE approximation [10, 17]. So far, to our best knowledge there are few studies on the optimal order error estimates except for [23], in which a family of characteristic MFEM with arbitrary degree of Raviart-Thomas-Nédélec space in [24, 25] for transient convection diffusion equations was studied.

Recently, based on the low regularity requirement of the flux variable in practical problems, a new mixed variational form for second elliptic problem was proposed in [26]. It has two typical advantages: the flux space belongs to the square integrable space instead of the traditional $H(\operatorname{div};\Omega)$, which makes the choices of MFE spaces sufficiently simple and easy; the LBB condition is automatically satisfied when the gradient of approximation space for the original variable is included in approximation space for the flux variable. Motivated by this idea, this paper will construct a characteristic nonconforming MFE scheme for (1) with a new mixed variational formulation. Similar to the expanded characteristic MFEM, the coefficient b of (1) in this proposed scheme does not need to be inverted; therefore, it is also suitable for the case when b is small. By employing some distinct characters of the interpolation operators on the element instead of the mixed or expanded mixed elliptic projection used in [1, 17, 20] which is an indispensable tool in the traditional characteristic MFEM analysis, the $O(h^2)$ order error estimate in L^2-norm for original variable u, which is one order higher than [1, 20] and half order higher than [18], is derived, and the optimal error estimates with order $O(h)$ for auxiliary variable σ in L^2-norm and for u in broken H^1-norm are obtained, respectively. It seems that the result for u in broken H^1-norm has never been seen in the existing literature by making full use of the high-accuracy estimates of the lowest order Raviart-Thomas element proved by the technique of integral identities in [27] and the special properties of nonconforming EQ_1^{rot} element (see Lemma 1 below).

The paper is organized as follows. Section 2 is devoted to the introduction of the nonconforming FE approximation spaces and their corresponding interpolation operators. In Section 3, we will give the construction of the new characteristic nonconforming MFE scheme and two important lemmas, and the existence and uniqueness of the discrete scheme solution will be proved. In Section 4, the convergence analysis and optimal error estimates for both the original variable u and the flux variable σ are obtained. In Section 5, some numerical results are provided to illustrate the effectiveness of our proposed method.

Throughout this paper, C denotes a generic positive constant independent of the mesh parameters h and Δt with respect to domain Ω and time t.

2. Construction of Nonconforming MFEs

As in [28], we frequently employ the space $L^2(\Omega)$ of square integrable functions with scalar product and norm

$$(u,v) = (u,v)_{L^2(\Omega)} = \left(\int_\Omega uv\,dx\,dy\right)^{1/2},$$
$$\|v\| = \|v\|_{L^2(\Omega)} = \left(\int_\Omega v^2\,dx\,dy\right)^{1/2}. \tag{3}$$

We also employ the Sobolev space $H^m(\Omega)$, $m \geq 1$, of functions v such that $D^\beta v \in L^2(\Omega)$ for all $|\beta| \leq m$, equipped with the norm and seminorm

$$\|v\|_{m,\Omega} = \|v\|_{H^m(\Omega)} = \left(\sum_{|\beta|\leq m} \left\|D^\beta v\right\|^2\right)^{1/2},$$
$$|v|_{m,\Omega} = |v|_{H^m(\Omega)} = \left(\sum_{|\beta|=m} \left\|D^\beta v\right\|^2\right)^{1/2}. \tag{4}$$

The space $H_0^1(\Omega)$ denotes the closure of the set of infinitely differentiable functions with compact supports in Ω. For any Sobolev space Y, $L^p(0,T;Y)$ is the space of measurable Y-valued functions Φ of $t \in (0,T)$, such that $\int_0^T \|\Phi(\cdot,t)\|_Y^p dt < \infty$ if $1 \leq p < \infty$, or such that $\operatorname{ess\,sup}_{0<t<T}\|\Phi(\cdot,t)\|_Y < \infty$ if $p = \infty$.

We now introduce the nonconforming MFE space described in [29] for and summarize it as follows.

Let $\Omega \subset \mathbb{R}^2$ be a polygon domain with edges parallel to the coordinate axes on xy plane, and let T_h be a rectangular subdivision of Ω satisfying the regular condition [30]. For a given element $e \in T_h$, denote the barycenter of element e by (x_e, y_e), denote the length of edges parallel to x-axis and y-axis by $2h_{x_e}$ and $2h_{y_e}$, respectively, $h_e = \max_{e\in T_h}\{h_{x_e}, h_{y_e}\}$, $h = \max_{e\in T_h} h_e$.

Let $\widehat{e} = [-1,1]\times[-1,1]$ be the reference element on $\widehat{x}\widehat{y}$ plane and four vertices $\widehat{d}_1 = (-1,-1)$, $\widehat{d}_2 = (1,-1)$, $\widehat{d}_3 = (1,1)$, and $\widehat{d}_4 = (-1,1)$, the four edges $\widehat{l}_1 = \overline{\widehat{d}_1\widehat{d}_2}$, $\widehat{l}_2 = \overline{\widehat{d}_2\widehat{d}_3}$, $\widehat{l}_3 = \overline{\widehat{d}_3\widehat{d}_4}$, and $\widehat{l}_4 = \overline{\widehat{d}_4\widehat{d}_1}$. Then there exists an affine mapping $F_e : \widehat{e} \to e$ as

$$x = x_e + h_{x_e}\widehat{x},$$
$$y = y_e + h_{y_e}\widehat{y}. \tag{5}$$

Define the FE spaces $(\widehat{e}, \widehat{P^i}, \widehat{\sum}^i)$, $(i = 1,2,3)$ by

$$\widehat{\sum}^1 = \{\widehat{v}_1, \widehat{v}_2, \widehat{v}_3, \widehat{v}_4, \widehat{v}_5\}, \quad \widehat{P}^1 = \operatorname{span}\{1, \widehat{x}, \widehat{y}, \phi(\widehat{x}), \phi(\widehat{y})\},$$
$$\widehat{\sum}^2 = \{\widehat{p}_1, \widehat{p}_2\}, \quad \widehat{P}^2 = \operatorname{span}\{1, \widehat{x}\},$$
$$\widehat{\sum}^3 = \{\widehat{q}_1, \widehat{q}_2\}, \quad \widehat{P}^3 = \operatorname{span}\{1, \widehat{y}\}, \tag{6}$$

where $\widehat{v}_i = (1/|\widehat{l}_i|)\int_{\widehat{l}_i} \widehat{v}\,d\widehat{s}$, $(i = 1,2,3,4)$, $\widehat{v}_5 = (1/|\widehat{e}|)\int_{\widehat{e}} \widehat{v}\,d\widehat{x}\,d\widehat{y}$, $\phi(t) = (1/2)(3t^2-1)$, $\widehat{p}_i = (1/|\widehat{l}_{2i}|)\int_{\widehat{l}_{2i}} \widehat{p}\,d\widehat{s}$, $\widehat{q}_i = (1/|\widehat{l}_{2i-1}|)\int_{\widehat{l}_{2i-1}} \widehat{q}\,d\widehat{s}$, $(i = 1,2)$.

The interpolation operators on $\widehat{e}$ are defined as follows:

$$\begin{aligned}
&\widehat{\Pi}^1 : \widehat{v} \in H^1(\widehat{e}) \longrightarrow \widehat{\Pi}^1 \widehat{v} \in \widehat{P}^1, \\
&\int_{\widehat{l}_i} \left(\widehat{\Pi}^1 \widehat{v} - \widehat{v}\right) d\widehat{s} = 0, \quad (i = 1,2,3,4), \\
&\int_{\widehat{e}} \left(\widehat{\Pi}^1 \widehat{v} - \widehat{v}\right) d\widehat{x}\, d\widehat{y} = 0, \\
&\widehat{\Pi}^2 : \widehat{p} \in H^1(\widehat{e}) \longrightarrow \widehat{\Pi}^2 \widehat{p} \in \widehat{P}^2, \\
&\int_{\widehat{l}_{2i}} \left(\widehat{\Pi}^2 \widehat{p} - \widehat{p}\right) d\widehat{s} = 0, \quad (i = 1,2), \\
&\widehat{\Pi}^3 : \widehat{q} \in H^1(\widehat{e}) \longrightarrow \widehat{\Pi}^3 \widehat{q} \in \widehat{P}^3, \\
&\int_{\widehat{l}_{2i-1}} \left(\widehat{\Pi}^3 \widehat{q} - \widehat{q}\right) d\widehat{s} = 0, \quad (i = 1,2).
\end{aligned} \tag{7}$$

Then the associated nonconforming EQ_1^{rot} element space M_h [29] and lowest order Raviart-Thomas element space $\mathbf{V}_h$ [25, 27] are defined as

$$\begin{aligned}
M_h = &\left\{ v_h : v_h|_e = \widehat{v} \circ F_e^{-1}, \widehat{v} \in \widehat{P}^1 \right., \\
&\left. \int_F [v_h]\, ds = 0, F \subset \partial e \right\}, \\
\mathbf{V}_h = &\left\{ \mathbf{w}_h = (w_{h1}, w_{h2}) : \right. \\
&\mathbf{w}_h|_e = \left(\widehat{w}_1 \circ F_e^{-1}, \widehat{w}_2 \circ F_e^{-1}\right), \\
&\left. \widehat{\mathbf{w}} = (\widehat{w}_1, \widehat{w}_2) \in \widehat{P}^2 \times \widehat{P}^3 \right\},
\end{aligned} \tag{8}$$

respectively, where $[\varphi]$ represents the jump value of φ across the boundary F, and $[\varphi] = \varphi$ if $F \subset \partial\Omega$.

Similarly, the interpolation operators π_h^1 and π_h^2 are defined as

$$\begin{aligned}
&\pi_h^1 : H^1(\Omega) \longrightarrow M_h, \qquad \pi_h^1\big|_e = \pi_e^1, \\
&\pi_e^1 v = \left(\widehat{\Pi}^1 \widehat{v}\right) \circ F_e^{-1}, \quad \forall v \in H^1(\Omega), \\
&\pi_h^2 : \left(H^1(\Omega)\right)^2 \longrightarrow \mathbf{V}_h, \quad \pi_h^2|_e = \pi_e^2, \\
&\pi_e^2 \mathbf{w} = \left(\left(\widehat{\Pi}^2 \widehat{w}_1\right) \circ F_e^{-1}, \left(\widehat{\Pi}^3 \widehat{w}_2\right) \circ F_e^{-1}\right), \\
&\qquad \forall \mathbf{w} = (w_1, w_2) \in \left(H^1(\Omega)\right)^2.
\end{aligned} \tag{9}$$

3. New Characteristic Nonconforming MFE Scheme and Two Lemmas

Let $\psi(x, y) = (1 + |\mathbf{a}(x, y)|^2)^{1/2}$ and $\tau = \tau(x, y)$ be the characteristic direction associated with $u_t + \mathbf{a}(x, y) \cdot \nabla u$, such that

$$\frac{\partial}{\partial \tau} = \frac{1}{\psi(x,y)} \frac{\partial}{\partial t} + \frac{\mathbf{a}(x,y)}{\psi(x,y)} \cdot \nabla. \tag{10}$$

Then (1) can be put in the following system:

$$\begin{aligned}
&\psi(x,y) \frac{\partial u}{\partial \tau} - \nabla \cdot (b(x,y) \nabla u) = f(x,y,t), \\
&\qquad \forall (x,y,t) \in \Omega \times (0,T], \\
&u(x,y,t) = 0, \quad (x,y,t) \in \partial\Omega \times (0,T], \\
&u(x,y,0) = u_0(x,y), \quad (x,y) \in \Omega.
\end{aligned} \tag{11}$$

By introducing $\sigma = -b(x, y)\nabla u$ and using Green's formula, we obtain the new characteristic mixed form of (11). Find $(u, \sigma) : (0, T] \rightarrow H_0^1(\Omega) \times (L^2(\Omega))^2$, such that

$$\begin{aligned}
&\left(\psi(x,y) \frac{\partial u}{\partial \tau}, v\right) - (\sigma, \nabla v) = (f(x,y,t), v) \quad \forall v \in H_0^1(\Omega), \\
&(\sigma, \mathbf{w}) + (b(x,y) \nabla u, \mathbf{w}) = 0, \quad \forall \mathbf{w} \in \left(L^2(\Omega)\right)^2.
\end{aligned} \tag{12}$$

Let $\Delta t > 0, N = T/\Delta t \in \mathbb{Z}, t^n = n\Delta t$, and $\phi^n = \phi(x, y, t^n)$. When solving u_h^{n+1}, we would like to make the scheme as implicit as possible by using of the characteristic vector τ. Denote $X = (x, y) \in \Omega$ and

$$\overline{X} = X - \mathbf{a}(x,y) \Delta t, \tag{13}$$

similar to [1, 3], and then we have the following approximation:

$$\begin{aligned}
\psi(x,y) \frac{\partial u}{\partial \tau}\bigg|_{t^n} &\approx \psi(x,y) \frac{u(X,t^n) - u\left(\overline{X}, t^{n-1}\right)}{\sqrt{\left(X - \overline{X}\right)^2 + (\Delta t)^2}} \\
&= \frac{u(X,t^n) - u\left(\overline{X}, t^{n-1}\right)}{\Delta t} = \frac{u^n - \overline{u}^{n-1}}{\Delta t}.
\end{aligned} \tag{14}$$

This leads to the following characteristic nonconforming MFE scheme. Find $(u_h, \sigma_h) : \{t^0, t^1, \ldots, t^N\} \rightarrow M_h \times \mathbf{V}_h$, such that

$$\left(\frac{u_h^n - \overline{u}_h^{n-1}}{\Delta t}, v_h\right) - (\sigma_h^n, \nabla v_h)_h = (f^n, v_h), \quad \forall v_h \in M_h, \tag{15a}$$

$$(\sigma_h^n, \mathbf{w}_h) + (b \nabla u_h^n, \mathbf{w}_h)_h = 0, \quad \forall \mathbf{w}_h \in \mathbf{V}_h, \tag{15b}$$

$$u_h^0 = \pi_h^1 u_0(x,y), \quad \sigma_h^0 = \pi_h^2(b \nabla u_0(x,y)), \quad \forall (x,y) \in \Omega, \tag{15c}$$

where $\overline{u}_h^n = u_h(\overline{X}, t^n)$, $(u, v)_h = \sum_{e \in T_h} \int_e uv\,dx\,dy$. Generally speaking, $\overline{u}_h^{n-1} (n = 2, \ldots, N)$ are not node values and should be derived by interpolation formulas on u_h^{n-1}.

Remark 1. In [1], the expanded characteristic MFE scheme was presented by introducing two new auxiliary variables which avoided the inversion of the coefficient b when b is small. The new mixed schemes (15a), (15b), and (15c) not only keep the advantage of expanded characteristic MFE scheme, but also donot need to solve three variables.

Now, we prove the existence and uniqueness of the solution of (15a), (15b), and (15c).

Theorem 1. *Under assumption (A3), there exists a unique solution* $(u_h, \sigma_h) \in M_h \times \mathbf{V}_h$ *to the schemes* (15a), (15b)*, and* (15c).

Proof. The linear system generated by (15a), (15b), and (15c) is square, so the existence of the solution is implied by its uniqueness. From (15a), (15b), and (15c), we have

$$\left(\frac{u_h^n}{\Delta t}, v_h\right) - \left(\sigma_h^n, \nabla v_h\right)_h = \left(\frac{\overline{u}_h^{n-1}}{\Delta t}, v_h\right) + \left(f^n, v_h\right), \quad \forall v_h \in M_h,$$
$$\left(\sigma_h^n, \mathbf{w}_h\right) + \left(b\nabla u_h^n, \mathbf{w}_h\right)_h = 0, \quad \forall \mathbf{w}_h \in \mathbf{V}_h. \tag{16}$$

Let u_h^n and f be zero, and thus $\overline{u}_h^n$ is zero too; taking $v_h = u_h^n$, $\mathbf{w}_h = (1/b)\sigma_h^n$ in (16) and adding them together, we have

$$\frac{1}{\Delta t}\left\|u_h^n\right\|^2 + \left(\frac{1}{b}\sigma_h^n, \sigma_h^n\right) = 0. \tag{17}$$

Thus assumption (A3) implies that $u_h^n = \sigma_h^n = 0$. The proof is complete. □

To get error estimates, we state the following two important lemmas.

Lemma 1 (see [27, 29, 31]). *Assume that* $u \in H^1(\Omega)$, $\mathbf{p} \in (H^2(\Omega))^2$, *for all* $v_h \in M_h$, $\mathbf{w}_h \in \mathbf{V}_h$, *and then there hold*

$$\left(\nabla\left(u - \pi_h^1 u\right), \nabla v_h\right)_h = 0, \quad \left(\nabla\left(u - \pi_h^1 u\right), \mathbf{w}_h\right)_h = 0, \tag{18}$$

$$\left(\mathbf{p} - \pi_h^2 \mathbf{p}, \mathbf{w}_h\right) \le Ch^2 |\mathbf{p}|_{2,\Omega} \left\|\mathbf{w}_h\right\|, \tag{19}$$

$$\left|\sum_{e\in T_h}\int_{\partial e} \mathbf{p} v_h \cdot \mathbf{n}\, ds\right| \le Ch^2 |\mathbf{p}|_{2,\Omega} \left\|v_h\right\|_{1,h}, \tag{20}$$

where $\|\cdot\|_{1,h} = (\sum_{e\in T_h} |\cdot|_{1,e})^{1/2}$ *is a norm on* M_h, *and* $\mathbf{n}$ *denotes the outward unit normal vector on* ∂e.

Lemma 2 (see [1, 3]). *Let* $\varphi \in L^2(\Omega)$, *and* $\overline{\varphi} = \varphi(X - g(X)\Delta t)$, *where function* g *and its gradient* ∇g *are bounded, then*

$$\left\|\varphi - \overline{\varphi}\right\|_{-1} \le C \left\|\varphi\right\| \Delta t, \tag{21}$$

where $\|\varphi\|_{-1} = \sup_{\phi\in H^1(\Omega)}((\varphi, \phi)/\|\phi\|_{1,\Omega})$.

4. Convergence Analysis and Optimal Order Error Estimates

In this section, we aim to analyze the convergence analysis and error estimates of characteristic nonconforming MFEM. In order to do this, let

$$u_h - u = u_h - \pi_h^1 u + \pi_h^1 u - u = e + \rho,$$
$$\sigma_h - \sigma = \sigma_h - \pi_h^2 \sigma + \pi_h^2 \sigma - \sigma = \xi + \eta. \tag{22}$$

Taking $t = t^n$ in (12) yields

$$\left(\psi\frac{\partial u^n}{\partial \tau}, v_h\right) - \left(\sigma^n, \nabla v_h\right)_h + \sum_{e\in T_h}\int_{\partial e} \sigma^n v_h \cdot \mathbf{n} ds = \left(f^n, v_h\right), \quad \forall v_h \in M_h, \tag{23a}$$

$$\left(\sigma^n, \mathbf{w}_h\right) + \left(b\nabla u^n, \mathbf{w}_h\right)_h = 0, \quad \forall \mathbf{w}_h \in \mathbf{V}_h. \tag{23b}$$

From (23a), (23b), (15a), (15b), and (15c) we get

$$\begin{aligned}\left(\frac{e^n - \overline{e}^{n-1}}{\Delta t}, v_h\right) - \left(\xi^n, \nabla v_h\right)_h &= \left(\psi\frac{\partial u^n}{\partial \tau} - \frac{u^n - \overline{u}^{n-1}}{\Delta t}, v_h\right) - \left(\frac{\rho^n - \overline{\rho}^{n-1}}{\Delta t}, v_h\right) \\ &\quad + \left(\eta^n, \nabla v_h\right)_h + \sum_{e\in T_h}\int_{\partial e} \sigma^n v_h \cdot \mathbf{n}\, ds, \quad \forall v_h \in M_h,\end{aligned} \tag{24a}$$

$$\left(\xi^n, \mathbf{w}_h\right) + \left(b\nabla e^n, \mathbf{w}_h\right)_h = -\left(\eta^n, \mathbf{w}_h\right) - \left(b\nabla \rho^n, \mathbf{w}_h\right)_h, \quad \forall \mathbf{w}_h \in \mathbf{V}_h. \tag{24b}$$

We are now in a position to prove the optimal order error estimates.

Theorem 2. *Let* (u, σ) *and* (u_h^n, σ_h^n) *be the solutions of* (12), (15a), (15b), *and* (15c), *respectively,* $(\partial^2 u/\partial \tau^2) \in L^2(0, T; L^2(\Omega))$, $u_t \in L^2(0, T; H^2(\Omega))$, $u \in L^\infty(0, T; H^2(\Omega))$, $\sigma \in L^\infty(0, T; H^2(\Omega))$ *and assume that* $\Delta t = O(h^2)$. *Then under assumption (A3), we have*

$$\max_{0\le n\le N} \left\|\left(u_h - u\right)\left(t^n\right)\right\|_{1,h} \le C\left(\Delta t + h\right), \tag{25}$$

$$\max_{0\le n\le N} \left\|\left(u_h - u\right)\left(t^n\right)\right\| \le C\left(\Delta t + h^2\right), \tag{26}$$

$$\max_{0\le n\le N} \left\|\left(\sigma_h - \sigma\right)\left(t^n\right)\right\| \le C\left(\Delta t + h\right). \tag{27}$$

Proof. Taking $v_h = e^n$ in (24a) and $\mathbf{w}_h = \nabla e^n$ in (24b), and adding them, we have

$$\begin{aligned}\left(\frac{e^n - \overline{e}^{n-1}}{\Delta t}, e^n\right) + \left(b\nabla e^n, \nabla e^n\right)_h &= \left(\psi\frac{\partial u^n}{\partial \tau} - \frac{u^n - \overline{u}^{n-1}}{\Delta t}, e^n\right) - \left(\frac{\rho^n - \rho^{n-1}}{\Delta t}, e^n\right) \\ &\quad - \left(\frac{\rho^{n-1} - \overline{\rho}^{n-1}}{\Delta t}, e^n\right) \\ &\quad + \sum_{e\in T_h}\int_{\partial e} \sigma^n e^n \cdot \mathbf{n}\, ds - \left(b\nabla\rho^n, \nabla e^n\right)_h \\ &= \sum_{i=1}^{5} (\mathrm{Err})_i.\end{aligned} \tag{28}$$

On the one hand, we consider the right hand of (28).

Using the method similar to [3], we have

$$(\mathrm{Err})_1 \le C\left\|\psi\frac{\partial u^n}{\partial\tau}-\frac{u^n-\overline{u}^{n-1}}{\Delta t}\right\|^2+\frac{\varepsilon_1}{2}\|e^n\|^2 \le C\Delta t\left\|\frac{\partial^2 u}{\partial\tau^2}\right\|^2_{L^2(t^{n-1},t^n;L^2(\Omega))}+\frac{\varepsilon_1}{2}\|e^n\|^2. \tag{29}$$

$(\mathrm{Err})_2$ can be estimated as

$$\begin{aligned}\left|(\mathrm{Err})_2\right| &\le \frac{1}{\Delta t}\left(\int_\Omega\left(\int_{t^{n-1}}^{t^n}\rho_t ds\right)^2 dx\,dy\right)^{1/2}\|e^n\| \\ &\le \frac{1}{\sqrt{\Delta t}}\left(\int_\Omega\int_{t^{n-1}}^{t^n}\rho_t^2 ds\,dx\,dy\right)^{1/2}\|e^n\| \\ &\le \frac{C}{\Delta t}\int_{t^{n-1}}^{t^n}\|\rho_t\|^2 ds+\frac{\varepsilon_1}{2}\|e^n\|^2 \\ &\le \frac{Ch^4}{\Delta t}\int_{t^{n-1}}^{t^n}\|u_t\|^2_{2,\Omega}ds+\frac{\varepsilon_1}{2}\|e^n\|^2.\end{aligned} \tag{30}$$

By Lemma 2, we obtain

$$\begin{aligned}\left|(\mathrm{Err})_3\right| &\le \frac{1}{\Delta t}\left\|\rho^{n-1}-\overline{\rho}^{n-1}\right\|_{-1}\|e^n\|_{1,h} \\ &\le C\left\|\rho^{n-1}\right\|^2+\frac{b_1}{6}\|e^n\|^2_{1,h} \\ &\le Ch^4\left\|u^{n-1}\right\|^2_{2,\Omega}+\frac{b_1}{6}\|e^n\|^2_{1,h}.\end{aligned} \tag{31}$$

It follows from Lemma 1 that

$$\left|(\mathrm{Err})_4\right| \le Ch^4\|\sigma^n\|^2_{2,\Omega}+\frac{b_1}{6}\|e^n\|^2_{1,h}. \tag{32}$$

Let $\overline{b}=(1/|e|)\int_e b(x,y)dx\,dy$. By Lemma 1, we have

$$\begin{aligned}\left|(\mathrm{Err})_5\right| &= \left|-\left(\left(b-\overline{b}\right)\nabla\rho^n,\nabla e^n\right)_h\right| \\ &\le Ch|b|_{W^{1,\infty}(\Omega)}\|\rho^n\|_{1,h}\|e^n\|_{1,h} \\ &\le Ch^4\|u^n\|^2_{2,\Omega}+\frac{b_1}{6}\|e^n\|^2_{1,h}.\end{aligned} \tag{33}$$

On the other hand, the left hand of (28) can be bounded by

$$\begin{aligned}&\left(\frac{e^n-\overline{e}^{n-1}}{\Delta t},e^n\right)+\left(b\nabla e^n,\nabla e^n\right)_h \\ &\quad\ge \frac{1}{2\Delta t}\left(\left(e^n,e^n\right)-\left(\overline{e}^{n-1},\overline{e}^{n-1}\right)\right)+b_1\|e^n\|^2_{1,h} \\ &\quad\ge \frac{1}{2\Delta t}\left(\|e^n\|^2-(1+C\Delta t)\left\|e^{n-1}\right\|^2\right)+b_1\|e^n\|^2_{1,h},\end{aligned} \tag{34}$$

where the inequality $\|\overline{e}^{n-1}\|^2 \le (1+C\Delta t)\|e^{n-1}\|^2$ proved in [3] is used in the last step.

Combining (29)–(34) with (28) gives

$$\begin{aligned}&\frac{1}{2\Delta t}\left(\|e^n\|^2-\left\|e^{n-1}\right\|^2\right)+b_1\|e^n\|^2_{1,h} \\ &\quad\le C\Bigg(\Delta t\left\|\frac{\partial^2 u}{\partial\tau^2}\right\|^2_{L^2(t^{n-1},t^n;L^2(\Omega))}+\frac{h^4}{\Delta t}\int_{t^{n-1}}^{t^n}\|u_t\|^2_{2,\Omega}ds \\ &\qquad+h^4\left(\left\|u^{n-1}\right\|^2_{2,\Omega}+\|u^n\|^2_{2,\Omega}+\|\sigma^n\|^2_{2,\Omega}\right)\Bigg) \\ &\qquad+\varepsilon_1\|e^n\|^2+C\left\|e^{n-1}\right\|^2+\frac{b_1}{2}\|e^n\|_{1,h}.\end{aligned} \tag{35}$$

Taking $1-2\Delta t\varepsilon_1>0$, multiplying (35) by $2\Delta t$, summing over from $i=1$ to $i=n$, and noticing that $e^0=0$, we obtain

$$\begin{aligned}&\|e^n\|^2+\Delta t\sum_{i=1}^n\left\|e^i\right\|^2_{1,h} \\ &\quad\le C\Bigg((\Delta t)^2\left\|\frac{\partial^2 u}{\partial\tau^2}\right\|^2_{L^2(0,t^n;L^2(\Omega))}+h^4\int_0^{t^n}\|u_t\|^2_{2,\Omega}ds \\ &\qquad+\Delta t h^4\sum_{i=1}^n\left(\left\|u^i\right\|^2_{2,\Omega}+\left\|\sigma^i\right\|^2_{2,\Omega}\right)\Bigg)+C\sum_{i=1}^{n-1}\left\|e^i\right\|^2.\end{aligned} \tag{36}$$

It follows from discrete Gronwall's lemma that

$$\begin{aligned}&\|e^n\|^2+\Delta t\sum_{i=1}^n\left\|e^i\right\|^2_{1,h} \\ &\quad\le C\Bigg((\Delta t)^2\left\|\frac{\partial^2 u}{\partial\tau^2}\right\|^2_{L^2(0,t^n;L^2(\Omega))} \\ &\qquad+h^4\bigg(\|u_t\|^2_{L^2(0,t^n;H^2(\Omega))}+\|u\|^2_{L^\infty(0,t^n;H^2(\Omega))} \\ &\qquad+\|\sigma\|^2_{L^\infty\left(0,t^n;\left(H^2(\Omega)\right)^2\right)}\bigg)\Bigg).\end{aligned} \tag{37}$$

From (37) we get the optimal order error estimate of $\|e^n\|$ rather than $\|e^n\|_{1,h}$. So we start to reestimate $\|e^n\|_{1,h}$ in the following manner and derive the estimation of $\|\xi^n\|$ simultaneously.

Firstly, choosing $v_h=((e^n-e^{n-1})/\Delta t)$ in (24a) and $\mathbf{w}_h=\nabla((e^n-e^{n-1})/\Delta t)$ in (24b), and adding them, we have

$$\begin{aligned}&\left(\frac{e^n-\overline{e}^{n-1}}{\Delta t},\frac{e^n-e^{n-1}}{\Delta t}\right)+\left(b\nabla e^n,\nabla\frac{e^n-e^{n-1}}{\Delta t}\right)_h \\ &\quad=\left(\psi\frac{\partial u^n}{\partial\tau}-\frac{u^n-\overline{u}^{n-1}}{\Delta t},\frac{e^n-e^{n-1}}{\Delta t}\right) \\ &\qquad-\left(\frac{\rho^n-\rho^{n-1}}{\Delta t},\frac{e^n-e^{n-1}}{\Delta t}\right)\end{aligned}$$

$$-\left(\frac{\rho^{n-1}-\overline{\rho}^{n-1}}{\Delta t},\frac{e^n-e^{n-1}}{\Delta t}\right)$$
$$+\sum_{e\in T_h}\int_{\partial e}\sigma^n\frac{e^n-e^{n-1}}{\Delta t}\cdot\mathbf{n}\,ds-\left(b\nabla\rho^n,\nabla\frac{e^n-e^{n-1}}{\Delta t}\right)_h$$
$$=\sum_{i=1}^{5}(\mathrm{Err})_i'. \tag{38}$$

The left hand can be estimated as

$$\left(\frac{e^n-\overline{e}^{n-1}}{\Delta t},\frac{e^n-e^{n-1}}{\Delta t}\right)+\left(b\nabla e^n,\nabla\frac{e^n-e^{n-1}}{\Delta t}\right)_h$$
$$\geq\left\|\frac{e^n-e^{n-1}}{\Delta t}\right\|^2+\frac{1}{2\Delta t}\left[\left(b\nabla e^n,\nabla e^n\right)-\left(b\nabla e^{n-1},\nabla e^{n-1}\right)\right]$$
$$+\left(\frac{e^{n-1}-\overline{e}^{n-1}}{\Delta t},\frac{e^n-e^{n-1}}{\Delta t}\right), \tag{39}$$

and $(\mathrm{Err})_i'$, $(i=1,2,3,4,5)$ can be bounded by

$$\left|(\mathrm{Err})_1'\right|\leq C\Delta t\left\|\frac{\partial^2u}{\partial\tau^2}\right\|^2_{L^2(t^{n-1},t^n;L^2(\Omega))}+\frac{1}{4}\left\|\frac{e^n-e^{n-1}}{\Delta t}\right\|^2,$$
$$\left|(\mathrm{Err})_2'\right|\leq\frac{Ch^4}{\Delta t}\int_{t^{n-1}}^{t^n}\|u_t\|^2_{2,\Omega}ds+\frac{1}{4}\left\|\frac{e^n-e^{n-1}}{\Delta t}\right\|^2,$$
$$\left|(\mathrm{Err})_3'\right|\leq\frac{Ch^4}{\Delta t}\left\|u^{n-1}\right\|^2_{2,\Omega}+\frac{\varepsilon}{3}\Delta t\left\|\frac{e^n-e^{n-1}}{\Delta t}\right\|^2_{1,h},$$
$$\left|(\mathrm{Err})_4'\right|\leq\frac{Ch^4}{\Delta t}\left\|\sigma^n\right\|^2_{2,\Omega}+\frac{\varepsilon}{3}\Delta t\left\|\frac{e^n-e^{n-1}}{\Delta t}\right\|^2_{1,h},$$
$$\left|(\mathrm{Err})_5'\right|\leq\frac{Ch^4}{\Delta t}\left\|u^n\right\|^2_{2,\Omega}+\frac{\varepsilon}{3}\Delta t\left\|\frac{e^n-e^{n-1}}{\Delta t}\right\|^2_{1,h}. \tag{40}$$

From (38)–(40), we get

$$\frac{1}{2}\left\|\frac{e^n-e^{n-1}}{\Delta t}\right\|^2+\frac{1}{2\Delta t}\left[\left(b\nabla e^n,\nabla e^n\right)_h-\left(b\nabla e^{n-1},\nabla e^{n-1}\right)_h\right]$$
$$\leq C\left[\Delta t\left\|\frac{\partial^2u}{\partial\tau^2}\right\|^2_{L^2(t^{n-1},t^n;L^2(\Omega))}\right.$$
$$+\frac{h^4}{\Delta t}\left(\int_{t^{n-1}}^{t^n}\|u_t\|^2_{2,\Omega}ds+\|u^n\|^2_{2,\Omega}+\left\|u^{n-1}\right\|^2_{2,\Omega}\right.$$
$$\left.\left.+\|\sigma^n\|^2_{2,\Omega}\right)\right]+\varepsilon\Delta t\left\|\frac{e^n-e^{n-1}}{\Delta t}\right\|^2_{1,h}$$
$$+\left(\frac{e^{n-1}-\overline{e}^{n-1}}{\Delta t},\frac{e^n-e^{n-1}}{\Delta t}\right). \tag{41}$$

Multiplying (41) by $2\Delta t$ and summing over in time from $i=1$ to $i=n$ yield

$$\Delta t\left\|\frac{e^n-e^{n-1}}{\Delta t}\right\|^2+b_1\|e^n\|^2_{1,h}$$
$$\leq C\left[(\Delta t)^2\left\|\frac{\partial^2u}{\partial\tau^2}\right\|^2_{L^2(0,t^n;L^2(\Omega))}+h^4\|u_t\|^2_{L^2(0,t^n;H^2(\Omega))}\right.$$
$$\left.+h^4\sum_{i=1}^{n}\left(\left\|u^i\right\|^2_{2,\Omega}+\left\|\sigma^i\right\|^2_{2,\Omega}\right)\right]$$
$$+\varepsilon(\Delta t)^2\sum_{i=1}^{n}\left\|\frac{e^i-e^{i-1}}{\Delta t}\right\|^2_{1,h}+\sum_{i=1}^{n}\left(\frac{e^{i-1}-\overline{e}^{i-1}}{\Delta t},e^i-e^{i-1}\right). \tag{42}$$

Secondly, we take $\Delta t\to 0$ and Δt must approach zero in such a way that Δt and h satisfy

$$\Delta t=O\left(h^2\right), \tag{43}$$

and by inverse inequality, we have

$$(\Delta t)^2\sum_{i=1}^{n}\left\|\frac{e^i-e^{i-1}}{\Delta t}\right\|^2_{1,h}\leq C\Delta t\sum_{i=1}^{n}\left\|\frac{e^i-e^{i-1}}{\Delta t}\right\|^2. \tag{44}$$

At the same time, using Lemma 2, we obtain

$$\sum_{i=1}^{n}\left(\frac{e^{i-1}-\overline{e}^{i-1}}{\Delta t},e^i-e^{i-1}\right)$$
$$=\left(\frac{e^{n-1}-\overline{e}^{n-1}}{\Delta t},e^n\right)+\sum_{i=1}^{n-1}\left(\frac{e^{i-1}-e^i-\left(\overline{e}^{i-1}-\overline{e}^i\right)}{\Delta t},e^i\right)$$
$$\leq C\left\|e^{n-1}\right\|\|e^n\|_{1,h}+\sum_{i=1}^{n-1}\left\|e^i-e^{i-1}\right\|\left\|e^i\right\|_{1,h}$$
$$\leq C\left\|e^{n-1}\right\|^2+\frac{b_1}{2}\|e^n\|^2_{1,h}+\Delta t\sum_{i=1}^{n-1}\left\|\frac{e^i-e^{i-1}}{\Delta t}\right\|^2$$
$$+C\Delta t\sum_{i=1}^{n-1}\left\|e^i\right\|^2_{1,h}. \tag{45}$$

From (42)–(45), taking suitable small ε such that $1-\varepsilon C>0$, we have

$$\Delta t\left\|\frac{e^n-e^{n-1}}{\Delta t}\right\|^2+\|e^n\|^2_{1,h}$$
$$\leq C\left[(\Delta t)^2\left\|\frac{\partial^2u}{\partial\tau^2}\right\|^2_{L^2(0,t^n;L^2(\Omega))}+h^4\|u_t\|^2_{L^2(0,t^n;H^2(\Omega))}\right.$$
$$\left.+h^4\sum_{i=1}^{n}\left(\left\|u^i\right\|^2_{2,\Omega}+\left\|\sigma^i\right\|^2_{2,\Omega}\right)\right]$$
$$+\left\|e^{n-1}\right\|^2+C\Delta t\sum_{i=1}^{n-1}\left\|\frac{e^i-e^{i-1}}{\Delta t}\right\|^2+C\Delta t\sum_{i=1}^{n-1}\left\|e^i\right\|^2_{1,h}. \tag{46}$$

Finally, applying discrete Gronwall's lemma yields

$$\|e^n\|_{1,h}^2 \le C\left[(\Delta t)^2\left\|\frac{\partial^2 u}{\partial \tau^2}\right\|_{L^2(0,t^n;L^2(\Omega))}^2 + h^4\|u_t\|_{L^2(0,t^n;H^2(\Omega))}^2 + h^2\left(\|u\|_{L^\infty(0,t^n;H^2(\Omega))}^2 + \|\sigma\|_{L^\infty(0,t^n;H^2(\Omega))}^2\right)\right]. \tag{47}$$

In order to derive (27), set $\mathbf{w}_h = \xi^n$ in (24b) and employ Lemma 1 and assumption (A3) to give

$$\begin{aligned}\|\xi^n\|^2 &= -(b\nabla e^n, \xi^n)_h - (\eta^n, \xi^n) - (b\nabla\rho^n, \xi^n)_h \\ &\le C\left(\|e^n\|_{1,h}^2 + h^4\|\sigma^n\|_{2,\Omega}\right) \\ &\quad - \left(\left(b-\bar{b}\right)\nabla\rho^n, \xi^n\right)_h + \frac{1}{4}\|\xi^n\|^2 \\ &\le C\left(\|e^n\|_{1,h}^2 + h^4\left(\|\sigma^n\|_{2,\Omega} + \|u^n\|_{2,\Omega}\right)\right) + \frac{1}{2}\|\xi^n\|^2.\end{aligned} \tag{48}$$

Combining (47) with (48) yields

$$\|\xi^n\|^2 \le C\left[(\Delta t)^2\left\|\frac{\partial^2 u}{\partial \tau^2}\right\|_{L^2(0,t^n;L^2(\Omega))}^2 + h^4\|u_t\|_{L^2(0,t^n;H^2(\Omega))}^2 + h^2\left(|u|_{L^\infty(0,t^n;H^2(\Omega))}^2 + \|\sigma\|_{L^\infty(0,t^n;H^2(\Omega))}^2\right)\right]. \tag{49}$$

By using of interpolation theory and the triangle inequality, (37), (47), and (49) lead to (25), (26), and (27), respectively, which are the desired results. □

Remark 2. From (37), we have

$$\begin{aligned}\Delta t\sum_{i=1}^{n}\left\|e^i\right\|_{1,h}^2 &= \Delta t\sum_{i=1}^{n}\left\|\left(\pi_h^1 u - u_h\right)^i\right\|_{1,h}^2 \\ &\le C\left((\Delta t)^2\left\|\frac{\partial^2 u}{\partial \tau^2}\right\|_{L^2(0,t^n;L^2(\Omega))}^2 + h^4\left(\|u_t\|_{L^2(0,t^n;H^2(\Omega))}^2 + \|u\|_{L^\infty(0,t^n;H^2(\Omega))}^2 + \|\sigma\|_{L^\infty(0,t^n;(H^2(\Omega))^2)}^2\right)\right).\end{aligned} \tag{50}$$

This byproduct can be regarded as the superclose result between $\pi_h^1 u$ and u_h in mean broken H^1-norm. It seems that both (25) and (50) have never been seen in the existing studies. At the same time, by employing the new characteristic nonconforming MFE scheme, we can also obtain the same error estimate of (27) as traditional characteristic MFEM [10].

Remark 3. From the analysis of Theorem 2 in this paper, we may see that Lemma 1 is the key result leading to the successful optimal order error estimations. If we want to get higher order accuracy, similar to Lemma 1, the nonconforming finite elements for approximating u should also possess a very special property, that is, the consistency error estimates with $O(h^2)$ order, and satisfy (18). For the famous nonconforming Wilson element [32] whose shape function is span$\{1, x, y, x^2, y^2\}$, by a counter-example, it has been proven in [32] that its consistency error estimate is of $O(h)$ order and cannot be improved any more. For the rotated bilinear Q_1 element [33] whose shape function is span$\{1, x, y, x^2 - y^2\}$, although its consistency error with $O(h^2)$ order and $(\nabla(u - \pi_h^1 u), \nabla v_h)_h = 0$ on square meshes is satisfied, the second term of (18) is not valid. Thus when they are applied to (1) on new characteristic mixed finite element scheme, up to now, the optimal order error estimates of (25), (26), and (27) cannot be obtained directly.

Table 1: Numerical results of $\|u - u_h\|_{1,h}$.

$m\times n$	$t = 0.2$	α	$t = 0.3$	α	$t = 0.4$	α
8×8	0.75277	/	0.75017	/	0.66433	/
16×16	0.42984	0.81	0.41849	0.84	0.35474	0.91
32×32	0.21758	0.99	0.21412	0.97	0.17552	1.02
$m\times n$	$t = 0.5$	α	$t = 0.8$	α	$t = 0.9$	α
8×8	0.55291	/	0.42211	/	0.40937	/
16×16	0.29234	0.92	0.23117	0.87	0.21120	0.96
32×32	0.14466	1.02	0.10807	1.10	0.09343	1.18

Table 2: Numerical results of $\|u - u_h\|$.

$m\times n$	$t = 0.4$	α	$t = 0.5$	α	$t = 0.7$	α
8×8	0.0298190	/	0.0276370	/	0.0223240	/
16×16	0.0073087	2.03	0.0062445	2.15	0.0048038	2.22
32×32	0.0020769	1.82	0.0017926	1.80	0.0013309	1.85
$m\times n$	$t = 0.8$	α	$t = 0.9$	α	$t = 1.0$	α
8×8	0.0198730	/	0.0175900	/	0.0154090	/
16×16	0.0044472	2.16	0.0041982	2.07	0.0039150	1.98
32×32	0.0011894	1.90	0.0010738	1.97	0.0009466	2.05

5. Numerical Example

In order to verify our theoretical analysis in previous sections, we consider the convection-dominated diffusion problem (1) as follows:

$$\begin{gathered}u_t + u_x + u_y - 10^{-4}\left(u_{xx} + u_{yy}\right) = f(x, y, t), \quad (x, y, t) \in \Omega\times(0, T), \\ u(x, y, t) = 0, \quad (x, y, t) \in \partial\Omega\times(0, T), \\ u(x, y, 0) = u_0(x, y), \quad (x, y) \in \Omega\end{gathered} \tag{51}$$

with $\Omega = [0, 1]\times[0, 1]$, $\mathbf{a}(x, y) = (1, 1)$, and $b(x, y) = 10^{-4}$.

The right hand term $f(x, y, t)$ is taken such that $u = e^{-t}\sin(\pi x)\sin(2\pi y)$, $\sigma = -10^{-4}e^{-t}(\pi\cos(\pi x)\sin(2\pi y), 2\pi\sin(\pi x)\cos(2\pi y))$ are the exact solutions.

Table 3: Numerical results of $\|\sigma - \sigma_h\|$.

$m \times n$	$t = 0.1$	α	$t = 0.4$	α	$t = 0.5$	α
8×8	$4.9528e-005$	/	$4.2661e-005$	/	$3.8292e-005$	/
16×16	$2.3945e-005$	1.05	$1.8843e-005$	1.18	$1.6806e-005$	1.19
32×32	$1.1749e-005$	1.03	$9.0029e-006$	1.07	$8.0521e-006$	1.06
$m \times n$	$t = 0.7$	α	$t = 0.8$	α	$t = 0.9$	α
8×8	$3.0714e-005$	/	$2.7735e-005$	/	$2.524e-005$	/
16×16	$1.3326e-005$	1.20	$1.224e-005$	1.18	$1.1443e-005$	1.14
32×32	$6.455e-006$	1.05	$5.8353e-006$	1.07	$5.3751e-006$	1.09

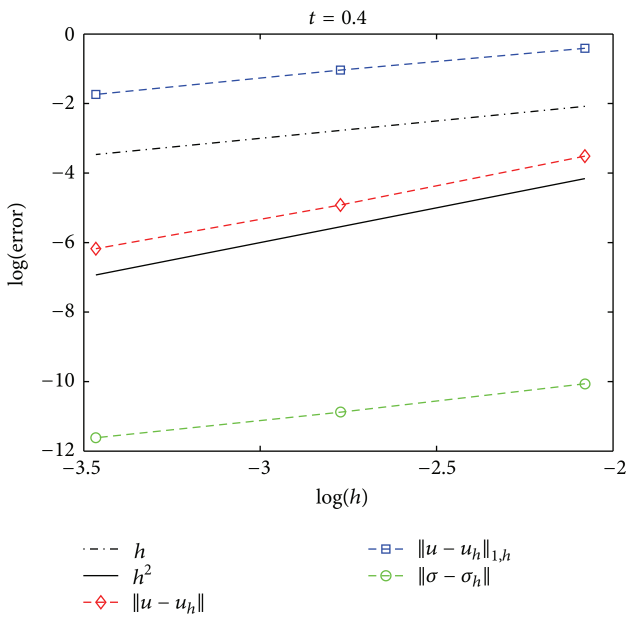

Figure 1: Errors at $t = 0.4$.

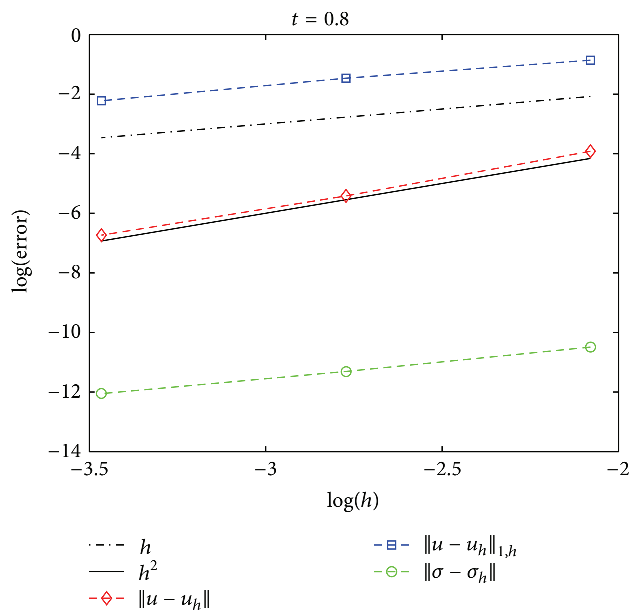

Figure 3: Errors at $t = 0.8$.

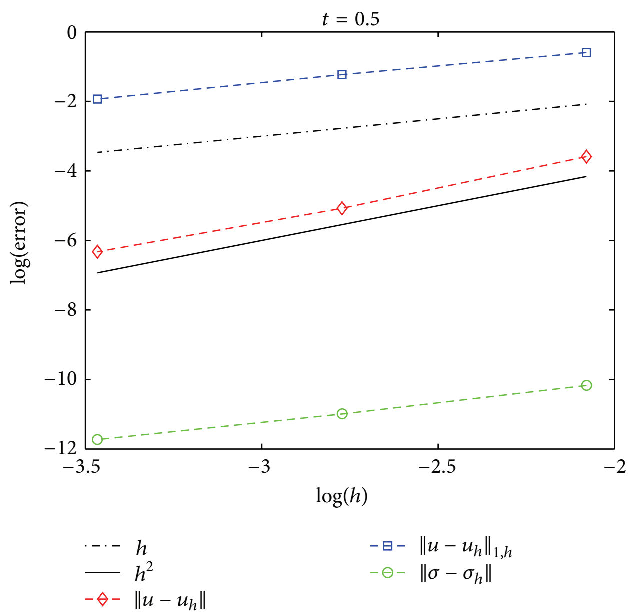

Figure 2: Errors at $t = 0.5$.

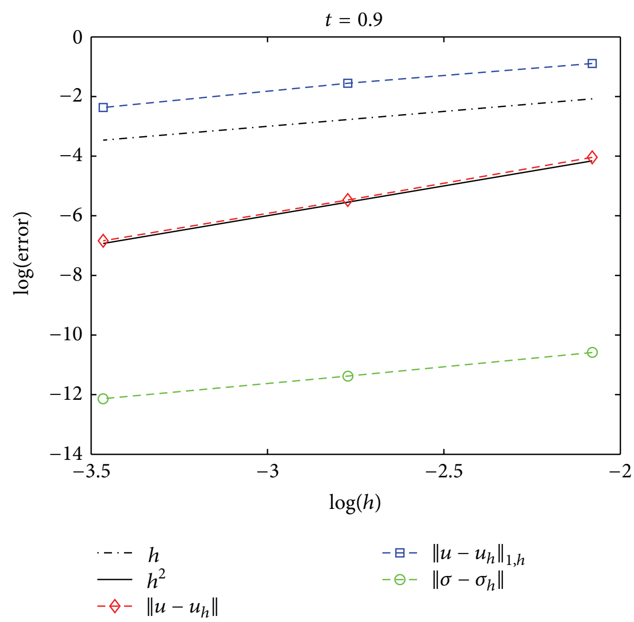

Figure 4: Errors at $t = 0.9$.

We first divide the domain Ω into m and n equal intervals along x-axis and y-axis and the numerical results at different times are listed in Tables 1, 2, and 3 and pictured in Figures 1, 2, 3, and 4, respectively. $(u_h, \mathbf{p}_h)$ denotes the characteristic nonconforming MFE solution of the problem (15a), (15b), and (15c). Δt represents the time step and the experiment is done with $\Delta t = h^2$. α stands for the convergence order.

It can be seen from the above Tables 1, 2, and 3 that $\|u - u_h\|_{1,h}$ and $\|\sigma - \sigma_h\|$ are convergent at optimal rate of $O(h)$ and $\|u - u_h\|$ is convergent at optimal rate of $O(h^2)$, respectively, which coincide with our theoretical investigation in Section 4.

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