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Research Article

Fite-Wintner-Leighton-Type Oscillation Criteria for Second-Order Differential Equations with Nonlinear Damping

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Some new oscillation criteria for a general class of second-order differential equations with nonlinear damping are shown. Except some general structural assumptions on the coefficients and nonlinear terms, we additionally assume only one sufficient condition (of Fite-Wintner-Leighton type). It is different compared to many early published papers which use rather complex sufficient conditions. Our method contains three items: classic Riccati transformations, a pointwise comparison principle, and a blow-up principle for sub- and supersolutions of a class of the generalized Riccati differential equations associated to any nonoscillatory solution of the main equation.

1. Introduction

In the paper, we develop some new oscillation criteria for the following class of second-order differential equations with nonlinear damping:

$$\begin{aligned}\left(r(t)k_1\left(x,x'\right)\right)' + p(t)k_2\left(x,x'\right)x' \\ + q(t)f(x) = 0, \quad t \ge t_0 > 0,\end{aligned} \tag{1}$$

where the coefficients $r \in C^1([t_0,\infty),(0,\infty))$, $p,q \in C([t_0,\infty),\mathbb{R})$, and the functions $k_1(u,v)$, $k_2(u,v)$ are continuous in all their variables, $k_1 \in C^1(\mathbb{R}^2,\mathbb{R})$ and solution $x = x(t)$, $x \in C^2((t_0,\infty),\mathbb{R}) \cap C([t_0,\infty),\mathbb{R})$. A function $x(t)$ is said to be oscillatory if there is a sequence $t_n \ge t_0$ such that $x(t_n) = 0$ and $t_n \to \infty$ as $n \to \infty$.

In Section 2, we present some basic structural assumptions on the coefficients: $r(t)$, $p(t)$, and $q(t)$ and on the nonlinear functions: $k_1(u,v)$ and $k_2(u,v)$, which are slightly more general than those of the previously published results such as in Zhao et al. [1, Theorem 2.1] (see also Theorem A, Section 2), [1, Theorems 2.2–2.8], [2, Theorem 2], [3, Theorem 2.1]. In Section 3, we study some new oscillation criteria for (1) based on an additional sufficient condition of Fite-Wintner-Leighton type, which is rather simpler than Kamenev-type conditions or related complex ones. Equation (1) in various different forms has been considered in many several published papers, see, for instance, [4–12] and references therein. In Section 4, we state and prove a pointwise comparison principle between all sub- and supersolutions of the corresponding generalized Riccati differential equation associated with every nonoscillatory solution $x(t)$ of (1). Furthermore, under the main assumption of Fite-Wintner-Leighton type, we construct a subsolution of the Riccati differential equation which blows up in time. It together with classic Riccati transformation gives the proof of the main result.

2. Main Assumptions and Remarks

In particular, for $m = n = 1$, in [1] authors firstly supposed the next five basic conditions on the coefficients $p(t)$, $q(t)$ and the functions $f(u)$, $k_1(u,v)$, and $k_2(u,v)$:

$$p(t) \ge 0 \quad \forall t \ge t_0, \tag{2}$$

$$\frac{f(u)}{u} \ge K \quad \text{for some } K > 0 \text{ and all } u \in \mathbb{R}, u \ne 0, \tag{3}$$

$$\begin{gathered} q(t) \ge 0 \quad \forall t \ge t_0, \\ q(t) \not\equiv 0 \ \text{ on } \ [t_*,\infty) \text{ for any } t_* \ge t_0, \end{gathered} \tag{4}$$

$$k_1^{2m}(u,v) \le \alpha_1 u^{2m-2} v k_1(u,v) \quad \text{for some } m \in \mathbb{N}, \alpha_1 > 0 \text{ and all } (u,v) \in \mathbb{R}^2,\ u \ne 0, \tag{5}$$

$$u^{2n-1} v k_2(u,v) \ge \alpha_2 k_1^{2n}(u,v) \quad \text{for some } n \in \mathbb{N}, \alpha_2 > 0 \text{ and all } (u,v) \in \mathbb{R}^2,\ u \ne 0. \tag{6}$$

Such a set of assumptions, with slightly different (6) and $m = n = 1$, was introduced for the first time in [2], see also [3]. Just the same as in [1], besides (6) we also consider a similar assumption:

$$uvk_2(u,v) \ge \alpha_2 u k_1(u,v) \quad \text{for some } \alpha_2 > 0 \text{ and all } (u,v) \in \mathbb{R}^2. \tag*{(6)$_1$}$$

And, in the case when $p(t)$ and $q(t)$ may change the sign, instead of (2)–(4) and (6), one considers also:

$$f \in C^1(\mathbb{R},\mathbb{R}), \qquad uf(u) \ne 0, \qquad f'(u) \ge K > 0 \quad \forall u \in \mathbb{R},\ u \ne 0, \tag*{(3)$_1$}$$

$$vk_2(u,v) = \alpha_2 k_1(u,v) \quad \text{for some } \alpha_2 > 0 \text{ and all } (u,v) \in \mathbb{R}^2. \tag*{(6)$_2$}$$

Here, assumptions (5) and (6) are generalized in the following sense, see Theorem 5—(ii) and (iii), respectively,

$$vk_1(u,v) \ge 0 \quad \forall (u,v) \in \mathbb{R}^2, \tag*{(5)$_w$}$$

$$uvk_2(u,v) \ge 0 \quad \forall (u,v) \in \mathbb{R}^2, \tag*{(6)$_w$}$$

which are weaker than (5) and (6), respectively. One of the reasons for that is presented in the next remarks.

Remark 1.

(1) The most simple second-order differential operator which satisfies assumption (5) for $m = 1$ is linear in variable v; that is,

$$\left(r(t)\, k_1(x,x')\right)' = \left(A(x)\, x'\right)', \tag{7}$$

where $r(t) \equiv 1$, $k_1(u,v) = A(u)v$, and $A(u)$ is an arbitrary function satisfying $0 \le A(u) \le \alpha_1$. It is because $k_1^2(u,v) = A^2(u)v^2 \le \alpha_1 A(u)v^2 = \alpha_1 v k_1(u,v)$ for all $(u,v) \in \mathbb{R}^2$ and $\alpha_1 \ge 0$, see also Corollary 6. However, it is easy to check that the differential operator from (7) does not satisfy assumption (5) for every $m > 1$.

(2) Next, we consider the corresponding second-order quasilinear differential operator:

$$\left(r(t)\, k_1(x,x')\right)' = \left(A(x)\left|x'\right|^{\beta-1} x'\right)', \tag{8}$$

where $r(t) \equiv 1$, $k_1(u,v) := A(u)|v|^{\beta-1}v$, and $A(u)$ is an arbitrary function satisfying $0 \le A(u) \le \alpha_1$ and in order to ensure that $k_1 \in C^1(\mathbb{R}^2,\mathbb{R})$, we take $\beta \ge 1$ since $\partial k_1/\partial v = \beta|v|^{\beta-1}$. Unfortunately, the differential operator from (8) does not satisfy assumption (5) for every $m \in \mathbb{N}$, $\beta > 1$. It is because $k_1^2(u,v) = A^2(u)v^{2\beta-2}v^2 = A^2(u)v^{2\beta} \le \alpha_1 A(u)v^{2\beta} = \alpha_1 v k_1(u,v)|v|^{\beta-1}$, which is different from (5).

(3) Unlike (5), the differential operator from (8) satisfies assumption $(5)_w$, and hence, (8) is also included in our study of the oscillation of (1), see Corollary 11.

(4) Although both differential operators from (7) and (8) do not satisfy assumption (5) for every $m > 1$, the so-called generalized prescribed mean curvature-like differential operator:

$$\left(r(t)\, k_1(x,x')\right)' = \left(A(x)\frac{x'}{\left(1+x'^2\right)^{\alpha/2}}\right)' \tag{9}$$

satisfies assumption (5) for every $m \ge 1$, where $r(t) \equiv 1$, $k_1(u,v) := A(u)v/(1+v^2)^{\alpha/2}$, $\alpha \ge 1$, and $A(u)$ is an arbitrary function satisfying $0 \le A^{2m-1}(u) \le \alpha_1 u^{2m-2}$, see Corollary 9.

(5) The simple case $k_2(u,v) \equiv 0$ is involved in $(6)_w$ unlike (6), and hence, the nonlinear equation $x'' + q(t)f(x) = 0$ can be considered as a special case of (1).

We pay attention to the recently published paper [13] in which authors show that any generalization of the assumptions (2)–(6) should be done very carefully.

Now, we can recall [1, Theorems 2.5].

Theorem A. *Let (2)–(6) hold. Assume that there exist $\rho \in C^1([t_0,\infty),(0,\infty))$, $H \in \mathbb{H}$, $g \in C^1([t_0,\infty),\mathbb{R})$, and some $t_1 \ge t_0$ such that for all $T \ge t_1$:*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\,\gamma_1(s) - \frac{\alpha_1 \rho(s)\, v(s)\, r^2(s)}{4\left(\alpha_2 p(s) + r(s)\right)} Q_1^2(t,s) \right] ds = \infty, \tag{10}$$

where $v(t)$ and $\gamma_1(t)$ are defined, respectively, by

$$v(t) = \exp\left(-\frac{2}{\alpha_1}\int_{t_1}^t g(s)\,ds\right),$$
$$\gamma_1(t) = \rho(t)\, v(t)\left[\frac{1}{\alpha_1} r(t)\, g^2(t) + Kq(t) + \alpha_2 g^2(t)\, p(t) - \left(r(t)\, g(t)\right)'\right], \tag{11}$$

and $Q_1 \in C(D, \mathbb{R})$ *satisfies*

$$-\frac{\partial H(t,s)}{\partial s} = \left(\frac{\rho'(s)}{\rho(s)} + \frac{2\alpha_2 p(s) g(s)}{r(s)}\right) H(t,s) + Q_1(t,s)\sqrt{H(t,s)}. \tag{12}$$

Then, (1) *is oscillatory.*

In Theorem A, the set $D = \{(t,s) : t \geq s \geq t_0\}$. And the assumption $H \in \mathbb{H}$ means that $H \in C(D, \mathbb{R}_+)$, $\partial H(t,s)/\partial s$ is continuous on $D_0 = \{(t,s) : t > s \geq t_0\}$, $H(t,t) = 0$ for all $t \geq t_0$ and $H(t,s) > 0$ for all $(t,s) \in D_0$. It is easy to see that the coefficients: $r(t)$, $p(t)$, and $q(t)$ are involved in the assumptions (10)–(12), often called the general Kamenev-type conditions, about the Kamenev-type conditions and their several generalization we refer the reader, for instance, to [14–18]. The main purpose of supposing the existence of the functions: $\rho(t)$, $H(t,s)$, and $g(t)$ satisfying the corresponding assumptions (10)–(12) is to ensure the nonexistence of continuous function $w(t)$ which satisfies the corresponding Riccati differential inequality:

$$w' \geq \alpha(t)\omega^2 + \beta(t)\omega + \gamma(t), \quad t \geq T, \tag{13}$$

where $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ depend on $r(t)$, $p(t)$, α_1, and α_2, and $T \geq t_0$.

Instead of Kamenev-type conditions (10)–(12), we consider the next one (which can be called the Fite-Wintner-Leighton-type condition by a reason given in Remark 2): for the explicitly given two functions $a(t)$ and $b(t)$ which depend on the data $r(t)$, $p(t)$, m, n, α_1, α_2, and $q(t)$, (see Theorems 5 and 15), let there be a function $E(t)$ and a point $T_1 \geq t_0$ such that

$$E \in C([T_1, \infty)), \quad \limsup_{t\to\infty} \int_{T_1}^{t} E(\tau)\,d\tau = \infty,$$

$$\forall m, n \in \mathbb{N}, \text{we have}$$

$$E(t) \leq \begin{cases} \min\{a(t), b(t)\}, \ t \geq T_1, & \text{if } \min\{m,n\} = 1, \\ \min\{a(t), b(t) - a(t)\}, \ t \geq T_1, & \text{if } \min\{m,n\} > 1. \end{cases} \tag{14}$$

Combining a pointwise comparison principle and a blow-up argument, which is a different method than that in the case of Kamenev-type conditions, we are able to prove the nonexistence of any continuous function $\psi(t)$ which satisfies the corresponding Riccati differential inequality:

$$\psi' \geq a_1(t)\psi^{2m} + a_2(t)\psi^{2n} + b(t), \quad t \geq T, \tag{15}$$

where $a_1(t)$, $a_2(t)$, and $b(t)$ are arbitrary functions. On the various aspects of the comparison principles, we refer the reader, for instance, to [19, 20]—the comparison principles for Volterra integral operators, [21, 22]—the pointwise comparison principle for ODEs and [23]—the abstract form of comparison principles.

Remark 2. It is simple to check that in particular for $k_1(u,v) \equiv v$, $k_2(u,v) \equiv 0$, and $f(u) \equiv u$, the conditions (3), (5) with $m = 1$ and $(6)_w$ still hold where the inequality "≥" is replaced by "=." Then (1) becomes the linear second-order differential equation (LEq): $(r(t)x')' + q(t)x = 0$. Hence, the inequality in (14) for $m = n = 1$ can be replaced by the corresponding equality, where $a(t) = 1/r(t)$ and $b(t) = q(t)$ (see the case (iii) of Theorem 5), and so, we conclude that in this case, (14) is equivalent to:

$$\limsup_{t\to\infty} \int_{T_1}^{t} \frac{1}{r(\tau)}\,d\tau = \limsup_{t\to\infty} \int_{T_1}^{t} q(\tau)\,d\tau = \infty, \tag{16}$$

which presents the classic Fite-Wintner-Leighton oscillation criterion for linear second-order differential equation (LEq), where "lim" appears instead of "lim sup." In Fite [24], Wintner [25], and Leighton [26] equation (LEq) was considered, respectively, with $r(t) \equiv 1$ and $q(t) > 0$, $r(t) \equiv 1$ and $q(t)$ may change sign, and arbitrary $r(t) > 0$ and $q(t)$ may change sign. Nonlinear version of such a class of oscillation criteria was due to Wong [27], and Nth-order extension for linear equations can be found in Travis [28].

In order to simplify notation, we firstly introduce the following definition for the pointwise comparison principle of the corresponding Riccati differential equation:

$$w' = a_1(t)w^{2m} + a_2(t)w^{2n} + b(t), \quad t \geq T, \tag{17}$$

where $a_1(t)$, $a_2(t)$, and $b(t)$ are three arbitrary functions, and $T \geq t_0$.

Definition 3. Let T_0 and T^* be two arbitrary real numbers, $T \leq T_0 < T^*$. Two functions, $\varphi(t)$ and $\psi(t), \varphi, \psi \in C^1((T_0, T^*), \mathbb{R}) \cap C([T_0, T^*), \mathbb{R})$, are said to be, respectively, subsolution and supersolution of the Riccati differential equation (17) provided that

$$\begin{aligned} \varphi' &\leq a_1(t)\varphi^{2m} + a_2(t)\varphi^{2n} + b(t), \\ \psi' &\geq a_1(t)\psi^{2m} + a_2(t)\psi^{2n} + b(t), \quad t \in (T_0, T^*). \end{aligned} \tag{18}$$

Moreover, if the statement:

$$\varphi(T_0) \leq \psi(T_0) \ \text{ implying } \varphi(t) \leq \psi(t) \quad \forall t \in [T_0, T^*) \tag{19}$$

is fulfilled for all sub- and supersolutions $\varphi, \psi \in C^1((T_0, T^*), \mathbb{R}) \cap C([T_0, T^*), \mathbb{R})$ of (17), then we say that comparison principle (19) holds for (17) with arbitrary T_0 and T^*, $T \leq T_0 < T^*$.

Remark 4. The possibilty that (19) holds for all sub- and supersolutions and with arbitrary T_0 and T^*, $T \leq T_0 < T^*$ plays an essential role in some concrete situations. According to it, when the comparison principle (19) holds for the Riccati differential equation (17) with arbitrary T_0 and T^*, $T \leq T_0 < T^*$, then we can choose some concrete sub- and supersolutions as well as T_0 and T^* with some suitable properties.

Our method contains the next three steps:

(i) at the first step, we give a sufficient condition on $a_1(t), a_2(t)$ such that comparison principle (19) holds for the Riccati differential equation (17) with arbitrary $b(t)$, T_0 and T^*, $T \le T_0 < T^*$;

(ii) at the second step, for a supersolution $\psi \in C^1((T,\infty),\mathbb{R}) \cap C([T,\infty),\mathbb{R})$ of (17), where $a_1(t)$, $a_2(t)$, and $b(t)$ are three arbitrary functions, and under assumption (14), we find two real numbers T_0 and T^*, $T \le T_0 < T^*$, and construct a subsolution $\varphi(t)$, $\varphi \in C^1((T_0,T^*),\mathbb{R}) \cap C([T_0,T^*),\mathbb{R})$ of (17) such that the following initial and blow-up arguments are satisfied:

$$\varphi(T_0) \le \psi(T_0), \qquad \lim_{t\to T^*} \varphi(t) = \infty; \tag{20}$$

(iii) at the third step, under conditions (2)–(6) or related ones such as $(5)_w$ and $(6)_w$, we show that if the main equation (1) allows a nonoscillatory solution $x(t)$, then the function:

$$\psi(t) = -\frac{r(t)\, k_1(x(t), x'(t))}{x(t)}, \qquad t \ge T, \tag{21}$$

is well defined for some $T \ge t_0$, $\psi \in C^1((T,\infty),\mathbb{R}) \cap C([T,\infty),\mathbb{R})$, and $\psi(t)$ is a supersolution of (17) with some concrete $a_1(t)$, $a_2(t)$, and $b(t)$; in the case when $p(t)$ and $q(t)$ change the sign, instead of (21), we consider the function:

$$\psi(t) = -\frac{r(t)\, k_1(x(t), x'(t))}{f(x(t))}, \qquad t \ge T. \tag{22}$$

In conclusion, combining (19) and (20), we obtain the nonexistence of any continuous supersolution of the Riccati differential equation (17), and hence, the function $\psi(t)$ given by (21) or (22) is not possible. Therefore, (1) does not allow any nonoscillatory solution.

3. Main Results and Examples

As usual, we recognize two main different cases: the first one is when $p(t)$ and $q(t)$ are positive and the second one is when they may change the sign. Moreover, in the first case, depending on the combination of assumptions (5), (6), $(5)_w$, $(6)_w$, and $(6)_1$, we consider five subcases such as is done in our first oscillation criterion for (1).

Theorem 5 (positive coefficients). *Let assumptions* (2)–(4) *be fulfilled. Then,* (1) *is oscillatory if one of the next five cases is met.*

(i) *Let $m, n \in \mathbb{N}$ and* (5), (6) *hold. One supposes* (14) *with respect to $a(t) := a_1(t) + a_2(t)$ provided that $m = n = 1$ or $a(t) := \min\{a_1(t), a_2(t)\}$, otherwise,*

$$a_1(t) = \frac{1}{\alpha_1 r^{2m-1}(t)}, \qquad a_2(t) = \frac{p(t)\,\alpha_2}{r^{2n}(t)}, \qquad b(t) = Kq(t). \tag{23}$$

(ii) *Let $n \in \mathbb{N}$ and $(5)_w$,* (6) *hold. One supposes* (14) *with respect to $a(t)$ and $b(t)$ given by*

$$a(t) = \frac{p(t)\,\alpha_2}{r^{2n}(t)}, \qquad b(t) = Kq(t). \tag{24}$$

(iii) *Let $m \in \mathbb{N}$ and* (5), $(6)_w$ *hold. One supposes* (14) *with respect to $a(t)$ and $b(t)$ given by*

$$a(t) = \frac{1}{\alpha_1 r^{2m-1}(t)}, \qquad b(t) = Kq(t). \tag{25}$$

(iv) *Let $m = 1$ and* (5), $(6)_1$ *hold. One supposes* (14) *with respect to $a(t)$ and $b(t)$ given by*

$$a(t) = \frac{1}{\alpha_1 r(t)} e^{-\alpha_2 \int (p(\tau)/r(\tau))d\tau}, \qquad b(t) = Kq(t)\, e^{\alpha_2 \int (p(\tau)/r(\tau))d\tau}. \tag{26}$$

(v) *Let $p \in C^1((t_0,\infty),\mathbb{R})$, $m = 1$, and* (5), $(6)_1$ *hold. One supposes* (14) *with respect to $a(t)$ and $b(t)$ given by*

$$a(t) = \frac{1}{\alpha_1 r(t)}, \qquad b(t) = Kq(t) - \frac{\alpha_1\alpha_2 p'(t)}{2} - \frac{\alpha_1\alpha_2^2 p^2(t)}{4r(t)}. \tag{27}$$

For each of the cases (i)–(v) of Theorem 5, we derive some consequences and examples, which show the importance of our oscillation criterion.

The case (i) of Theorem 5 for $m = n = 1$ allows us to consider the following class of equations:

$$\left(r(t)\, A(x)\, x'\right)' + p(t)\, B(x)\, x'^2 + q(t)\, f(x) = 0, \qquad t \ge t_0 > 0, \tag{28}$$

where the functions $A = A(u)$, $A \in C^1(\mathbb{R})$, and $B(u)$ satisfy

$$0 \le A(u) \le \alpha_1, \quad uB(u) \ge \alpha_2 A^2(u) \quad \forall (u,v) \in \mathbb{R}^2, \tag{29}$$

for some $\alpha_1 > 0$ and $\alpha_2 > 0$. Under assumption (29), it is easy to see that the functions $k_1(u,v) := A(u)v$ and $k_2(u,v) := B(u)v$ satisfy both required assumptions (5) and (6) with $m = n = 1$. Hence, as an easy consequence of Theorem 5, we obtain the following result.

Corollary 6. *Let* (2)–(4) *and* (14) *hold with respect to $a(t)$ and $b(t)$ given in case (i) of Theorem 5 with $m = n = 1$. If $A(u)$ and $B(u)$ satisfy* (29), *then* (28) *is oscillatory.*

Example 7. Let $K > 0$, $\mu \le 1$ or $\nu \ge 2\mu - 1$, and $\sigma \le 1$. Then, the equation:

$$\left(t^\mu \frac{x^2}{1+x^2}\, x'\right)' + t^\nu x^3 x'^2 + Kt^{-\sigma} x = 0, \quad t \ge t_0 > 0 \tag{30}$$

is oscillatory. Indeed, it is enough to check that the coefficients $r(t) = t^\mu$, $p(t) = t^\nu$, and $q(t) = t^{-\sigma}$ and the functions $f(u) = Ku$, $A(u) = u^2/(1+u^2)$, and $B(u) = u^3$ satisfy all the assumptions of Corollary 6 with respect to $\alpha_1 = \alpha_2 = 1$ and $E(t) = c/t$ for some $c > 0$.

Example 8. Let $K > 0$, $\mu \le 1$ or $\nu \ge 2\mu - 1$, and $\sigma \le 1$. Then, the equation:

$$\left(t^{\mu}(\sin x)^2 x'\right)' + t^{\nu} x^3 x'^2 + Kt^{-\sigma} x = 0, \qquad t \ge t_0 > 0 \tag{31}$$

is oscillatory. In fact, it is easy to check that the coefficients $r(t) = t^{\mu}$, $p(t) = t^{\nu}$, and $q(t) = t^{-\sigma}$ and the functions $f(u) = Ku$, $A(u) = \sin^2 u$, and $B(u) = u^3$ satisfy all the assumptions of Corollary 6 with respect to $\alpha_1 = \alpha_2 = 1$ and $E(t) = c/t$ for some $c > 0$.

The case (i) of Theorem 5 for $m, n \in \mathbb{N}$ proposes the following class of differential equations:

$$\left(r(t) A(x) \frac{x'}{\left(1 + x'^2\right)^{\alpha/2}}\right)' + p(t) B(x) \left(\frac{x'}{\left(1 + x'^2\right)^{\alpha/2}}\right)^{2n} + q(t) f(x) = 0, \quad t \ge t_0 > 0, \tag{32}$$

where $\alpha \ge 1$, $n \in \mathbb{N}$, and the functions $A = A(u)$, $A \in C^1(\mathbb{R})$, and $B(u)$ satisfy

$$0 \le A^{2m-1}(u) \le \alpha_1 u^{2m-2}, \quad u^{2n-1} B(u) \ge \alpha_2 A^{2n}(u), \quad u \in \mathbb{R}. \tag{33}$$

As a consequence of Theorem 5, we derive the next interesting corollary.

Corollary 9. *Let* (2)–(4) *and* (14) *hold with respect to $a(t)$ and $b(t)$ given in case (i) of Theorem 5. If $A(u)$ and $B(u)$ satisfy* (33), *then* (32) *is oscillatory.*

Example 10. Let $\alpha \ge 1$, $n \in \mathbb{N}$, $K > 0$, $\mu \le 1$ or $\nu \ge 2\mu - 1$, and $\sigma \le 1$. Then, according to Corollary 9, we conclude that the equation:

$$\left(t^{\mu} \frac{x^2}{1 + x^2} \frac{x'}{(1 + x'^2)^{\alpha/2}}\right)' + t^{\nu} x \left(\frac{x x'}{(1 + x^2)\left(1 + x'^2\right)^{\alpha/2}}\right)^{2n} + Kt^{-\sigma} x = 0, \quad t \ge t_0 > 0 \tag{34}$$

is oscillatory.

We have pointed out in Remark 1 that assumption $(5)_w$ unlike (5) allows to consider the oscillation of the following quasilinear differential equation:

$$\left(r(t) A(x) \left|x'\right|^{\beta-1} x'\right)' + p(t) B(x) x'^{2\beta} + q(t) f(x) = 0, \qquad t \ge t_0 > 0, \tag{35}$$

where $\beta \ge 1$ and the functions $A = A(u)$, $A \in C^1(\mathbb{R})$, and $B(u)$ satisfy

$$0 \le A(u), \quad uB(u) \ge \alpha_2 A^2(u) \quad \forall (u, v) \in \mathbb{R}^2, \tag{36}$$

for some $\alpha_2 > 0$. It is clear that (28) is a particular case of (35) for $\beta = 1$. Under assumption (36), the functions

$$k_1(u, v) := A(u) |v|^{\beta-1} v, \qquad k_2(u, v) := B(u) v^{2\beta-1} \tag{37}$$

satisfy both required assumptions $(5)_w$ and (6) with $n = 1$. Therefore, we can derive the following easy consequence of the case (ii) of Theorem 5.

Corollary 11. *Let* (2)–(4) *and* (14) *hold with respect to $a(t)$ and $b(t)$ given in case (ii) of Theorem 5 with $n = 1$. If $A(u)$ and $B(u)$ satisfy* (36), *then* (35) *is oscillatory.*

Example 12. Let $\beta \ge 1$, $K > 0$, $\nu \ge 2\mu - 1$ and $\sigma \le 1$. Then the equation:

$$\left(t^{\mu}(\sin x)^2 x'^{\beta}\right)' + t^{\nu} x^3 x'^{2\beta} + Kt^{-\sigma} x = 0, \quad t \ge t_0 > 0, \tag{38}$$

is oscillatory. In fact, it is enough to check that the coefficients $r(t) = t^{\mu}$, $p(t) = t^{\nu}$, $q(t) = t^{-\sigma}$ and the functions $f(u) = Ku$, $A(u) = \sin^2 u$ and $B(u) = u^3$ satisfy all assumptions of Corollary 11 with respect to $E(t) = c/t$ for some $c > 0$.

The case (iii) of Theorem 5 allows us to consider the following class of equations:

$$\left(r(t) A(x) x'\right)' + p(t) B(x) C\left(x'\right) x' + q(t) f(x) = 0, \quad t \ge t_0 > 0, \tag{39}$$

where the functions $A = A(u)$, $A \in C^1(\mathbb{R})$, $B(u)$ and $C(v)$ satisfy:

$$0 \le A(u) \le \alpha_1, \quad uB(u) \ge 0, \quad vC(v) \ge 0 \quad \forall (u, v) \in \mathbb{R}^2, \tag{40}$$

for some $\alpha_1 > 0$. Under this assumption, it is easy to see that the functions $k_1(u, v) := A(u)v$ and $k_2(u, v) := B(u)C(v)$ satisfy both required assumptions (5) and $(6)_w$. Hence, as an easy consequence of case (iii) of Theorem 5, we obtain the following result.

Corollary 13. *Let* (2), (3), *and* (14) *hold with respect to $a(t)$ given in case (iii) of Theorem 5. If $A(u)$, $B(u)$, and $C(v)$ satisfy* (40), *then* (39) *is oscillatory.*

Example 14. Let $K > 0$, $\mu \le 1$, $\nu \ge 0$, $\lambda \ge 0$, and $\sigma \le 1$. Then, the equation:

$$\left(t^{\mu} \frac{x^2}{1 + x^2} x'\right)' + t^{\nu} |x|^{\lambda} x \,\mathrm{sh}\left(x'\right) x' + Kt^{-\sigma} x = 0, \quad t \ge t_0 > 0 \tag{41}$$

is oscillatory. In order to show that, it is enough to check that the coefficients: $r(t) = t^{\mu}$, $q(t) = t^{-\sigma}$ and the functions: $f(u) = Ku$, $A(u) = u^2/(1 + u^2)$, $B(u) = |u|^{\lambda} u$, and $C(v) = \mathrm{sh}(v)$ satisfy all the assumptions of Corollary 13 with respect to $\alpha_1 = \alpha_2 = 1$ and $E(t) = c/t$ for some $c > 0$.

Next, we consider the oscillation of (1) in the case when the coefficients $p(t)$ and $q(t)$ may change the sign.

Theorem 15 (coefficients may change the sign). *Let $m = 1$ and assumptions* $(3)_1$, (5), *and* $(6)_2$ *hold. Then,* (1) *is oscillatory provided that one of the following two cases is met. (vi) One assumes* (14) *with respect to $a(t)$ and $b(t)$ given by*

$$a(t) = \frac{K}{\alpha_1 r(t)} e^{-\alpha_2 \int_T^t (p(\tau)/r(\tau))d\tau}, \qquad b(t) = q(t)\, e^{\alpha_2 \int_T^t (p(\tau)/r(\tau))d\tau}, \quad t \geq T. \tag{42}$$

(vii) Let $p \in C^1((t_0, \infty), \mathbb{R})$. One assumes (14) *with respect to $a(t)$ and $b(t)$ given by*

$$a(t) = \frac{K}{\alpha_1 r(t)}, \quad b(t) = q(t) - \frac{\alpha_1 \alpha_2 p'(t)}{2K} - \frac{\alpha_1 \alpha_2^2 p^2(t)}{4Kr(t)}, \quad t \geq T. \tag{43}$$

The case (vi) of Theorem 15 allows us to consider the following class of equations:

$$\left(r(t) A(x) x'\right)' + \alpha_2 p(t) A(x) x' + q(t) f(x) = 0, \quad t \geq t_0 > 0, \tag{44}$$

where $\alpha_2 \in \mathbb{R}$, and the function $A(u)$ satisfies

$$0 \leq A(u) \leq \alpha_1 \quad \forall (u, v) \in \mathbb{R}^2 \text{ and some } \alpha_1 > 0. \tag{45}$$

Under (45), one can easily check that the functions $k_1(u, v) := A(u)v$ and $k_2(u, v) := \alpha_2 A(u)$ satisfy both required assumptions (5) and $(6)_2$. Hence, as an easy consequence of case (vi) of Theorem 5, we conclude the next consequence.

Corollary 16. *Let* $(3)_1$ *and* (14) *hold with respect to $a(t)$ and $b(t)$ given in case (vi) of Theorem 5. If $A(u)$ satisfies* (45), *then* (44) *is oscillatory.*

Example 17. Let $\mu \geq 2$ and $q_0 \in \mathbb{R}$. Then, the equations:

$$\begin{aligned} &\left(t^{-\mu} \frac{x^2}{1 + x^2} x'\right)' + t^{-\mu-1} \frac{x^2}{1 + x^2} x' + q_0 (\sin t)\, x = 0, \quad t \geq t_0 > 0, \\ &\left(t^{-\mu} (\sin x)^2 x'\right)' + t^{-\mu-1} (\sin x)^2 x' + q_0 (\sin t)\, x = 0, \quad t \geq t_0 > 0 \end{aligned} \tag{46}$$

are oscillatory. In order to show that, it is enough to check that the coefficients: $r(t) = t^{-\mu}$, $p(t) = t^{-\mu-1}$, and $q(t) = q_0 \sin t$ and the functions: $f(u) = u$, $A(u) = u^2/(1 + u^2)$, and $A(u) = \sin^2 u$ satisfy all the assumptions of Corollary 16 with respect to $\alpha_1 = \alpha_2 = 1$ and $E(t) = ct \sin t$ for some $c \in \mathbb{R}$, $c \neq 0$.

4. Proofs of the Main Results

In this section, we study the oscillation of (1) in the view of a pointwise comparison principle presented below, which will be shown for the corresponding Riccati differential equation.

Definition 18. A function $h(t, u)$ is said to be locally Lipschitz in the second variable if for any bounded interval $I_0 \subseteq [T, \infty)$ and $M > 0$ there is a constant $L > 0$ depending on I_0, M, h such that

$$\left|h(t, u_1) - h(t, u_2)\right| \leq L \left|u_1 - u_2\right| \quad \forall t \in I_0, \quad u_1, u_2 \in [-M, M]. \tag{47}$$

Now, we state and use the following general comparison principle, which will be proved at the end of this section.

Lemma 19. *Let T_0 and T^* be two arbitrary real numbers such that $T_0 < T^*$. Let $\tilde{\varphi}(t)$ and $\tilde{\psi}(t)$, $\tilde{\varphi}, \tilde{\psi} \in C^1((T_0, T^*), \mathbb{R}) \cap C([T_0, T^*), \mathbb{R})$, be two functions satisfying:*

$$\tilde{\varphi}' \leq h(t, \tilde{\varphi}), \quad \tilde{\psi}' \geq h(t, \tilde{\psi}), \quad t \in (T_0, T^*), \tag{48}$$

where $h(t, u)$ is a locally Lipschitz function in the second variable. Then, we have

$$\tilde{\varphi}(T_0) \leq \tilde{\psi}(T_0) \text{ implies } \tilde{\varphi}(t) \leq \tilde{\psi}(t) \quad \forall t \in [T_0, T^*). \tag{49}$$

Definition 20. A function $a(t)$ is said to be locally bounded on $[T, \infty)$, if for any bounded interval $I_0 \subseteq [T, \infty)$ there is a constant $C > 0$ depending on I_0 such that $|a(t)| \leq C$ for all $t \in I_0$.

According to Lemma 19, we are able to give a sufficient condition on the functions: $a_1(t), a_2(t)$ such that the Riccati differential equation (17) satisfies the comparison principle (19).

Lemma 21. *If $a_1(t)$ and $a_2(t)$ are two locally bounded functions on $[T, \infty)$, then comparison principle* (19) *holds for the Riccati differential equation* (17) *with arbitrary $b(t)$, T_0, and T^*, where $T \leq T_0 < T^*$.*

Proof. Let $\varphi(t)$ and $\psi(t), \varphi, \psi \in C^1((T_0, T^*), \mathbb{R}) \cap C([T_0, T^*), \mathbb{R})$, be, respectively, sub- and supersolution of (17); that is, they satisfy (18). It is not difficult to check that $h(t, u) := a_1(t) u^{2m} + a_2(t) u^{2n} + b(t)$ is a locally Lipschitz function in the second variable. Indeed, for any bounded interval $I_0 \subseteq [t_0, \infty)$, $M > 0$, for all $t \in I_0$ and $u_1, u_2 \in [-M, M]$, we have

$$\begin{aligned} &\left|h(t, u_1) - h(t, u_2)\right| \\ &\quad \leq \left|a_1(t)\right| \left|u_1^{2m} - u_2^{2m}\right| + \left|a_2(t)\right| \left|u_1^{2n} - u_2^{2n}\right| \\ &\quad = \left|a_1(t)\right| \left|u_1 - u_2\right| \left|\sum_{j=1}^{2m} u_1^{2m-j} u_2^{j-1}\right| \\ &\qquad + \left|a_2(t)\right| \left|u_1 - u_2\right| \left|\sum_{j=1}^{2n} u_1^{2n-j} u_2^{j-1}\right| \\ &\quad \leq 2mM^{2m-1} \left|a_1(t)\right| \left|u_1 - u_2\right| + 2nM^{2n-1} \left|a_2(t)\right| \left|u_1 - u_2\right| \\ &\quad \leq 2C \left(mM^{2m-1} + nM^{2n-1}\right) \left|u_1 - u_2\right|, \end{aligned} \tag{50}$$

where $C = \max\{\sup_{I_0}|a_1(t)|, \sup_{I_0}|a_2(t)|\}$. Hence, Lemma 19 can be applied to $\varphi(t)$ and $\psi(t)$. If we set $\tilde{\varphi}(t) := \varphi(t)$, and $\tilde{\psi}(t) := \psi(t)$, then statement (48) is fulfilled because of assumption (18), and therefore, the desired conclusion (19) immediately follows from (49). □

Corollary 22. *If $a_1(t)$ and $a_2(t)$ are two continuous functions on $[T,\infty)$, then comparison principle* (19) *holds for the Riccati differential equation* (17) *with arbitrary $b(t)$, T_0, and T^*, where $T \le T_0 < T^*$.*

Proof. Since $a_1(t)$ and $a_2(t)$ are two continuous functions on $[T,\infty)$, they are also locally bounded functions on $[t_0,\infty)$, and hence, this corollary immediately follows from Lemma 21. □

Next, we present an essential lemma in which we construct a subsolution $\varphi(t)$ of (17) which has a blow-up desired property.

Lemma 23. *Let $a_1(t) \ge 0$, $a_2(t) \ge 0$, and $b(t)$ be three arbitrary functions, and let assumption* (14) *hold, where $a(t) = a_1(t) + a_2(t)$ if $m = n = 1$ and $a(t) = \min\{a_1(t), a_2(t)\}$ otherwise. Let $\psi \in C^1((T,\infty),\mathbb{R}) \cap C([T,\infty),\mathbb{R})$ be a supersolution of the Riccati differential equation* (17). *Then, there are two real numbers T_0 and T^*, $T \le T_0 < T^*$, and a subsolution $\varphi \in C^1((T_0,T^*),\mathbb{R}) \cap C([T_0,T^*),\mathbb{R})$ of* (17) *satisfying*

$$\varphi(T_0) \le \psi(T_0), \qquad \lim_{t\to T^*}\varphi(t) = \infty. \tag{51}$$

Proof. In particular from (14), we obtain a sequence $t_n \to \infty$ as $n \to \infty$ such that

$$\int_{T_1}^{t_n} E(\tau)\,d\tau \longrightarrow \infty \text{ as } n \longrightarrow \infty, \tag{52}$$

where T_1 (determined in (14)) can be chosen so that $T_1 \ge T$. From the previous statement, we conclude that there is a $T_2 > T_1$ such that

$$\int_{T_1}^{T_2} E(\tau)\,d\tau = \pi. \tag{53}$$

Since $\int_{T_1}^{t} E(\tau)d\tau$ is a continuous function in the variable t, there is a $T_0 \in [T_1, T_2)$ such that

$$\int_{T_1}^{T_0} E(\tau)\,d\tau = 0, \qquad \int_{T_1}^{t} E(\tau)\,d\tau \ge 0 \quad \forall t \in [T_0,T_2). \tag{54}$$

Consequently, we derive that

$$\begin{aligned}\int_{T_1}^{t} E(\tau)\,d\tau &= \int_{T_1}^{T_0} E(\tau)\,d\tau + \int_{T_0}^{t} E(\tau)\,d\tau \\ &= \int_{T_0}^{t} E(\tau)\,d\tau, \quad t \in [T_0,T_2),\end{aligned} \tag{55}$$

which together with (53) and (54) shows

$$\int_{T_0}^{t} E(\tau)\,d\tau \ge 0 \quad \forall t \in [T_0,T_2), \qquad \int_{T_0}^{T_2} E(\tau)\,d\tau = \pi. \tag{56}$$

Next, let $s_0 \in (-\pi/2, \pi/2)$ be such that $\tan(s_0) = \psi(T_0)$, where T_0 is from (54)-(56). Such s_0 exists since the tangent function is a bijection from $(-\pi/2,\pi/2)$ to $\mathbb{R}$. Let

$$V(t) := \int_{T_0}^{t} E(\tau)\,d\tau, \quad t \in [T_0,T_2]. \tag{57}$$

Because of (56), we have $s_0 + V(t) > -\pi/2$, $t \in [T_0,T_2)$, $s_0 + V(T_0) < \pi/2$ and $s_0 + V(T_2) > \pi/2$. Since $V(t)$ is a continuous function, it implies the existence of a $T^* \in (T_0,T_2)$ such that

$$s_0 + V(T^*) = \frac{\pi}{2}, \quad -\frac{\pi}{2} < s_0 + V(t) < \frac{\pi}{2} \quad \forall t \in [T_0,T^*). \tag{58}$$

As a consequence, the function

$$\varphi(t) = \tan(s_0 + V(t)), \quad t \in [T_0,T^*) \tag{59}$$

is well defined and obviously satisfies

$$\begin{gathered}\varphi(T_0) = \tan(s_0 + V(T_0)) = \tan(s_0) = \psi(T_0), \\ \lim_{t\to T^*}\varphi(t) = \tan(s_0 + V(T^*)) = \tan\left(\frac{\pi}{2}\right) = \infty.\end{gathered} \tag{60}$$

Also, since $E(t)$ is continuous on $[t_0,\infty)$, we have $\varphi \in C^1((T_0,T^*),\mathbb{R}) \cap C([T_0,T^*),\mathbb{R})$. Now, by taking the derivative of $\varphi(t)$ for every $t \in [T_0,T^*)$, we obtain

$$\begin{aligned}\varphi' &= \frac{1}{\cos^2\left(s_0 + \int_{T_0}^{t} E(\tau)\,d\tau\right)} \cdot E(t) \\ &= E(t)\left(1 + \tan^2\left(s_0 + \int_{T_0}^{t} E(\tau)\,d\tau\right)\right) \\ &= E(t)\varphi^2 + E(t).\end{aligned} \tag{61}$$

According to (14), we observe that

(1) if $m = n = 1$, then

$$\begin{aligned}E(t)\varphi^2 + E(t) &\le (a_1(t) + a_2(t))\varphi^2 + b(t) \\ &= a_1(t)\varphi^{2m} + a_2(t)\varphi^{2n} + b(t);\end{aligned} \tag{62}$$

(2) if $1 = \min\{m,n\} < \max\{m,n\}$, then

$$\begin{aligned}E(t)\varphi^2 + E(t) &\le a(t)\varphi^2 + b(t) \le a(t)\left(\varphi^{2m} + \varphi^{2n}\right) + b(t) \\ &\le a_1(t)\varphi^{2m} + a_2(t)\varphi^{2n} + b(t);\end{aligned} \tag{63}$$

(3) if $\min\{m,n\} > 1$, then

$$\begin{aligned}E(t)\varphi^2 + E(t) &\le a(t)\varphi^2 + b(t) - a(t) \\ &\le a(t)\left(\varphi^{2m} + \varphi^{2n} + 1\right) + b(t) - a(t) \\ &\le a_1(t)\varphi^{2m} + a_2(t)\varphi^{2n} + b(t).\end{aligned} \tag{64}$$

Thus, in all three cases of $m, n \in \mathbb{N}$, we have

$$E(t)\varphi^2 + E(t) \le a_1(t)\varphi^{2m} + a_2(t)\varphi^{2n} + b(t), \quad t \in [T_0, T^*). \tag{65}$$

Putting the previous inequality into (61) and taking into account of (60), we conclude that

$$\varphi(T_0) \le \psi(T_0), \quad \varphi' \le a_1(t)\varphi^{2m} + a_2(t)\varphi^{2n} + b(t), \quad t \in [T_0, T^*), \quad \lim_{t \to T^*} \varphi(t) = \infty. \tag{66}$$

It proves that $\varphi(t)$ is a subsolution of the Riccati differential equation (17) which satisfies the statement (51). □

Next, we are concerned with the following technical but crucial lemma.

Lemma 24. *Let the assumptions of Theorem 5 in the cases (i)–(iii) hold. If the main equation* (1) *allows a nonoscillatory solution $x(t)$, then the function $\psi(t)$ given by* (21) *is well-defined with respect to such an $x(t)$ and some $T \ge t_0$, $\psi \in C^1((T,\infty),\mathbb{R}) \cap C([T,\infty),\mathbb{R})$, and $\psi(t)$ is a supersolution of the Riccati differential equation* (17).

Proof. If the main equation (1) allows a nonoscillatory solution $x(t)$, then there is a $T \ge t_0$ such that $x(t) \ne 0$ for all $t \ge T$. Hence, the function $\psi(t)$ given by (21) is well defined for such an $x(t)$. Next, making the derivative of $\psi(t)$, using that $x(t)$ satisfies (1) and taking common assumptions of Theorem 5 for the functions $p(t)$, $r(t)$, $q(t)$, $k_1(u,v)$, and $k_2(u,v)$, we obtain

$$\begin{aligned}\psi'(t) &= -\frac{1}{x(t)}\left(r(t)k_1\left(x(t),x'(t)\right)\right)' + \frac{1}{x^2(t)} r(t)k_1\left(x(t),x'(t)\right)x'(t) \\ &= \frac{p(t)}{x(t)} k_2\left(x(t),x'(t)\right)x'(t) + \frac{r(t)}{x^2(t)} k_1\left(x(t),x'(t)\right)x'(t) + q(t)\frac{f(x(t))}{x(t)} \\ &= \frac{p(t)}{x^{2n}(t)}\left[k_2\left(x(t),x'(t)\right)x^{2n-1}(t)x'(t)\right] + \frac{r(t)}{x^{2m}(t)}\left[k_1\left(x(t),x'(t)\right)x^{2m-2}(t)x'(t)\right] + q(t)\frac{f(x(t))}{x(t)}.\end{aligned} \tag{67}$$

Depending on each of the three cases (i)–(iii) of Theorem 5, from the previous equality, we obtain

$$\psi'(t) \ge \begin{cases} \dfrac{\alpha_2 p(t)}{x^{2n}(t)} k_1^{2n}\left(x(t),x'(t)\right) + \dfrac{r(t)}{\alpha_1 x^{2m}(t)} k_1^{2m}\left(x(t),x'(t)\right) + Kq(t), & \text{in (i)—Theorem 5,} \\ \dfrac{\alpha_2 p(t)}{x^{2n}(t)} k_1^{2n}\left(x(t),x'(t)\right) + Kq(t), & \text{in (ii)—Theorem 5,} \\ \dfrac{r(t)}{\alpha_1 x^{2m}(t)} k_1^{2m}\left(x(t),x'(t)\right) + Kq(t), & \text{in (ii)—Theorem 5,} \end{cases} \tag{68}$$

Next from (21), we also have

$$k_1\left(x(t),x'(t)\right) = -\frac{x(t)}{r(t)}\psi(t). \tag{69}$$

Now, from (68) and (69), we immediately obtain: $\psi' \ge a_1(t)\psi^{2m} + a_2(t)\psi^{2n} + b(t)$, $t \ge T$. According to the definition of a supersolution, the previous inequality shows this lemma. □

Lemma 25. *Let the assumptions of Theorem 5 in the cases (iv)-(v) hold. If the main equation* (1) *allows a nonoscillatory solution $x(t)$, then the function $\psi(t)$ given by*

$$\psi(t) = \begin{cases} -\dfrac{r(t)k_1\left(x(t),x'(t)\right)}{x(t)} \times e^{\alpha_2\int(p(\tau)/r(\tau))d\tau}, \quad t \ge T, & \text{in the case (iv),} \\ -\dfrac{r(t)k_1\left(x(t),x'(t)\right)}{x(t)} - \dfrac{\alpha_1\alpha_2 p(t)}{2}, \quad t \ge T, & \text{in the case (v)} \end{cases} \tag{70}$$

is well defined with respect to such an $x(t)$ and some $T \ge t_0$ such that $\psi \in C^1((T,\infty),\mathbb{R}) \cap C([T,\infty),\mathbb{R})$, and $\psi(t)$ is a supersolution of the Riccati differential equation (17).

The proof of Lemma 25 is omitted because it is very similar to the proof of the following lemma.

Lemma 26. *Let assumptions of Theorem 15 hold. If the main equation* (1) *allows a nonoscillatory solution $x(t)$, then the function $\psi(t)$ given by*

$$\psi(t) = \begin{cases} -\dfrac{r(t)k_1\left(x(t),x'(t)\right)}{f(x(t))} \times e^{\alpha_2\int(p(\tau)/r(\tau))d\tau}, \quad t \ge T, & \text{in the case (vi),} \\ -\dfrac{r(t)k_1\left(x(t),x'(t)\right)}{f(x(t))} - \dfrac{\alpha_1\alpha_2 p(t)}{2K}, \quad t \ge T, & \text{in the case (vii)} \end{cases} \tag{71}$$

is well defined with respect to such an $x(t)$ and some $T \ge t_0$, $\psi \in C^1((T,\infty),\mathbb{R}) \cap C([T,\infty),\mathbb{R})$, and $\psi(t)$ is a supersolution of the Riccati differential equation (17), *where $a(t) = a_1(t) + a_2(t)$ and $a(t)$, $b(t)$ are given in the case (vi) of Theorem 5.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1), and thus, we can take a $T \geq t_0$ such that $x(t) \neq 0$ on $[T, \infty)$. Let $\psi_0(t)$ be a function defined by

$$\psi_0(t) = -\frac{r(t) k_1(x(t), x'(t))}{f(x(t))}, \quad t \geq T. \tag{72}$$

From the assumptions of Theorem 15 and from equalities (1) and (72), we can easily make the following computation:

$$\begin{aligned}
\psi_0'(t) = & -\frac{1}{f(x(t))}\left(r(t) k_1(x(t), x'(t))\right)' \\
& + \frac{1}{f^2(x(t))} r(t) k_1(x(t), x'(t)) x'(t) f'(x(t)) \\
= & \frac{p(t)}{f(x(t))}\left[k_2(x(t), x'(t)) x'(t)\right] \\
& + \frac{r(t)}{f^2(x(t))}\left[k_1(x(t), x'(t)) x'(t)\right] f'(x(t)) + q(t) \\
\geq & \alpha_2 p(t)\left[\frac{k_1(x(t), x'(t))}{f(x(t))}\right] \\
& + \frac{K r(t)}{\alpha_1}\left[\frac{k_1(x(t), x'(t))}{f(x(t))}\right]^2 + q(t) \\
= & -\frac{\alpha_2 p(t)}{r(t)} \psi_0(t) + \frac{K}{\alpha_1 r(t)} \psi_0^2(t) + q(t);
\end{aligned} \tag{73}$$

that is,

$$\psi_0'(t) \geq \frac{K}{\alpha_1 r(t)} \psi_0^2(t) - \frac{\alpha_2 p(t)}{r(t)} \psi_0(t) + q(t), \quad t > T. \tag{74}$$

Now, if the middle term on the right-hand side of (74) is moved into the left-hand side, and multiplying such equality by $e^{\alpha_2 \int (p(\tau)/r(\tau)) d\tau}$, we conclude that the function

$$\psi(t) = \psi_0(t) e^{\alpha_2 \int p(\tau)/r(\tau) d\tau}, \quad t \geq T \tag{75}$$

satisfies the Riccati differential equation (17) with respect to $a(t)$ and $b(t)$ given in the case (vi) of Theorem 15, which proves the first statement of this lemma.

However, if we group the first two terms on the right-hand side of (74) by the purpose of getting the complete square, then from (74) we easily conclude that the function:

$$\psi(t) = \psi_0(t) - \frac{\alpha_1 \alpha_2 p(t)}{2K}, \quad t \geq T \tag{76}$$

satisfies the Riccati differential equation (17) with respect to $a(t)$ and $b(t)$ given in the case (vii) of Theorem 15, which proves the second statement of this lemma. □

Now, we are able to present a common proof of the main results of the paper.

Proof of Theorems 5 and 15. At first, it is worth pointing out that the functions: $a(t)$, $a_1(t)$, $a_2(t)$, and $b(t)$, which are appearing at the same time in the main assumption (14) and the Riccati differential equation (17), only depend on the appropriate combination of basic assumptions on the coefficients: $r(t)p(t)$, and $q(t)$ and the functions: $k_1(u, v)$ and $k_2(u, v)$, which are formulated in one of the five cases of Theorem 5 and one of the two cases of Theorem 15.

Now, if we assume the contrary to the main assertion of the theorem; that is, if (1) is not oscillatory, then there is a nonoscillatory solution $x(t)$ of (1) and a point $T \geq t_0$ and $T \geq T_1$, where T_1 is appearing in (14), such that $x(t) \neq 0$ for all $t \in [T, \infty)$. Then by Lemmas 24, 25 and 26, the function $\psi(t)$ given by (21) or (70), and (71) is well defined with respect to such an $x(t)$, smooth enough on (T, ∞), and it is a supersolution of the Riccati differential equation (17). Taking into account the main results of Lemma 23, we obtain the two numbers T_0 and T^*, $T \leq T_0 < T^*$, and a subsolution $\varphi(t)$ of (17) such that the blow-up argument (51) is satisfied. By Corollary 22, we can apply the comparison principle (19) to (17) with arbitrary T_0 and T^*, where $T \leq T_0 < T^*$. Hence, combining (19) and (51), we get $\psi(t) \to \infty$ as $t \to T^*$, which contradicts the fact that $\psi \in C^1((T_0, \infty), \mathbb{R}) \cap C([T_0, \infty), \mathbb{R})$. Thus, $\psi(t)$ is not possible, and therefore, (1) does not allow any nonoscillatory solution. □

Proof of Lemma 19. Let $\tilde{d}(t) = \tilde{\psi}(t) - \tilde{\varphi}(t)$ and $\tilde{\varphi}(T_0) \leq \tilde{\psi}(T_0)$; that is,

$$\tilde{d}(T_0) \geq 0. \tag{77}$$

If statement (49) does not hold, then there is a point $T_* \in (T_0, T^*)$ such that $\tilde{\varphi}(T_*) > \tilde{\psi}(T_*)$; that is,

$$\tilde{d}(T_*) < 0. \tag{78}$$

Moreover, since $\tilde{d} \in C^1((T_0, T^*), \mathbb{R}) \cap C([T_0, T^*), \mathbb{R})$ from (77) and (78), we obtain a $T_1 \in [T_0, T_*)$ such that

$$\tilde{d}(T_1) = 0, \quad \tilde{d}(t) < 0 \quad \forall t \in (T_1, T_*]. \tag{79}$$

Since $\tilde{\varphi}, \tilde{\psi} \in C([T_1, T_*])$, we may use (47) in particular for

$$I_0 = [T_1, T_*], \qquad M = \max\left\{\max_{t \in I_0} |\tilde{\varphi}(t)|, \max_{t \in I_0} |\tilde{\psi}(t)|\right\}. \tag{80}$$

Hence, from (47), (48), and (79), we get

$$\begin{aligned}
\tilde{d}'(t) = \tilde{\psi}'(t) - \tilde{\varphi}'(t) &\geq h(t, \tilde{\psi}(t)) - h(t, \tilde{\varphi}(t)) \\
&\geq -L|\tilde{d}(t)| = L\tilde{d}(t), \quad t \in (T_1, T_*).
\end{aligned} \tag{81}$$

Multiplying this inequality by e^{-Lt} and denoting by $\theta(t) := \tilde{d}(t)\, e^{-Lt}$, we get

$$\theta'(t) = e^{-Lt}\left(\tilde{d}'(t) - L\tilde{d}(t)\right) \geq 0, \quad t \in (T_1, T_*). \tag{82}$$

Thus, according to (79) and (82), we have that $\theta(T_1) = 0$, $\theta(t) < 0$ and $\theta'(t) \geq 0$ on (T_1, T_*), which is not possible. Hence, the hypothesis (78) yields to a contradiction and, thus, $\tilde{\varphi}(t) \leq \tilde{\psi}(t)$ for all $t \in [T_0, T^*)$. □

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