Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 838302, 6 pages
http://dx.doi.org/10.1155/2013/838302

Research Article

The Global Weak Solution for a Generalized Camassa-Holm Equation

Shaoyong Lai

Department of Applied Mathematics, Southwestern University of Finance and Economics, Chengdu, Sichuan 610074, China

Correspondence should be addressed to Shaoyong Lai; laishaoy@swufe.edu.cn

Received 25 October 2012; Accepted 24 December 2012

Academic Editor: Yong Hong Wu

A nonlinear generalization of the famous Camassa-Holm model is investigated. Provided that initial value $u_0 \in H^s(R)(1 \le s \le 3/2)$ and $(1-\partial_x^2)u_0$ satisfies an associated sign condition, it is shown that there exists a unique global weak solution to the equation in space $u(t,x) \in L^2([0,+\infty), H^s(R))$ in the sense of distribution, and $u_x \in L^\infty([0,+\infty)\times R)$.

1. Introduction

In recent years, a lot of works have been carried out to investigate the Camassa-Holm equation [1],

$$u_t - u_{txx} + ku_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad (1)$$

which is a completely integrable equation. In fact, the Camassa-Holm equation arises as a model describing the unidirectional propagation of shallow water waves over a flat bottom [1–3]. The equation was originally derived much earlier as a bi-Hamiltonian generalization of the Korteweg-de Vries equation (see [4]). Johnson [2], Constantin and Lannes [5] derived models which include the Camassa-Holm equation (1). It has been found that (1) conforms with many conservation laws (see [6, 7]) and possesses smooth solitary wave solutions if $k > 0$ [3, 8] or peakons if $k = 0$ [3, 9]. Equation (1) is also regarded as a model of the geodesic flow for the H^1 right invariant metric on the Bott-Virasoro group if $k > 0$ and on the diffeomorphism group if $k = 0$ (see [10–14]). The well-posedness of local strong solutions for generalized forms of (1) has been given in [15–17]. The sharpest results for the global existence and blow-up solutions are found in Bressan and Constantin [18, 19].

Recently, Li et al. [20] studied the following generalized Camassa-Holm equation:

$$\begin{aligned} u_t - u_{txx} + ku^m u_x + (m+3)\, u^{m+1} u_x \\ = (m+2)\, u^m u_x u_{xx} + u^{m+1} u_{xxx}, \end{aligned} \quad (2)$$

where $m \ge 0$ is a natural number. Obviously, (2) reduces to (1) if $m = 0$. The authors applied the pseudoparabolic regularization technique to build the local well-posedness for (2) in Sobolev space $H^s(R)$ with $s > 3/2$ via a limiting procedure. Provided that the initial value u_0 satisfies a sign condition and $u_0 \in H^s(R)(s > 3/2)$, it is shown that there exists a unique global strong solution for (2) in space $C([0,\infty); H^s(R)) \bigcap C^1([0,\infty); H^{s-1}(R))$. However, the existence and uniqueness of the global weak solution for (2) is not investigated in [20].

The objective of this paper is to establish the well-posedness of global weak solutions for (2). Using the estimates in $H^q(R)$ with $0 \le q \le 1/2$, which are derived from the equation itself, we prove that there exists a unique global weak solution to (2) in space $H^s(R)$ with $1 \le s \le 3/2$ if $u_0 \in H^s(R)$, and $(1-\partial_x^2)u_0$ satisfies an associated sign condition.

The structure of this paper is as follows. The main result is given in Section 2. Several lemmas are given in Section 3. Section 4 establishes the proof of the main result.

2. Main Results

Firstly, we give some notations.

The space of all infinitely differentiable functions $\phi(t,x)$ with compact support in $[0,+\infty)\times R$ is denoted by C_0^∞. $L^p = L^p(R)$ $(1 \le p < +\infty)$ is the space of all measurable functions h such that $\|h\|_{L^p}^p = \int_R |h(t,x)|^p dx < \infty$. We define $L^\infty = L^\infty(R)$ with the standard norm

$\|h\|_{L^\infty} = \inf_{m(e)=0} \sup_{x\in R\setminus e} |h(t,x)|$. For any real number s, we let $H^s = H^s(R)$ denote the Sobolev space with the norm defined by

$$\|h\|_{H^s} = \left(\int_R \left(1+|\xi|^2\right)^s \left|\widehat{h}(t,\xi)\right|^2 d\xi \right)^{1/2} < \infty, \tag{3}$$

where $\widehat{h}(t,\xi) = \int_R e^{-ix\xi} h(t,x)dx$.

For $T>0$ and nonnegative number s, let $C([0,T); H^s(R))$ denote the Frechet space of all continuous H^s-valued functions on $[0,T)$. We set $\Lambda = (1-\partial_x^2)^{1/2}$.

Defining

$$\phi(x) = \begin{cases} e^{1/(x^2-1)}, & |x|<1, \\ 0, & |x|\geq 1, \end{cases} \tag{4}$$

and letting $\phi_\varepsilon(x) = \varepsilon^{-(1/4)}\phi(\varepsilon^{-(1/4)}x)$ with $0<\varepsilon<1/4$ and $u_{\varepsilon 0} = \phi_\varepsilon \star u_0$ (convolution of ϕ_ε and u_0), we know that $u_{\varepsilon 0} \in C^\infty$ for any $u_0 \in H^s$ with $s>0$. Notation $(1-\partial_x^2)u + k/2(m+1) \in N^+(R)$ (or equivalently $(1-\partial_x^2)u + k/2(m+1) \in N^-(R)$) means that $(1-\partial_x^2)u \star \phi_\varepsilon + k/2(m+1) \geq 0$ (or equivalently $(1-\partial_x^2)u \star \phi_\varepsilon + k/2(m+1) \leq 0$) for an arbitrary sufficiently small $\varepsilon > 0$.

For the equivalent form of (2), we consider its Cauchy problem

$$\begin{aligned} u_t - u_{txx} = & -\frac{k}{m+1}\left(u^{m+1}\right)_x - \frac{m+3}{m+2}\left(u^{m+2}\right)_x \\ & + \frac{1}{m+2}\partial_x^3\left(u^{m+2}\right) - (m+1)\partial_x\left(u^m u_x^2\right) \\ & + u^m u_x u_{xx}, \\ & u(0,x) = u_0(x). \end{aligned} \tag{5}$$

Definition 1. A function $u(t,x) \in L^2([0,+\infty), H^s(R))$ is called a global weak solution to problem (5) if for every $T>0$, $u(t,x) \in H^s(R)$, $u_t(t,x) \in H^{s-1}(R)$, and all $\psi(t,x) \in C_0^\infty$, it holds that

$$\begin{aligned} \int_0^T \int_R \big[& u_t - u_{txx} + ku^m u_x + (m+3)u^{m+1}u_x \\ & - (m+2)u^m u_x u_{xx} - u^{m+1}u_{xxx} \big] \psi(t,x)\, dx\, dt = 0 \end{aligned} \tag{6}$$

with $u(0,x) = u_0(x)$.

Now, we give the main result of this work.

Theorem 2. *Let $u_0(x) \in H^s(R)$, $1 \leq s \leq 3/2$, $(1-\partial_x^2)u_0 + k/2(m+1) \in N^+(R)$, and $k \geq 0$ (or equivalently $(1-\partial_x^2)u_0 + k/2(m+1) \in N^-(R)$, $k \leq 0$). Then, problem (5) has a unique global weak solution $u(t,x) \in L^2([0,+\infty), H^s(R))$ in the sense of distribution, and $u_x \in L^\infty([0,+\infty)\times R)$.*

3. Several Lemmas

Lemma 3 (see [20]). *Let $u_0(x) \in H^s(R)$ with $s > 3/2$. Then, the Cauchy problem (5) has a unique solution*

$$u(t,x) \in C\left([0,T); H^s(R)\right) \bigcap C^1\left([0,T); H^{s-1}(R)\right), \tag{7}$$

where $T>0$ depends on $\|u_0\|_{H^s(R)}$.

Lemma 4 (see [20]). *Let $u_0(x) \in H^s$, $s > 3/2$, and $k \geq 0$, $(1-\partial_x^2)u_0 + k/2(m+1) \geq 0$ (or equivalently $k \leq 0$, $(1-\partial_x^2)u_0 + k/2(m+1) \leq 0$). Then, problem (5) has a unique solution satisfying*

$$u(t,x) \in C\left([0,\infty); H^s(R)\right) \bigcap C^1\left([0,\infty); H^{s-1}(R)\right). \tag{8}$$

Using the first equation of system (5) derives

$$\frac{d}{dt}\int_R \left(u^2 + u_x^2\right) dx = 0, \tag{9}$$

from which one has the conservation law

$$\int_R \left(u^2+u_x^2\right) dx = \int_R \left(u_0^2 + u_{0x}^2\right) dx. \tag{10}$$

Lemma 5 (see [20]). *Let $s > 3/2$, and the function $u(t,x)$ is a solution of problem (5) and the initial data $u_0(x) \in H^s$. Then, the following inequality holds:*

$$\|u\|_{H^1}^2 \leq \int_R \left(u^2+u_x^2\right) dx = \int_R \left(u_0^2+u_{0x}^2\right) dx. \tag{11}$$

For $q \in (0, s-1]$, there is a constant c such that

$$\begin{aligned} \int_R \left(\Lambda^{q+1}u\right)^2 dx \leq & \int_R \left(\Lambda^{q+1}u_0\right)^2 dx \\ & + c\int_0^t \|u\|_{H^{q+1}}^2 \left(\|u_x\|_{L^\infty}\|u\|_{L^\infty}^m \right. \\ & \left. + \|u\|_{L^\infty}^{m-1}\|u_x\|_{L^\infty}^2\right) d\tau. \end{aligned} \tag{12}$$

For $q \in [0, s-1]$, there is a constant c such that

$$\begin{aligned} \|u_t\|_{H^q} \leq c\|u\|_{H^{q+1}} & \left(\|u\|_{L^\infty}^m \|u\|_{H^1} + \|u\|_{L^\infty}^m \|u_x\|_{L^\infty} \right. \\ & \left. + \|u\|_{L^\infty}^{m-1}\|u_x\|_{L^\infty}^2\right). \end{aligned} \tag{13}$$

For (2), consider the problem

$$\begin{aligned} p_t &= u^{m+1}(t,p), \quad t \in [0,T), \\ p(0,x) &= x. \end{aligned} \tag{14}$$

Lemma 6 (see [20]). *Let $u_0 \in H^s$, $s \geq 3$, and let $T > 0$ be the maximal existence time of the solution to problem (5). Then, problem (14) has a unique solution $p \in C^1([0,T)\times R)$. Moreover, the map $p(t,\cdot)$ is an increasing diffeomorphism of R with $p_x(t,x) > 0$ for $(t,x) \in [0,T)\times R$.*

Differentiating (14) with respect to x yields

$$\frac{d}{dt}p_x = (m+1)\,u^m u_x(t,p)\,p_x, \quad t \in [0,T), \qquad p_x(0,x) = 1, \tag{15}$$

which leads to

$$p_x(t,x) = \exp\left(\int_0^t (m+1)\,u^m u_x(\tau, p(\tau,x))\,d\tau\right). \tag{16}$$

The next lemma is reminiscent of a strong invariance property of the Camassa-Holm equation (the conservation of momentum [21]).

Lemma 7 (see [20]). *Let $u_0 \in H^s$ with $s \geq 3$, and let $T > 0$ be the maximal existence time of the problem* (5). *It holds that*

$$y(t,p(t,x))\,p_x^2(t,x) = y_0(x)\,e^{\int_0^t m u^m u_x d\tau}, \tag{17}$$

where $(t,x) \in [0,T) \times R$ and $y := u - u_{xx} + k/2(m+1)$.

Lemma 8. *If $u_0 \in H^s$, $s \geq 3$, such that $(1-\partial_x^2)u_0 + k/2(m+1) \geq 0$, $k \geq 0$ (or equivalently, $(1-\partial_x^2)u_0 + k/2(m+1) \leq 0$, $k \leq 0$), then the solution of problem* (5) *satisfies*

$$\|u_x\|_{L^\infty} \leq \|u\|_{L^\infty} + \frac{|k|}{2(m+1)} \leq c. \tag{18}$$

Proof. Using $u_0 - u_{0xx} + k/2(m+1) \geq 0$, it follows from Lemma 7 that $u - u_{xx} + k/2(m+1) \geq 0$. Letting $Y_1 = u - u_{xx}$, we have

$$u = \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\eta} Y_1(t,\eta)\,d\eta + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-\eta} Y_1(t,\eta)\,d\eta, \tag{19}$$

from which we obtain

$$\begin{aligned}
\partial_x u(t,x) &= -\frac{1}{2}\left(e^{-x}\int_{-\infty}^{x} e^{\eta} Y_1(t,\eta)\,d\eta + e^{x}\int_{x}^{\infty} e^{-\eta} Y_1(t,\eta)\,d\eta\right) \\
&\quad + e^{x}\int_{x}^{\infty} e^{-\eta} Y_1(t,\eta)\,d\eta \\
&= -u(t,x) + e^{x}\int_{x}^{\infty} e^{-\eta} Y_1(t,\eta)\,d\eta \\
&= -u(t,x) + e^{x}\int_{x}^{\infty} e^{-\eta}\left(Y_1(t,\eta) + \frac{k}{2(m+1)}\right) d\eta \\
&\quad - \frac{k}{2(m+1)}e^{x}\int_{x}^{\infty} e^{-\eta}\,d\eta \\
&= -u(t,x) + e^{x}\int_{x}^{\infty} e^{-\eta}(y(t,\eta))\,d\eta - \frac{k}{2(m+1)} \\
&\geq -u(t,x) - \frac{k}{2(m+1)}.
\end{aligned} \tag{20}$$

On the other hand, we have

$$\begin{aligned}
\partial_x u(t,x) &= \frac{1}{2}\left(e^{-x}\int_{-\infty}^{x} e^{\eta} Y_1(t,\eta)\,d\eta + e^{x}\int_{x}^{\infty} e^{-\eta} Y_1(t,\eta)\,d\eta\right) \\
&\quad - e^{-x}\int_{-\infty}^{x} e^{\eta} Y_1(t,\eta)\,d\eta \\
&= u(t,x) - e^{-x}\int_{-\infty}^{x} e^{\eta} Y_1(t,\eta)\,d\eta \\
&= u(t,x) - e^{-x}\int_{-\infty}^{x} e^{\eta}\left(Y_1(t,\eta) + \frac{k}{2(m+1)}\right) d\eta \\
&\quad + \frac{k}{2(m+1)}e^{-x}\int_{-\infty}^{x} e^{\eta}\,d\eta \\
&= u(t,x) - e^{-x}\int_{-\infty}^{x} e^{\eta} y(t,\eta)\,d\eta + \frac{k}{2(m+1)} \\
&\leq u(t,x) + \frac{k}{2(m+1)}.
\end{aligned} \tag{21}$$

The inequalities (19), (20), and (21) derive that inequality (18) is valid. Similarly, if $(1-\partial_x^2)u_0 + k/2(m+1) \leq 0, k \leq 0$, we still know that (18) is valid. □

Lemma 9. *For $s > 0$, $u_0 \in H^s$, it holds that*

$$\begin{gathered}
\|u_{\varepsilon 0x}\|_{L^\infty} \leq c\|u_{0x}\|_{L^\infty}, \\
\|u_{\varepsilon 0}\|_{H^q} \leq c, \quad \text{if } q \leq s, \\
\|u_{\varepsilon 0}\|_{H^q} \leq c\varepsilon^{s-q/4}, \quad \text{if } q > s, \\
\|u_{\varepsilon 0} - u_0\|_{H^q} \leq c\varepsilon^{s-q/4}, \quad \text{if } q \leq s, \\
\|u_{\varepsilon 0} - u_0\|_{H^s} = o(1),
\end{gathered} \tag{22}$$

where c is a constant independent of ε.

The proof of this lemma can be found in Lai and Wu [15]. From Lemma 3, it derives that the Cauchy problem

$$\begin{aligned}
u_t - u_{txx} &= -\frac{m+3}{m+2}\left(u^{m+2}\right)_x + \frac{1}{m+2}\partial_x^3\left(u^{m+2}\right) \\
&\quad - (m+1)\,\partial_x\left(u^m u_x^2\right) + u^m u_x u_{xx}, \\
u(0,x) &= u_{\varepsilon 0}(x), \quad x \in R,
\end{aligned} \tag{23}$$

has a unique solution u depending on the parameter ε. We write $u_\varepsilon(t,x)$ to represent the solution of problem (23). Using Lemma 3 derives that $u_\varepsilon(t,x) \in C^\infty([0,T), H^\infty(R))$ since $u_{\varepsilon 0}(x) \in C_0^\infty(R)$.

Lemma 10. *Provided that* $u_0 \in H^s$, $1 \le s \le 3/2$, $k \ge 0$, *and* $(1-\partial_x^2)u_0 + k/2(m+1) \in N^+(R)$ *(or equivalently* $(1-\partial_x^2)u_0 + k/2(m+1) \in N^-(R)$, $k \le 0$*), then there exists a constant* $c_0 > 0$ *independent of* ε *such that the solution of problem (23) satisfies*

$$\|u_{\varepsilon x}\|_{L^\infty} \le \|u_x\|_{L^\infty} + \frac{|k|}{2(m+1)} \le c_0. \tag{24}$$

Proof. Using identity (10) and Lemma 9, if $u_0 \in H^s(R)$ with $1 \le s \le 3/2$, we have

$$\|u_\varepsilon\|_{L^\infty} \le \|u_\varepsilon\|_{H^1} = \|u_{\varepsilon 0}\|_{H^1} \le c, \tag{25}$$

where c is independent of ε.

From Lemma 8, we have

$$\|u_{\varepsilon x}\|_{L^\infty} \le \|u_\varepsilon\|_{L^\infty} + \frac{|k|}{2(m+1)} \le c + \frac{|k|}{2(m+1)}, \tag{26}$$

which completes the proof. □

Lemma 11. *For any* $f_1 \in L^\infty$, $f_2 \in H^z$ *with* $z \le 0$, *it holds that*

$$\|f_1 f_2\|_{H^z} \le c\|f_1\|_{L^\infty}\|f_2\|_{H^z} \quad \text{for any } z \le 0. \tag{27}$$

The proof of this lemma can be found in [15].

4. Existence and Uniqueness of Global Weak Solution

Provided that $1 \le s \le 3/2$, for problem (23), applying Lemmas 5, 9, and 10, and the Gronwall's inequality, we obtain the inequalities

$$\begin{gathered}
\|u_\varepsilon\|_{H^1} \le \|u_{\varepsilon 0}\|_{H^1} \le c, \\
\|u_\varepsilon\|_{H^q} \le c\|u_{\varepsilon 0}\|_{H^q} \exp\left[\int_0^t \left(\|u_{\varepsilon x}\| + \|u_{\varepsilon x}\|_{L^\infty}^2\right) d\tau\right] \le c e^{ct}, \\
\|u_{\varepsilon t}\|_{H^r} \le \|u_\varepsilon\|_{H^{r+1}}\left(1+e^{ct}\right) \le c\left(1+e^{ct}\right),
\end{gathered} \tag{28}$$

where $q \in (0, s]$, $r \in [0, s-1]$, and c is a constant independent of ε. It follows from the Aubin's compactness theorem that there is a subsequence of $\{u_\varepsilon\}$, denoted by $\{u_{\varepsilon_n}\}$, such that $\{u_{\varepsilon_n}\}$ and their temporal derivatives $\{u_{\varepsilon_n t}\}$ are weakly convergent to a function $u(t,x)$ and its derivative u_t in $L^2([0,T], H^s)$ and $L^2([0,T], H^{s-1})$, respectively, where T is an arbitrary fixed positive number. Moreover, for any real number $R_1 > 0$, $\{u_{\varepsilon_n}\}$ is convergent to the function u strongly in the space $L^2([0,T], H^q(-R_1, R_1))$ for $q \in (0, s]$ and $\{u_{\varepsilon_n t}\}$ converges to u_t strongly in the space $L^2([0,T], H^r(-R_1, R_1))$ for $r \in [0, s-1]$.

4.1. The Proof of Existence for Global Weak Solution. For an arbitrary fixed $T > 0$, from Lemma 10, we know that $\{u_{\varepsilon_n x}\}(\varepsilon_n \to 0)$ is bounded in the space L^∞. Thus, the sequences $\{u_{\varepsilon_n}\}$, $\{u_{\varepsilon_n x}\}$, $\{u_{\varepsilon_n x}^2\}$, and $\{u_{\varepsilon_n x}^3\}$ are weakly convergent to u, u_x, u_x^2, and u_x^3 in $L^2([0,T], H^r(-R_1, R_1))$ for any $r \in [0, s-1)$, separately. Using $u^m(u_x^2)_x = (u^m u_x^2)_x - (u^m)_x u_x^2$, we know that u satisfies the equation

$$\begin{aligned}
&-\int_0^T \int_R u\left(g_t - g_{xxt}\right) dx\, dt \\
&\quad = \int_0^T \int_R \Bigg[\left(\frac{m+3}{m+2}u^{m+2} + (m+1)u^m u_x^2\right) g_x \\
&\qquad - \frac{1}{m+2}u^{m+2} g_{xxx} - \frac{1}{2}u^m u_x^2 g_x \\
&\qquad - \frac{m}{2}u^{m-1}u_x^3 g\Bigg] dx\, dt,
\end{aligned} \tag{29}$$

with $u(0,x) = u_0(x)$ and $g \in C_0^\infty$. Since $X = L^1([0,T] \times R)$ is a separable Banach space and $\{u_{\varepsilon_n x}\}$ is a bounded sequence in the dual space $X^* = L^\infty([0,T] \times R)$ of X, there exists a subsequence of $\{u_{\varepsilon_n x}\}$, still denoted by $\{u_{\varepsilon_n x}\}$, weakly star convergent to a function v in $L^\infty([0,T] \times R)$. As $\{u_{\varepsilon_n x}\}$ weakly converges to u_x in $L^2([0,T] \times R)$, it results that $u_x = v$ almost everywhere. Thus, we obtain $u_x \in L^\infty([0,T] \times R)$. Since T > 0 is an arbitrary number, we complete the global existence of weak solutions to problem (5).

Proof of Uniqueness. Suppose that there exist two global weak solutions $u(t,x)$ and $v(t,x)$ to problem (5) with the same initial value $u(0,x) \in H^s(R)$, $1 \le s \le 3/2$, we consider its associated regularized problem (23). Letting $w_\varepsilon = u_\varepsilon(t,x) - v_\varepsilon(t,x)$, from Lemma 10, we get $\|\partial u_{\varepsilon(t,x)}/\partial x\|_{L^\infty} \le c$ and $\|\partial v_{\varepsilon(t,x)}/\partial x\|_{L^\infty} \le c$ which is independent of ε. Still denoting $u = u_\varepsilon$, $v = v_\varepsilon$, and $w = w_\varepsilon$, it holds that

$$\begin{aligned}
w_t = {}&\left(1-\partial_x^2\right)^{-1}\Big[-\partial_x\left(u^{m+2} - v^{m+2}\right) \\
&- \partial_x\Big[\partial_x\left(u^{m+1}\right)\partial_x w \\
&+ \partial_x\left(u^{m+1} - v^{m+1}\right)\partial_x v\Big] \\
&+ \left[u^m u_x u_{xx} - v^m v_x v_{xx}\right]\Big] \\
&- \frac{1}{m+2}\partial_x\left(u^{m+2} - v^{m+2}\right), \\
&w(0,x) = 0.
\end{aligned} \tag{30}$$

Multiplying both sides of (30) by w, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_R w^2dx &\le c\left|\int_R w\left(u^{m+2}-v^{m+2}\right)_x dx\right| \\ &+\left|\int_R w\Lambda^{-2}\left(u^{m+2}-v^{m+2}\right)_x dx\right| \\ &+\left|\int_R w\Lambda^{-2}\left[\partial_x\left(u^{m+1}\right)\partial_x w\right]_x dx\right| \\ &+\left|\int_R w\Lambda^{-2}\left[\partial_x\left(u^{m+1}-v^{m+1}\right)\partial_x v\right]_x dx\right| \\ &+\left|\int_R w\Lambda^{-2}\left[u^m u_x u_{xx}-v^m v_x v_{xx}\right]dx\right| \\ &= I_1+I_2+I_3+I_4+I_5. \end{aligned} \tag{31}$$

Using $\|u\|_{L^\infty}\le c$, $\|v\|_{L^\infty}\le c$, $\|u_x\|_{L^\infty}\le c$, $\|v_x\|_{L^\infty}\le c$, we have

$$\begin{aligned} I_1 &\le c\left|\int_R w\left[w\sum_{j=0}^{m+1}u^j v^{m+1-j}\right]_x dx\right| \\ &= c\left|\int_R w\left[w_x\sum_{j=0}^{m+1}u^j v^{m+1-j}+w\sum_{j=0}^{m+1}\left(u^j v^{m+1-j}\right)_x\right]dx\right| \\ &= c\left|\int_R\left(\frac{1}{2}w^2\right)_x\sum_{j=0}^{m+1}u^j v^{m+1-j}+w^2\sum_{j=0}^{m+1}\left(u^j v^{m+1-j}\right)_x dx\right| \\ &= c\left|\int_R\left(\frac{-1}{2}w^2\right)\sum_{j=0}^{m+1}\left(u^j v^{m+1-j}\right)_x+w^2\sum_{j=0}^{m+1}\left(u^j v^{m+1-j}\right)_x dx\right| \\ &= c\left|\int_R\left(\frac{1}{2}\mathrm{w}^2\right)\sum_{j=0}^{m+1}\left(u^j v^{m+1-j}\right)_x dx\right| \\ &\le c\|w\|_{L^2}^2\sum_{j=0}^{m+1}\left\|\left(u^j v^{m+1-j}\right)_x\right\|_{L^\infty} \\ &\le c\|w\|_{L^2}^2. \end{aligned} \tag{32}$$

Applying Lemma 11 repeatedly, we have

$$\begin{aligned} I_2 &\le c\|w\|_{L^2}\left\|\Lambda^{-2}\left(u^{m+2}-v^{m+2}\right)_x\right\|_{L^2} \\ &\le c\|w\|_{L^2}\left\|w\sum_{j=0}^{m+1}u^j v^{m+1-j}\right\|_{L^2} \\ &\le c\|w\|_{L^2}^2\sum_{j=0}^{m+1}\|u\|_{L^\infty}^j\|v\|_{L^\infty}^{m+1-j} \\ &\le c\|w\|_{L^2}^2, \\ I_3 &\le c\|w\|_{L^2}\left\|\Lambda^{-2}\left[\partial_x\left(u^{m+1}\right)\partial_x w\right]_x\right\|_{L^2} \\ &\le c\|w\|_{L^2}\left\|\partial_x\left(u^{m+1}\right)\partial_x w\right\|_{H^{-1}} \\ &\le c\|w\|_{L^2}\|\partial_x w\|_{H^{-1}}\left\|\partial_x\left(u^{m+1}\right)\right\|_{L^\infty} \\ &\le c\|w\|_{L^2}^2, \\ I_4 &\le c\|w\|_{L^2}\left\|\partial_x\left(u^{m+1}-v^{m+1}\right)\partial_x v\right\|_{H^{-1}} \\ &\le c\|w\|_{L^2}\|\partial_x v\|_{L^\infty}\left\|\partial_x\left(u^{m+1}-v^{m+1}\right)\right\|_{H^{-1}} \\ &\le c\|w\|_{L^2}\left\|u^{m+1}-v^{m+1}\right\|_{H^0} \\ &\le c\|w\|_{L^2}\left\|w\sum_{j=0}^{m}u^j v^{m-j}\right\|_{L^2} \\ &\le c\|w\|_{L^2}^2\sum_{j=0}^{m}\|u\|_{L^\infty}^j\|v\|_{L^\infty}^{m-j} \\ &\le c\|w\|_{L^2}^2. \end{aligned} \tag{33}$$

For I_5, using Lemma 11 derives

$$\begin{aligned} I_5 &\le c\|w\|_{L^2}\left\|\left(u^m-v^m\right)\left(u_x^2\right)_x+v^m\left[u_x^2-v_x^2\right]_x\right\|_{H^{-2}} \\ &\le c\|w\|_{L^2}\left\|\left(u^m-v^m\right)\left(u_x^2\right)_x\right\|_{H^{-2}}+\left\|v^m\left[u_x^2-v_x^2\right]_x\right\|_{H^{-2}} \\ &\le c\|w\|_{L^2}\left(\left\|\left[\left(u^m-v^m\right)\left(u_x^2\right)\right]_x-\left(u^m-v^m\right)_x u_x^2\right\|_{H^{-2}}\right. \\ &\qquad\left.+\|v\|_{L^\infty}^m\left\|(u-v)_x\left(u_x+v_x\right)\right\|_{H^{-1}}\right) \\ &\le c\|w\|_{L^2}\left(\left\|\left(u^m-\mathrm{v}^m\right)u_x^2\right\|_{H^{-1}}+\left\|\left(u^m-v^m\right)_x u_x^2\right\|_{H^{-2}}+c\|w\|_{L^2}\right) \\ &\le c\|w\|_{L^2}\left(\|u_x\|_{L^\infty}^2\|w\|_{L^2}\sum_{j=0}^{m-1}\|u\|_{L^\infty}^j\|v\|_{L^\infty}^{m-1-j}+c\|w\|_{L^2}\right) \\ &\le c\|w\|_{L^2}^2. \end{aligned} \tag{34}$$

Using (32)–(34), we get

$$\frac{1}{2}\frac{d}{dt}\int_R w^2dx \le c\|w\|_{L^2}^2. \tag{35}$$

Applying $w(0)=0$ results in $\|w\|_{L^2}^2=0$. Consequently, we know that the global weak solution is unique. □

Acknowledgment

This work is supported by the Fundamental Research Funds for the Central Universities (JBK120504).

References

[1] R. Camassa and D. D. Holm, "An integrable shallow water equation with peaked solitons," *Physical Review Letters*, vol. 71, no. 11, pp. 1661–1664, 1993.

[2] R. S. Johnson, "Camassa-Holm, Korteweg-de Vries and related models for water waves," *Journal of Fluid Mechanics*, vol. 455, no. 1, pp. 63–82, 2002.

[3] R. S. Johnson, "On solutions of the Camassa-Holm equation," *Proceedings of the Royal Society A*, vol. 459, no. 2035, pp. 1687–1708, 2003.

[4] A. Fokas and B. Fuchssteiner, "Symplectic structures, their Bäcklund transformations and hereditary symmetries," *Physica D*, vol. 4, no. 1, pp. 47–66, 1981.

[5] A. Constantin and D. Lannes, "The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations," *Archive for Rational Mechanics and Analysis*, vol. 192, no. 1, pp. 165–186, 2009.

[6] A. Constantin, "On the scattering problem for the Camassa-Holm equation," *Proceedings of the Royal Society A*, vol. 457, no. 2008, pp. 953–970, 2001.

[7] J. Lenells, "Conservation laws of the Camassa-Holm equation," *Journal of Physics A*, vol. 38, no. 4, pp. 869–880, 2005.

[8] H. P. McKean, "Fredholm determinants and the Camassa-Holm hierarchy," *Communications on Pure and Applied Mathematics*, vol. 56, no. 5, pp. 638–680, 2003.

[9] A. Constantin and J. Escher, "Global existence and blow-up for a shallow water equation," *Annali della Scuola Normale Superiore di Pisa*, vol. 26, no. 2, pp. 303–328, 1998.

[10] A. Constantin, "Existence of permanent and breaking waves for a shallow water equation: a geometric approach," *Annales de l'Institut Fourier*, vol. 50, no. 2, pp. 321–362, 2000.

[11] A. Constantin, "On the inverse spectral problem for the Camassa-Holm equation," *Journal of Functional Analysis*, vol. 155, no. 2, pp. 352–363, 1998.

[12] A. Constantin and J. Escher, "Global weak solutions for a shallow water equation," *Indiana University Mathematics Journal*, vol. 47, no. 4, pp. 1527–1545, 1998.

[13] A. Constantin and B. Kolev, "Geodesic flow on the diffeomorphism group of the circle," *Commentarii Mathematici Helvetici*, vol. 78, no. 4, pp. 787–804, 2003.

[14] A. Constantin, T. Kappeler, B. Kolev, and P. Topalov, "On geodesic exponential maps of the Virasoro group," *Annals of Global Analysis and Geometry*, vol. 31, no. 2, pp. 155–180, 2007.

[15] S. Y. Lai and Y. H. Wu, "The local well-posedness and existence of weak solutions for a generalized Camassa-Holm equation," *Journal of Differential Equations*, vol. 248, no. 8, pp. 2038–2063, 2010.

[16] S. Y. Lai and Y. H. Wu, "A model containing both the Camassa-Holm and Degasperis-Procesi equations," *Journal of Mathematical Analysis and Applications*, vol. 374, no. 2, pp. 458–469, 2011.

[17] S. Y. Lai and Y. H. Wu, "Existence of weak solutions in lower order Sobolev space for a Camassa-Holm-type equation," *Journal of Physics A*, vol. 43, no. 9, Article ID 095205, 13 pages, 2010.

[18] A. Bressan and A. Constantin, "Global conservative solutions of the Camassa-Holm equation," *Archive for Rational Mechanics and Analysis*, vol. 183, no. 2, pp. 215–239, 2007.

[19] A. Bressan and A. Constantin, "Global dissipative solutions of the Camassa-Holm equation," *Analysis and Applications*, vol. 5, no. 1, pp. 1–27, 2007.

[20] N. Li, S. Y. Lai, S. Li, and M. Wu, "The local and global existence of solutions for a generalized Camassa-Holm equation," *Abstract and Applied Analysis*, vol. 2012, Article ID 532369, 26 pages, 2012.

[21] B. Kolev, "Poisson brackets in hydrodynamics," *Discrete and Continuous Dynamical Systems A*, vol. 19, no. 3, pp. 555–574, 2007.