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Letter to the Editor

A Note on the Semi-Inverse Method and a Variational Principle for the Generalized KdV-mKdV Equation

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Ji-Huan He systematically studied the inverse problem of calculus of variations. This note reveals that the semi-inverse method also works for a generalized KdV-mKdV equation with nonlinear terms of any orders.

1. Introduction

In [1], the semi-inverse method is systematically studied and many examples are given to show how to establish a variational formulation for a nonlinear equation. From the given examples, we found that it is difficult to find a variational principle for nonlinear evolution equations with nonlinear terms of any orders.

For example, consider the following generalized KdV-mKdV equation:

$$u_t + \left(\alpha + \beta u^p + \gamma u^{2p}\right) u_x + u_{xxx} + \eta u_{xxxxx} + g(t)\, u = 0, \tag{1}$$

where α, β, γ, and η are constant coefficients, while p is a positive number. Equation (1) is an important model in plasma physics and solid state physics.

2. Variational Principle by He's Semi-Inverse Method

For (1), we introduce a potential function v defined as $u = v_x$; we have the following equation:

$$\begin{aligned} v_{xt} + \left(\alpha + \beta v_x^p + \gamma v_x^{2p}\right) v_{xx} + v_{xxxx} + \eta v_{xxxxxx} \\ + g(t)\, v_x = 0. \end{aligned} \tag{2}$$

In order to use the semi-inverse method [1–4] to establish a Lagrangian for (2), we first check some simple cases:

$$\begin{aligned} L &= -\frac{v_x v_t}{2} \quad \text{for } v_{xt} = 0, \\ L &= \frac{\left(v_{xx}\right)^2}{2} \quad \text{for } v_{xxxx} = 0, \\ L &= -\frac{v_x^3}{6} \quad \text{for } \frac{\left(v_x^2\right)_x}{2} = v_x v_{xx} = 0, \\ L &= -\frac{v_x^n}{n(n-1)} \quad \text{for } v_x^{n-2} v_{xx} = 0. \end{aligned} \tag{3}$$

We can easily obtain a variational principle for (2) for $g(t) \equiv 0$, which is

$$\begin{aligned} J(v) = \iint \Big\{ & -\frac{1}{2} v_x v_t - \frac{1}{2}\alpha v_x^2 - \frac{\beta}{(p+2)(p+1)} v_x^{p+2} \\ & - \frac{\gamma}{(2p+2)(2p+1)} v_x^{2p+2} + v_{xx}^2 - \frac{\eta}{2} v_{xxx}^2 \Big\}\, dx\, dt, \end{aligned} \tag{4}$$

Now, according to the semi-inverse method [1–4], we construct a trial functional for (2):

$$J(v) = \iint \left\{ f(t) \left[-\frac{1}{2} v_x v_t - \frac{1}{2} \alpha v_x^2 - \frac{\beta}{(p+2)(p+1)} v_x^{p+2} - \frac{\gamma}{(2p+2)(2p+1)} v_x^{2p+2} + v_{xx}^2 - \frac{\eta}{2} v_{xxx}^2 \right] + F \right\} dx dt, \tag{5}$$

where F is an unknown function of u and/or its derivatives.

Making the trial-functional, (5), stationary with respect to v results in the following Euler-Lagrange equation:

$$\frac{1}{2}(f v_x)_t + \frac{1}{2}(f v_t)_x + \alpha (f v_x)_x + \frac{\beta}{(p+1)} \left(f v_x^{p+1} \right)_x + \frac{\gamma}{(2p+1)} \left(f v_x^{2p+1} \right)_x + (f v_{xx})_{xx} - \eta (f v_{xxx})_{xxx} + \frac{\delta F}{\delta v} = 0, \tag{6}$$

where $\delta F / \delta v$ is called variational differential with respect to v, defined as

$$\frac{\delta F}{\delta v} = \frac{\partial F}{\partial v} - \frac{\partial}{\partial t} \left(\frac{\partial F}{\partial v_t} \right) - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial v_x} \right) + \frac{\partial^2}{\partial t^2} \left(\frac{\partial F}{\partial v_{tt}} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial F}{\partial v_{xx}} \right) + \cdots . \tag{7}$$

We rewrite (6) in the form

$$\frac{f_t}{2f} v_x + v_{xt} + \alpha v_{xx} + \beta v_x^p v_{xx} + \gamma v_x^{2p} v_{xx} + v_{xxxx} + \eta v_{xxxxxx} + \frac{\delta F}{f \delta v} = 0. \tag{8}$$

Comparison of (8) and (2) leads to the following results:

$$\frac{f_t}{2f} = g(t), \qquad \frac{\delta F}{f \delta v} = 0, \tag{9}$$

from which we identify the unknown f and F as follows:

$$f(t) = e^{2 \int g(t) dt}, \qquad F = 0. \tag{10}$$

We, therefore, obtain the following needed variational principle:

$$J(v) = \iint \left\{ e^{2 \int g(t) dt} \left[-\frac{1}{2} v_x v_t - \alpha v_x^2 - \frac{\beta}{(p+2)(p+1)} v_x^{p+2} - \frac{\gamma}{(2p+2)(2p+1)} v_x^{2p+2} + v_{xx}^2 - \frac{\eta}{2} v_{xxx}^2 \right] \right\} dx dt. \tag{11}$$

3. Conclusion

This note shows that the semi-inverse method in [1] works also for the present problem, and it is concluded that the semi-inverse method is a powerful mathematical tool to the construction of a variational formulation for a nonlinear equation; illustrating examples are available in [5–10].

The semi-inverse method can be extended to fractional calculus [11–14].

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