

Research Article

Umbral Calculus and the Frobenius-Euler Polynomials

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We study some properties of umbral calculus related to the Appell sequence. From those properties, we derive new and interesting identities of the Frobenius-Euler polynomials.

1. Introduction

Let \mathbf{C} be the complex number field. For $\lambda \in \mathbf{C}$ with $\lambda \neq 1$, the Frobenius-Euler polynomials are defined by the generating function to be

$$\frac{1-\lambda}{e^t-\lambda} e^{xt} = e^{H(x|\lambda)t} = \sum_{n=0}^{\infty} H_n(x|\lambda) \frac{t^n}{n!}, \quad (1)$$

(see [1–5]) with the usual convention about replacing $H^n(x|\lambda)$ by $H_n(x|\lambda)$.

In the special case, $x = 0$, $H_n(0|\lambda) = H_n(\lambda)$ are called the n th Frobenius-Euler numbers. By (1), we get

$$H_n(x|\lambda) = \sum_{l=0}^n \binom{n}{l} H_{n-l}(\lambda) x^l = (H(\lambda) + x)^n, \quad (2)$$

(see [6–9]) with the usual convention about replacing $H^n(\lambda)$ by $H_n(\lambda)$.

Thus, from (1) and (2), we note that

$$(H(\lambda) + 1)^n - \lambda H_n(\lambda) = (1 - \lambda) \delta_{0,n}, \quad (3)$$

where $\delta_{n,k}$ is the kronecker symbol (see [1, 10, 11]).

For $r \in \mathbf{Z}_+$, the Frobenius-Euler polynomials of order r are defined by the generating function to be

$$\begin{aligned} \left(\frac{1-\lambda}{e^t-\lambda} \right)^r e^{xt} &= \underbrace{\left(\frac{1-\lambda}{e^t-\lambda} \right) \times \cdots \times \left(\frac{1-\lambda}{e^t-\lambda} \right)}_{r\text{-times}} e^{xt} \\ &= \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda) \frac{t^n}{n!}. \end{aligned} \quad (4)$$

In the special case, $x = 0$, $H_n^{(r)}(0|\lambda) = H_n^{(r)}(\lambda)$ are called the n th Frobenius-Euler numbers of order r (see [1, 10]).

From (4), we can derive the following equation:

$$\begin{aligned} H_n^{(r)}(x|\lambda) &= \sum_{l=0}^n \binom{n}{l} H_{n-l}^{(r)}(\lambda) x^l, \\ H_n^{(r)}(\lambda) &= \sum_{l_1+\cdots+l_r=n} \binom{n}{l_1, \dots, l_r} H_{l_1}(\lambda) \cdots H_{l_r}(\lambda). \end{aligned} \quad (5)$$

By (5), we see that $H_n^{(r)}(x|\lambda)$ is a monic polynomial of degree n with coefficients in $\mathbf{Q}(\lambda)$.

Let \mathbb{P} be the algebra of polynomials in the single variable x over \mathbf{C} and let \mathbb{P}^* be the vector space of all linear functionals on \mathbb{P} . As is known, $\langle L | p(x) \rangle$ denotes the action of the linear functional L on a polynomial $p(x)$ and we remind that

the addition and scalar multiplication on \mathbb{P}^* are, respectively, defined by

$$\begin{aligned} \langle L + M \mid p(x) \rangle &= \langle L \mid p(x) \rangle + \langle M \mid p(x) \rangle, \\ \langle cL \mid p(x) \rangle &= c \langle L \mid p(x) \rangle, \end{aligned} \tag{6}$$

where c is a complex constant (see [3, 12]).

Let \mathbf{F} denote the algebra of formal power series:

$$\mathbf{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \mid a_k \in \mathbf{C} \right\} \tag{7}$$

(see [3, 12]). The formal power series define a linear functional on \mathbb{P} by setting

$$\langle f(t) \mid x^n \rangle = a_n, \quad \forall n \geq 0. \tag{8}$$

Indeed, by (7) and (8), we get

$$\langle t^k \mid x^n \rangle = n! \delta_{n,k} \quad (n, k \geq 0) \tag{9}$$

(see [3, 12]). This kind of algebra is called an umbral algebra.

The order $O(f(t))$ of a nonzero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. A series $f(t)$ for which $O(f(t)) = 1$ is said to be an invertible series (see [2, 12]). For $f(t), g(t) \in \mathbf{F}$, and $p(x) \in \mathbb{P}$, we have

$$\begin{aligned} \langle f(t)g(t) \mid p(x) \rangle &= \langle f(t) \mid g(t)p(x) \rangle \\ &= \langle g(t) \mid f(t)p(x) \rangle \end{aligned} \tag{10}$$

(see [12]). One should keep in mind that each $f(t) \in \mathbf{F}$ plays three roles in the umbral calculus: a formal power series, a linear functional, and a linear operator. To illustrate this, let $p(x) \in \mathbb{P}$ and $f(t) = e^{yt} \in \mathbf{F}$. As a linear functional, e^{yt} satisfies $\langle e^{yt} \mid p(x) \rangle = p(y)$. As a linear operator, e^{yt} satisfies $e^{yt} p(x) = p(x + y)$ (see [12]). Let $s_n(x)$ denote a polynomial in x with degree n . Let us assume that $f(t)$ is a delta series and $g(t)$ is an invertible series. Then there exists a unique sequence $s_n(x)$ of polynomials such that $\langle g(t)f(t)^k \mid s_n(x) \rangle = n! \delta_{n,k}$ for all $n, k \geq 0$ (see [3, 12]). This sequence $s_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$ which is denoted by $s_n(x) \sim (g(t), f(t))$. If $s_n(x) \sim (1, f(t))$, then $s_n(x)$ is called the associated sequence for $f(t)$. If $s_n(x) \sim (g(t), t)$, then $s_n(x)$ is called the Appell sequence.

Let $s_n(x) \sim (g(t), f(t))$. Then we see that

$$\begin{aligned} h(t) &= \sum_{k=0}^{\infty} \frac{\langle h(t) \mid s_k(x) \rangle}{k!} g(t) f(t)^k, \quad h(t) \in \mathbf{F}, \\ p(x) &= \sum_{k=0}^{\infty} \frac{\langle g(t) f(t)^k \mid p(x) \rangle}{k!} s_k(x), \quad p(x) \in \mathbb{P}, \tag{11} \\ f(t) s_n(x) &= n s_{n-1}(x), \end{aligned}$$

$$\begin{aligned} \langle f(t) \mid p(\alpha x) \rangle &= \langle f(\alpha t) \mid p(x) \rangle, \\ \frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} &= \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} t^k, \quad \forall y \in \mathbf{C}, \end{aligned} \tag{12}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ (see [3]). In this paper, we study some properties of umbral calculus related to the Appell sequence. For those properties, we derive new and interesting identities of the Frobenius-Euler polynomials.

2. The Frobenius-Euler Polynomials and Umbral Calculus

By (4) and (12), we see that

$$H_n^{(r)}(x \mid \lambda) \sim \left(\left(\frac{e^t - \lambda}{1 - \lambda} \right)^r, t \right). \tag{13}$$

Thus, by (13), we get

$$\left\langle \left(\frac{e^t - \lambda}{1 - \lambda} \right)^r t^k \mid H_n^{(r)}(x \mid \lambda) \right\rangle = n! \delta_{n,k}. \tag{14}$$

Let

$$\mathbb{P}_n(\lambda) = \{ p(x) \in \mathbf{Q}(\lambda)[x] \mid \deg p(x) \leq n \}. \tag{15}$$

Then it is an $(n + 1)$ -dimensional vector space over $\mathbf{Q}(\lambda)$.

So we see that $\{H_0^{(r)}(x \mid \lambda), H_1^{(r)}(x \mid \lambda), \dots, H_n^{(r)}(x \mid \lambda)\}$ is a basis for $\mathbb{P}_n(\lambda)$. For $p(x) \in \mathbb{P}_n(\lambda)$, let

$$p(x) = \sum_{k=0}^n C_k H_k^{(r)}(x \mid \lambda), \quad (n \geq 0). \tag{16}$$

Then, by (13), (14), and (16), we get

$$\begin{aligned} &\left\langle \left(\frac{e^t - \lambda}{1 - \lambda} \right)^r t^k \mid p(x) \right\rangle \\ &= \sum_{l=0}^n C_l \left\langle \left(\frac{e^t - \lambda}{1 - \lambda} \right)^r t^k \mid H_l^{(r)}(x \mid \lambda) \right\rangle \\ &= \sum_{l=0}^n C_l l! \delta_{l,k} = k! C_k. \end{aligned} \tag{17}$$

From (17), we have

$$\begin{aligned} C_k &= \frac{1}{k!} \left\langle \left(\frac{e^t - \lambda}{1 - \lambda} \right)^r t^k \mid p(x) \right\rangle \\ &= \frac{1}{k!} \left\langle \left(\frac{e^t - \lambda}{1 - \lambda} \right)^r \mid D^k p(x) \right\rangle \\ &= \frac{1}{k!(1 - \lambda)^r} \sum_{j=0}^r \binom{r}{j} (-\lambda)^{r-j} \langle e^{jt} \mid D^k p(x) \rangle \\ &= \frac{1}{k!(1 - \lambda)^r} \sum_{j=0}^r \binom{r}{j} (-\lambda)^{r-j} \langle t^0 \mid e^{jt} D^k p(x) \rangle \\ &= \frac{1}{k!(1 - \lambda)^r} \sum_{j=0}^r \binom{r}{j} (-\lambda)^{r-j} \langle t^0 \mid D^k p(x + j) \rangle. \end{aligned} \tag{18}$$

Therefore, by (16) and (18), we obtain the following theorem.

Theorem 1. For $p(x) \in \mathbb{P}_n(\lambda)$, let

$$p(x) = \sum_{k=0}^n C_k H_k^{(r)}(x). \tag{19}$$

Then one has

$$C_k = \frac{1}{k!(1-\lambda)^r} \sum_{j=0}^r \binom{r}{j} (-\lambda)^{r-j} D^k p(j), \tag{20}$$

where $Dp(x) = dp(x)/dx$.

From Theorem 1, we note that

$$p(x) = \frac{1}{(1-\lambda)^r} \cdot \sum_{k=0}^n \left\{ \sum_{j=0}^r \frac{1}{k!} \binom{r}{j} (-\lambda)^{r-j} D^k p(j) \right\} H_k^{(r)}(x | \lambda). \tag{21}$$

Let us consider the operator $\tilde{\Delta}_\lambda$ with $\tilde{\Delta}_\lambda f(x) = f(x+1) - \lambda f(x)$ and let $J_\lambda = (1/(1-\lambda))\tilde{\Delta}_\lambda$. Then we have

$$J_\lambda(f)(x) = \frac{1}{1-\lambda} \{f(x+1) - \lambda f(x)\}. \tag{22}$$

Thus, by (22), we get

$$J_\lambda(H_n^{(r)}(x | \lambda)) = \frac{1}{1-\lambda} \{H_n^{(r)}(x+1 | \lambda) - \lambda H_n^{(r)}(x | \lambda)\}. \tag{23}$$

From (4), we can derive

$$\begin{aligned} & \sum_{n=0}^{\infty} \{H_n^{(r)}(x+1 | \lambda) - \lambda H_n^{(r)}(x | \lambda)\} \frac{t^n}{n!} \\ &= \left(\frac{1-\lambda}{e^t - \lambda}\right)^r e^{(x+1)t} - \lambda \left(\frac{1-\lambda}{e^t - \lambda}\right)^r e^{xt} \\ &= \left(\frac{1-\lambda}{e^t - \lambda}\right)^r e^{xt} (e^t - \lambda) = (1-\lambda) \left(\frac{1-\lambda}{e^t - \lambda}\right)^{r-1} e^{xt} \\ &= (1-\lambda) \sum_{n=0}^{\infty} H_n^{(r-1)}(x | \lambda) \frac{t^n}{n!}. \end{aligned} \tag{24}$$

By (23) and (24), we get

$$J_\lambda(H_n^{(r)}(x | \lambda)) = H_n^{(r-1)}(x | \lambda). \tag{25}$$

From (25), we have

$$\begin{aligned} J_\lambda^r(H_n^{(r)}(x | \lambda)) &= J_\lambda^{r-1}(H_n^{(r-1)}(x | \lambda)) \\ &= \dots = H_n^{(0)}(x | \lambda) = x^n, \end{aligned}$$

$$J_\lambda^r(x^n) = J_\lambda^r H_n^{(0)}(x | \lambda) = H_n^{(-r)}(x | \lambda) = J_\lambda^{2r} H_n^{(r)}(x | \lambda). \tag{26}$$

For $s \in \mathbb{Z}_+$, from (25), we have

$$J_\lambda^s(H_n^{(r)}(x | \lambda)) = H_n^{(r-s)}(x | \lambda). \tag{27}$$

On the other hand, by (12), (13), and (25),

$$\begin{aligned} J_\lambda^s(H_n^{(r)}(x | \lambda)) &= \left(\frac{e^t - \lambda}{1-\lambda}\right)^s (H_n^{(r)}(x | \lambda)) \\ &= \frac{1}{(1-\lambda)^s} \left((1-\lambda) + \sum_{k=1}^{\infty} \frac{t^k}{k!} \right)^s \\ &\quad \cdot (H_n^{(r)}(x | \lambda)). \end{aligned} \tag{28}$$

Thus, by (28), we get

$$\begin{aligned} & J_\lambda^s(H_n^{(r)}(x | \lambda)) \\ &= \sum_{m=0}^s \frac{\binom{s}{m}}{(1-\lambda)^m} \sum_{l=m}^{\infty} \left(\sum_{\substack{k_1+\dots+k_m=l \\ k_j \geq 1}} \frac{1}{k_1! \dots k_m!} \right) t^l (H_n^{(r)}(x | \lambda)) \\ &= \sum_{m=0}^s \frac{\binom{s}{m}}{(1-\lambda)^m} \sum_{l=m}^{\infty} \frac{1}{l!} \left(\sum_{\substack{k_1+\dots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \dots, k_m} D^l \right) \\ &\quad \cdot H_n^{(r)}(x | \lambda) \\ &= \sum_{m=0}^{\min\{s,n\}} \frac{\binom{s}{m}}{(1-\lambda)^m} \sum_{l=m}^n \binom{n}{l} \sum_{\substack{k_1+\dots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \dots, k_m} H_{n-l}^{(r)}(x | \lambda) \\ &= \sum_{l=0}^{\min\{s,n\}} \left\{ \binom{n}{l} \sum_{m=0}^l \frac{\binom{s}{m}}{(1-\lambda)^m} \right. \\ &\quad \cdot \left. \sum_{\substack{k_1+\dots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \dots, k_m} \right\} H_{n-l}^{(r)}(x | \lambda) \\ &\quad + \sum_{l=\min\{s,n\}+1}^n \left\{ \binom{n}{l} \sum_{m=0}^{\min\{s,n\}} \frac{\binom{s}{m}}{(1-\lambda)^m} \right. \\ &\quad \cdot \left. \sum_{\substack{k_1+\dots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \dots, k_m} \right\} H_{n-l}^{(r)}(x | \lambda). \end{aligned} \tag{29}$$

Therefore, by (27) and (29), we obtain the following theorem.

Theorem 2. For any $r, s \geq 0$, one has

$$\begin{aligned}
 & H_n^{(r-s)}(x | \lambda) \\
 &= \sum_{l=0}^{\min\{s,n\}} \left\{ \binom{n}{l} \sum_{m=0}^l \frac{\binom{s}{m}}{(1-\lambda)^m} \sum_{\substack{k_1+\dots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \dots, k_m} \right\} \\
 &\cdot H_{n-l}^{(r)}(x | \lambda) \\
 &+ \sum_{l=\min\{s,n\}+1}^n \left\{ \binom{n}{l} \sum_{m=0}^{\min\{s,n\}} \frac{\binom{s}{m}}{(1-\lambda)^m} \right. \\
 &\quad \left. \cdot \sum_{\substack{k_1+\dots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \dots, k_m} \right\} H_{n-l}^{(r)}(x | \lambda). \tag{30}
 \end{aligned}$$

Let us take $s = r - 1$ ($r \geq 1$) in Theorem 2. Then we obtain the following corollary.

Corollary 3. For $n \geq 0, r \geq 1$, one has

$$\begin{aligned}
 & H_n(x | \lambda) \\
 &= \sum_{l=0}^{\min\{r-1,n\}} \left\{ \binom{n}{l} \sum_{m=0}^l \frac{\binom{r-1}{m}}{(1-\lambda)^m} \sum_{\substack{k_1+\dots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \dots, k_m} \right\} \\
 &\cdot H_{n-l}^{(r)}(x | \lambda) \\
 &+ \sum_{l=\min\{r-1,n\}+1}^n \left\{ \binom{n}{l} \sum_{m=0}^{\min\{r-1,n\}} \frac{\binom{r-1}{m}}{(1-\lambda)^m} \right. \\
 &\quad \left. \cdot \sum_{\substack{k_1+\dots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \dots, k_m} \right\} H_{n-l}^{(r)}(x | \lambda). \tag{31}
 \end{aligned}$$

Let us take $s = r$ ($r \geq 1$) in Theorem 2. Then we obtain the following corollary.

Corollary 4. For $n \geq 0, r \geq 1$, one has

$$\begin{aligned}
 & x^n = \sum_{l=0}^{\min\{r,n\}} \left\{ \binom{n}{l} \sum_{m=0}^l \frac{\binom{r}{m}}{(1-\lambda)^m} \sum_{\substack{k_1+\dots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \dots, k_m} \right\} \\
 &\cdot H_{n-l}^{(r)}(x | \lambda)
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{l=\min\{r,n\}+1}^n \left\{ \binom{n}{l} \sum_{m=0}^{\min\{r,n\}} \frac{\binom{r}{m}}{(1-\lambda)^m} \right. \\
 &\quad \left. \cdot \sum_{\substack{k_1+\dots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \dots, k_m} \right\} H_{n-l}^{(r)}(x | \lambda). \tag{32}
 \end{aligned}$$

Now, we define the analogue of Stirling numbers of the second kind as follows:

$$S_\lambda(n, k) = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (-\lambda)^{k-j} j^n, \quad (n, k \geq 0). \tag{33}$$

Note that $S_1(n, k) = S(n, k)$ is the Stirling number of the second kind.

From the definition of $\tilde{\Delta}_\lambda$, we have

$$\tilde{\Delta}_\lambda^n f(0) = \sum_{k=0}^n \binom{n}{k} (-\lambda)^{n-k} f(k). \tag{34}$$

By (33) and (34), we get

$$S_\lambda(n, k) = \frac{1}{k!} \tilde{\Delta}_\lambda^k 0^n, \quad (n, k \geq 0). \tag{35}$$

Let us take $s = 2r$. Then we have

$$\begin{aligned}
 & J_\lambda^r x^n \\
 &= H_n^{(-r)}(x | \lambda) \\
 &= \sum_{l=0}^{\min\{2r,n\}} \left\{ \binom{n}{l} \sum_{m=0}^l \frac{\binom{2r}{m}}{(1-\lambda)^m} \sum_{\substack{k_1+\dots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \dots, k_m} \right\} \\
 &\cdot H_{n-l}^{(r)}(x | \lambda) \\
 &+ \sum_{l=\min\{2r,n\}+1}^n \left\{ \binom{n}{l} \sum_{m=0}^{\min\{2r,n\}} \frac{\binom{2r}{m}}{(1-\lambda)^m} \right. \\
 &\quad \left. \cdot \sum_{\substack{k_1+\dots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \dots, k_m} \right\} H_{n-l}^{(r)}(x | \lambda),
 \end{aligned}$$

$$\begin{aligned}
 & J_\lambda^r x^n = \left(\frac{1}{1-\lambda} \tilde{\Delta}_\lambda \right)^r x^n \\
 &= \frac{1}{(1-\lambda)^r} \sum_{j=0}^r \binom{r}{j} (-\lambda)^{r-j} (x+j)^n. \tag{36}
 \end{aligned}$$

By (36), we get

$$\begin{aligned}
 & \frac{1}{(1-\lambda)^r} \sum_{j=0}^r \binom{r}{j} (-\lambda)^{r-j} (x+j)^n \\
 &= \frac{1}{(1-\lambda)^r} \tilde{\Delta}_\lambda^r x^n \\
 &= \sum_{l=0}^{\min\{2r,n\}} \left\{ \binom{n}{l} \sum_{m=0}^l \frac{\binom{2r}{m}}{(1-\lambda)^m} \sum_{\substack{k_1+\dots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \dots, k_m} \right\} \\
 & \cdot H_{n-l}^{(r)}(x|\lambda) \\
 &+ \sum_{l=\min\{2r,n\}+1}^n \left\{ \binom{n}{l} \sum_{m=0}^{\min\{2r,n\}} \frac{\binom{2r}{m}}{(1-\lambda)^m} \right. \\
 & \left. \cdot \sum_{\substack{k_1+\dots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \dots, k_m} \right\} H_{n-l}^{(r)}(x|\lambda). \tag{37}
 \end{aligned}$$

Let us take $x = 0$ in (37). Then we obtain the following theorem.

Theorem 5. We have

$$\begin{aligned}
 & \frac{r!}{(1-\lambda)^r} S_\lambda(n, r) \\
 &= \frac{r!}{(1-\lambda)^r} \frac{\tilde{\Delta}_\lambda^r 0^n}{r!} \\
 &= \sum_{l=0}^{\min\{2r,n\}} \left\{ \binom{n}{l} \sum_{m=0}^l \frac{\binom{2r}{m}}{(1-\lambda)^m} \sum_{\substack{k_1+\dots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \dots, k_m} \right\} \\
 & \cdot H_{n-l}^{(r)}(\lambda) \\
 &+ \sum_{l=\min\{2r,n\}+1}^n \left\{ \binom{n}{l} \sum_{m=0}^{\min\{2r,n\}} \frac{\binom{2r}{m}}{(1-\lambda)^m} \right. \\
 & \left. \cdot \sum_{\substack{k_1+\dots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \dots, k_m} \right\} H_{n-l}^{(r)}(\lambda) \\
 &= \sum_{m=0}^{\min\{r,n\}} \frac{\binom{r}{m}}{(1-\lambda)^m} \sum_{\substack{k_1+\dots+k_m=n \\ k_j \geq 1}} \binom{n}{k_1, \dots, k_m}. \tag{38}
 \end{aligned}$$

Let us consider $s = 2r - 1$ in the identity of Theorem 2. Then we have

$$\begin{aligned}
 & J_\lambda^{r-1} x^n \\
 &= H_n^{-(r-1)}(x|\lambda) \\
 &= \sum_{l=0}^{\min\{2r-1,n\}} \left\{ \binom{n}{l} \sum_{m=0}^l \frac{\binom{2r-1}{m}}{(1-\lambda)^m} \sum_{\substack{k_1+\dots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \dots, k_m} \right\} \\
 & \cdot H_{n-l}^{(r)}(x|\lambda) \\
 &+ \sum_{l=\min\{2r-1,n\}+1}^n \left\{ \binom{n}{l} \sum_{m=0}^{\min\{2r-1,n\}} \frac{\binom{2r-1}{m}}{(1-\lambda)^m} \right. \\
 & \left. \cdot \sum_{\substack{k_1+\dots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \dots, k_m} \right\} H_{n-l}^{(r)}(x|\lambda) \\
 &= \frac{1}{(1-\lambda)^{r-1}} \sum_{j=0}^{r-1} \binom{r-1}{j} (-\lambda)^{r-1-j} (x+j)^n \\
 &= \frac{1}{(1-\lambda)^{r-1}} \tilde{\Delta}_\lambda^{r-1} x^n. \tag{39}
 \end{aligned}$$

Let us take $x = 0$ in (39). Then we obtain the following theorem.

Theorem 6. For $n \geq 0$ and $r \geq 1$, one has

$$\begin{aligned}
 & \frac{(r-1)!}{(1-\lambda)^{r-1}} S_\lambda(n, r-1) \\
 &= \frac{(r-1)!}{(1-\lambda)^{r-1}} \frac{\tilde{\Delta}_\lambda^{r-1} 0^n}{(r-1)!} \\
 &= \sum_{l=0}^{\min\{2r-1,n\}} \left\{ \binom{n}{l} \sum_{m=0}^l \frac{\binom{2r-1}{m}}{(1-\lambda)^m} \sum_{\substack{k_1+\dots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \dots, k_m} \right\} \\
 & \cdot H_{n-l}^{(r)}(\lambda) \\
 &+ \sum_{l=\min\{2r-1,n\}+1}^n \left\{ \binom{n}{l} \sum_{m=0}^{\min\{2r-1,n\}} \frac{\binom{2r-1}{m}}{(1-\lambda)^m} \right. \\
 & \left. \cdot \sum_{\substack{k_1+\dots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \dots, k_m} \right\} H_{n-l}^{(r)}(\lambda). \tag{40}
 \end{aligned}$$

Remark 7. Note that

$$\begin{aligned} & \frac{(r-1)!}{(1-\lambda)^{r-1}} S_\lambda(n, r-1) \\ &= \sum_{l=0}^{\min\{r,n\}} \left\{ \binom{n}{l} \sum_{m=0}^l \frac{\binom{r}{m}}{(1-\lambda)^m} \sum_{\substack{k_1+\dots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \dots, k_m} \right\} \\ & \quad \cdot H_{n-l}(\lambda) \\ & \quad + \sum_{l=\min\{r,n\}+1}^n \left\{ \binom{n}{l} \sum_{m=0}^{\min\{r,n\}} \frac{\binom{r}{m}}{(1-\lambda)^m} \right. \\ & \quad \left. \cdot \sum_{\substack{k_1+\dots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \dots, k_m} \right\} H_{n-l}(\lambda). \end{aligned} \tag{41}$$

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References

- [1] T. Kim, "Identities involving Frobenius–Euler polynomials arising from non-linear differential equations," *Journal of Number Theory*, vol. 132, no. 12, pp. 2854–2865, 2012.
- [2] T. Kim and J. Choi, "A note on the product of Frobenius-Euler polynomials arising from the p -adic integral on Z_p ," *Advanced Studies in Contemporary Mathematics*, vol. 22, no. 2, pp. 215–223, 2012.
- [3] S. Roman, *The Umbral Calculus*, Dover, New York, NY, USA, 2005.
- [4] Y. Simsek, O. Yurekli, and V. Kurt, "On interpolation functions of the twisted generalized Frobenius-Euler numbers," *Advanced Studies in Contemporary Mathematics*, vol. 15, no. 2, pp. 187–194, 2007.
- [5] K. Shiratani, "On Euler numbers," *Memoirs of the Faculty of Science. Kyushu University A*, vol. 27, pp. 1–5, 1973.
- [6] S. Araci and M. Acikgoz, "A note on the frobenius-euler numbers and polynomials associated with bernstein polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 22, no. 3, pp. 399–406, 2012.
- [7] L. Carlitz, "Some polynomials related to the Bernoulli and Euler polynomials," *Utilitas Mathematica*, vol. 19, pp. 81–127, 1981.
- [8] M. Can, M. Cenkci, V. Kurt, and Y. Simsek, "Twisted Dedekind type sums associated with Barnes' type multiple Frobenius-Euler l -functions," *Advanced Studies in Contemporary Mathematics*, vol. 18, no. 2, pp. 135–160, 2009.
- [9] I. N. Cangul, V. Kurt, H. Ozden, and Y. Simsek, "On the higher-order w - q -Genocchi numbers," *Advanced Studies in Contemporary Mathematics*, vol. 19, no. 1, pp. 39–57, 2009.
- [10] R. Dere and Y. Simsek, "Applications of umbral algebra to some special polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 22, no. 3, pp. 433–438, 2012.
- [11] K. Shiratani and S. Yamamoto, "On a p -adic interpolation function for the Euler numbers and its derivatives," *Memoirs of the Faculty of Science. Kyushu University A*, vol. 39, no. 1, pp. 113–125, 1985.
- [12] D. S. Kim and T. Kim, "Some identities of Frobenius-Euler polynomials arising from umbral calculus," *Advances in Difference Equations*, vol. 2012, article 196, 2012.