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Research Article

Applications of Hankel and Regular Matrices in Fourier Series

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Recently, Alghamdi and Mursaleen (2013) used the Hankel matrix to determine the necessary and suffcient condition to find the sum of the Walsh-Fourier series. In this paper, we propose to use the Hankel matrix as well as any general nonnegative regular matrix to obtain the necessary and sufficient conditions to sum the derived Fourier series and conjugate Fourier series.

1. Introduction and Preliminaries

Let X and Y be two sequence spaces and let $A = (a_{nk})_{n;k=1}^{\infty}$ be an infinite matrix of real or complex numbers. We write $Ax = (A_n(x))$ provided that $A_n(x) = \sum_k a_{nk}x_k$ converges for each n. A sequence $x = (x_k)$ is said to be A-summable to L if $\lim_n A_n(x) = L$. If $x = (x_k) \in X$ implies that $Ax \in Y$, then we say that A defines a matrix transformation from X into Y and by (X, Y) we denote the class of such matrices. If X and Y are equipped with the limits X-lim and Y-lim, respectively, $A \in (X, Y)$ and Y-$\lim_n A_n(x) = X$-$\lim_k x_k$ for all $x \in X$, then we say that A is a regular map from X into Y and in this case we write $A \in (X, Y)_{\text{reg}}$. The matrices $A \in (c, c)_{\text{reg}}$ are called *regular*, where c denotes the space of all convergent sequences.

The following are well-known Silverman-Töeplitz conditions for regularity of A.

Lemma 1. $A = (a_{nk})_{n,k=1}^{\infty}$ *is regular if and only if*

(i) $\|A\| = \sup_n \sum_k |a_{nk}| < \infty$,

(ii) $\lim_n a_{nk} = 0$ *for each* k,

(iii) $\lim_n \sum_k b_{nk} = 1$.

A *Hankel matrix* is a special case of the regular matrix; that is, if $a_{nk} = h_{n+k}$ then the matrix is known as the Hankel matrix. That is, a Hankel matrix is a square matrix (finite or infinite), constant on each diagonal orthogonal to the main diagonal. Its (n, k)th entry is a function of $n + k$. The Hankel transform of the sequence $x = (x_k)$ is defined as the sequence $y = (y_n)$, where $y_n = \sum_{k=0}^{\infty} h_{n+k}x_k$ provided that the series converges for each $n = 0, 1, 2, \ldots$. An operator T which transforms x into y as described is called the operator induced by the Hankel matrix H. In [1] we can find the applications of Hankel operators to approximation theory, prediction theory, and linear system theory. Hankel matrices have a number of applications in various fields.

Recently, Al-Homidan [2] proved that Hankel matrices are regular and obtained the sum of the conjugate Fourier series under certain conditions on the entries of Hankel matrix. Most recently, Alghamdi and Mursaleen [3] proved that Hankel matrices are strongly regular. Strongly regular matrices are those matrices which transform almost convergent sequences into convergent sequences leaving the limit invariant [4].

Our aim here is to find necessary and sufficient conditions for Hankel matrix as well as any arbitrary nonnegative regular matrix to sum the derived Fourier series and conjugate Fourier series.

2. Main Results

Let f be L-integrable and periodic with period 2π, and let the Fourier series of f be

$$\frac{1}{a_0} + \sum_{k=1}^{\infty} \left(a_k \cos kx + b_k \sin kx\right). \tag{1}$$

Then the series conjugated to it is

$$\sum_{k=1}^{\infty}\left(b_k \cos kx - a_k \sin kx\right), \tag{2}$$

and the derived series is

$$\sum_{k=1}^{\infty} k\left(b_k \cos kx - a_k \sin kx\right). \tag{3}$$

Let $S_n(x), \widetilde{S}_n(x)$, and $S_n'(x)$ denote the partial sums of series (1), (2), and (3) respectively. We write

$$\psi_x(t) = \psi(f,t) = \begin{cases} f(x+t) - f(x-t), & \text{for } 0 < t \leq \pi; \\ g(x), & \text{for } t = 0, \end{cases}$$
$$\beta_x(t) = \frac{\psi_x(t)}{4\sin(1/2)t}, \tag{4}$$

where $g(x) = f(x+0) - f(x-0)$.

We propose to prove the following results.

Theorem 2. *Let $f(x)$ be a function integrable in the sense of Lebesgue in $[0, 2\pi]$ and periodic with period 2π. Let $H = (h_{n+k})$ be a Hankel matrix. Then for each $\beta_x(t) \in BV[0, 2\pi]$, the Hankel matrix transform of the sequence $(S_k'(x))$ is $\beta_x(0_+)$; that is,*

$$\lim_n \sum_{k=1}^{\infty} h_{n+k} S_k'(x) = \beta_x(0+) \tag{5}$$

if and only if

$$\lim_{n\to\infty}\sum_{k=0}^{\infty} h_{n+k} \sin\left(k+\frac{1}{2}\right)t = 0 \tag{6}$$

for every $t \in (0, \pi]$, where $BV[0, 2\pi]$ denotes the set of all functions of bounded variations on $[0, 2\pi]$.

In the next result, we replace the Hankel matrix by an arbitrary nonnegative regular matrix in the result of Al-Homidan [2].

Theorem 3. *Let $f(x)$ be a function integrable in the sense of Lebesgue in $[0, 2\pi]$ and periodic with period 2π. Let $A = (a_{nk})$ be a nonnegative regular matrix. Then A-transform of the sequence $(k\widetilde{S}_k(x))$ converges to $g(x)/\pi$; that is*

$$\lim_n \sum_{k=1}^{\infty} k a_{nk} \widetilde{S}_k(x) = \frac{1}{\pi} g(x), \tag{7}$$

if and only if

$$\lim_{n\to\infty}\sum_{k=0}^{\infty} a_{nk} \cos kt = 0 \tag{8}$$

for every $t \in (0, \pi]$, where each $a_k, b_k \in BV[0, 2\pi]$.

3. Proofs

We will need the following lemma which is known as the Banach Weak Convergence Theorem [5].

Lemma 4. $\lim_{n\to\infty}\int_0^{\pi} g_n dh_x = 0$ *for all $h_x \in BV[0, \pi]$ if and only if $\|g_n\| < \infty$ for all n and $\lim_{n\to\infty} g_n = 0$.*

Proof of Theorem 2. We have

$$\begin{aligned} S_k'(x) &= \frac{1}{\pi}\int_0^{\pi} \psi_x(t)\left(\sum_{m=1}^{k} m \sin mt\right) dt \\ &= -\frac{1}{\pi}\int_0^{\pi} \psi_x(t) \frac{d}{dt}\left(\frac{\sin(k+1/2)t}{2\sin(t/2)}\right) dt, \\ &= I_k + \frac{2}{\pi}\int_0^{\pi} \sin\left(k+\frac{1}{2}\right) t \, d\beta_x(t), \end{aligned} \tag{9}$$

where

$$I_k = \frac{1}{\pi}\int_0^{\pi} \beta_x(t) \cos\frac{t}{2}\left(\frac{\sin(k+1/2)t}{\sin(t/2)}\right) dt. \tag{10}$$

Then

$$\sum_{k=1}^{\infty} h_{n+k} S_k'(x) = \sum_{k=1}^{\infty} h_{n+k} I_k + \frac{2}{\pi}\int_0^{\pi} L_n(t)\, d\beta_x(t), \tag{11}$$

where

$$L_n(t) = \sum_{k=1}^{\infty} h_{n+k} \sin\left(k+\frac{1}{2}\right) t. \tag{12}$$

Since $\beta_x(t)$ is of bounded variation on $[0, \pi]$ and $\beta_x(t) \to \beta_x(0+)$ as $t \to 0$, $\beta_x(t)\cos(t/2)$ has also the same property. Hence by Jordan's convergence criterion for Fourier series $I_k \to \beta_x(0+)$ as $k \to \infty$.

Since the Hankel matrix $H = (h_{n+k})$ is regular, we have

$$\lim_n \sum_{k=1}^{\infty} h_{n+k} I_k = \beta_x(0+). \tag{13}$$

Now, it is enough to show that (6) holds if and only if

$$\lim_n \int_0^{\pi} L_n(t)\, d\beta_x(t) = 0. \tag{14}$$

Hence, by Lemma 4, it follows that (14) holds if and only if

$$\|L_n(t)\| \leq M \quad \forall n,\ \forall t \in [0, \pi], \tag{15}$$

and (6) holds. Since (15) is satisfied by Lemma 1(i), it follows that (14) holds if and only if (6) holds. Hence the result follows immediately. □

Proof of Theorem 3. We have

$$\begin{aligned} \widetilde{S}_n(x) &= \frac{1}{\pi}\int_0^{\pi} \psi_x(t) \sin nt\, dt, \\ &= \frac{g(x)}{n\pi} + \frac{1}{n\pi}\int_0^{\pi} \cos nt\, d\psi_x(t). \end{aligned} \tag{16}$$

Therefore

$$\sum_{k=1}^{\infty} k a_{nk} \widetilde{S}_k (x) = \frac{g(x)}{\pi} \sum_{k=1}^{\infty} a_{nk} + \frac{1}{\pi} \int_0^{\pi} K_n (t) \, d\psi_x (t) , \qquad (17)$$

where

$$K_n (t) = \sum_{k=1}^{\infty} a_{nk} \cos kt. \qquad (18)$$

Now, taking limit as $n \to \infty$ on both sides of (17) and using Lemmas 1 and 4 as in the proof of Theorem 2, we get the required result. □

Remark 5. If we take $a_{nk} = h_{n+k}$, then Theorem 2 is reduced to Theorem 4.1 of [2].

Remark 6. If we replace the matrix H by an arbitrary nonnegative regular matrix $A = (a_{nk})$ in Theorem 2, we get Theorem 1 of Rao [6].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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