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Research Article

Variational Approximate Solutions of Fractional Nonlinear Nonhomogeneous Equations with Laplace Transform

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A novel modification of the variational iteration method is proposed by means of Laplace transform and homotopy perturbation method. The fractional lagrange multiplier is accurately determined by the Laplace transform and the nonlinear one can be easily handled by the use of He's polynomials. Several fractional nonlinear nonhomogeneous equations are analytically solved as examples and the methodology is demonstrated.

1. Introduction

Recently, systems of fractional nonlinear partial differential equations [1–3] have attracted much attention in a variety of applied sciences. With the development of nonlinear sciences, some numerical [4–6], semianalytical [7–12], and analytical methods [13–15] have been developed for fractional differential equations. So, the semianalytical methods have largely been used to solve fractional equations. Most of these methods have their inbuilt deficiencies like the calculation of Adomian's polynomials, the Lagrange multiplier, divergent results, and huge computational work. Recently, some improved homotopy perturbation methods [16, 17] and improved variational iteration methods, [18, 19] have been used by many researches.

The variational iteration method (VIM) [8, 9, 20] was extended to initial value problems of differential equations and has been one of the methods used most often. The key problem of the VIM is the correct determination of the Lagrange multiplier when the method is applied to fractional equations; combined with the Laplace transform, the crucial point of this method is solved efficiently by Wu and Baleanu [21, 22]. Laplace transform overcomes principle drawbacks in application of the VIM to fractional equations.

Motivated and inspired by the ongoing research in this field, we give a new modification of variational iteration method, combined with the Laplace transform and the homotopy perturbation method. The fractional Lagrange multiplier is accurately determined by the Laplace transform and the nonlinear one can be easily handed by the use of He's polynomials. In this work, we will use this new method to obtain approximate solutions of the fractional nonlinear equations, and the fractional derivatives are described in the Caputo sense.

2. Description of the Method

In order to illustrate the basic idea of the technique, consider the following general nonlinear system:

$$\frac{\partial^m u(x,t)}{\partial t^m} + R[u(x,t)] + N[u(x,t)] = g(x,t), \quad (1)$$

$$u^k(x,0^+) = a_k, \quad (2)$$

where $k = 0, \ldots, m-1$, $\partial^m u(x,t)/\partial t^m$ is the term of the highest-order derivative, $g(x,t)$ is the source term, N represents the general nonlinear differential operator, and R is the linear differential operator.

Now, we consider the application of the modified VIM [21, 22]. Taking the above Laplace transform to both sides of (1) and (2), then the linear part is transformed into an algebraic equation as follows:

$$s^m U(x,s) - u^{(m-1)}(x,0) - \cdots - s^{m-1} u(x,0) + L[R[u]] + L[N[u]] - L[g(x,t)] = 0, \tag{3}$$

where $U(x,s) = L[u(x,t)] = \int_0^\infty e^{-st} u(x,t)dt$. The iteration formula of (3) can be used to suggest the main iterative scheme involving the Lagrange multiplier as

$$U_{n+1}(x,s) = U_n(x,s) + \lambda(s) \times \left[s^m U_n(x,s) - \sum_{k=0}^{m-1} u^k(x,0^+) s^{m-1-k} + L[R[u_n(x,t)] + N[u_n(x,t)] - g(x,t)] \right]. \tag{4}$$

Considering $L[R[u_n(x,t)] + N[u_n(x,t)]]$ as restricted terms, one can derive a Lagrange multiplier as

$$\lambda = -\frac{1}{s^m}. \tag{5}$$

With (5) and the inverse-Laplace transform L^{-1}, the iteration formula (4) can be explicitly given as

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) - L^{-1} \times \left[\frac{1}{s^m} \left[s^m U_n(x,s) - \sum_{k=0}^{m-1} u^k(x,0^+) s^{m-1-k} + L[R[u_n(x,t)] + N[u_n(x,t)] - g(x,t)] \right] \right] \\ &= u_0(x,t) - L^{-1}\left[\frac{1}{s^m} [L[R[u_n(x,t)] + N[u_n(x,t)]]] \right]; \end{aligned} \tag{6}$$

$u_0(x,t)$ is an initial approximation of (1), and

$$\begin{aligned} u_0(x,t) &= L^{-1}\left(\sum_{k=0}^{m-1} u^k(x,0^+) s^{m-1-k} \right) + L^{-1}\left[\frac{1}{s^m} L[g(x,t)] \right] \\ &= u(x,0) + u'(x,0)t + \cdots + \frac{u^{m-1}(x,0)t^{m-1}}{(m-1)!} + L^{-1}\left[\frac{1}{s^m} L[g(x,t)] \right]. \end{aligned} \tag{7}$$

In order to deal with the nonlinear term in the iteration formula (6), combining with the homotopy perturbation method, we give a new modification of the above method [21, 22]. In the homotopy method, the basic assumption is that the solutions can be written as a power series in p:

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} p^n u_n(x,t) \\ &= u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \cdots, \end{aligned} \tag{8}$$

and the nonlinear term can be decomposed as

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n \mathscr{H}_n(u), \tag{9}$$

where $p \in [0,1]$ is an embedding parameter. $\mathscr{H}_n(u)$ is He's polynomials [16, 23] can be generated by

$$\mathscr{H}_n(u_0,\ldots,u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{n} p^i u_i \right) \right]_{p=0}, \quad n = 0,1,2,\ldots. \tag{10}$$

This new modified method is obtained by the elegant coupling of correction function (6) of variational iteration method with He's polynomials and is given by

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = u_0(x,t) - p\left(L^{-1}\left[\frac{1}{s^m} L\left[R\sum_{n=0}^{\infty} p^n u_n(x,t) \right] + \frac{1}{s^m} L\left[\sum_{n=0}^{\infty} p^n \mathscr{H}_n(u) \right] \right] \right), \tag{11}$$

$u_0(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Equating the terms with identical powers in p, we obtain the following approximations:

$$\begin{aligned} p^0 &: u_0(x,t) = u(x,0) + u'(x,0)t + \cdots + \frac{u^{m-1}(x,0)t^{m-1}}{(m-1)!} + L^{-1}\left[\frac{1}{s^m} L[g(x,t)] \right], \\ p^1 &: u_1(x,t) = -L^{-1}\left[\frac{1}{s^m} L[Ru_0(x,t)] + \frac{1}{s^m} L[\mathscr{H}_0(u)] \right], \\ p^2 &: u_2(x,t) = -L^{-1}\left[\frac{1}{s^m} L[Ru_1(x,t)] + \frac{1}{s^m} L[\mathscr{H}_1(u)] \right], \\ &\vdots \end{aligned} \tag{12}$$

The best approximations for the solution are

$$u(x,t) = \sum_{n=0}^{\infty} u_n. \quad (13)$$

This new modified method here transfers the problem into the partial differential equation in the Laplace s-domain, removes the differentiation with respect to time, and uses He's polynomials to deal with the nonlinear term. This new method basically illustrates how three powerful algorithms, variational iteration method, Laplace transform method, and homotopy perturbation method, can be combined and used to approximate the solutions of nonlinear equation. In this work, we will use this method to solve fractional nonlinear equations.

3. Illustrative Examples

We will apply the new modified VIM to both PDEs and FDEs. All the results are calculated by using the symbolic calculation software Mathematica.

3.1. Partial Differential Equations

Example 1. Consider the following nonhomogeneous nonlinear Gas Dynamic equation [24]

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} - u(1-u) = -e^{t-x} \quad (14)$$

with the initial condition

$$u(x,0) = 1 - e^{-x}. \quad (15)$$

After taking the Laplace transform to both sides of (14) and (15), we get the following iteration formula:

$$\begin{aligned} U_{n+1}(x,s) = U_n(x,s) + \lambda(s) \Big[& sU_n(x,s) - u(x,0) \\ & + L\left[\frac{1}{2}\frac{\partial u_n^2}{\partial x} - u_n + u_n^2 + e^{t-x}\right]\Big]. \end{aligned} \quad (16)$$

Considering $L[(1/2)(\partial u_n^2/\partial x) - u_n + u_n^2]$ as restricted terms, Lagrange multiplier can be defined as $\lambda(s) = -1/s$; with the inverse-Laplace transform, the approximate solution of (16) can be given as

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) - L^{-1} \\ &\quad \times \left[\frac{1}{s}\left[sU_n(x,s) - u(x,0) \right.\right. \\ &\qquad \left.\left. + L\left[\frac{1}{2}\frac{\partial u_n^2}{\partial x} - u_n + u_n^2 + e^{t-x}\right]\right]\right] \\ &= u_0(x,t) - L^{-1}\left[\frac{1}{s^\alpha}\left[L\left[\frac{1}{2}\frac{\partial u_n^2}{\partial x} - u_n + u_n^2\right]\right]\right], \end{aligned} \quad (17)$$

where $u_0(x,t)$ is an initial approximation of (14), and

$$u_0(x,t) = u(x,0) - L^{-1}\left[\frac{1}{s}L\left[e^{t-x}\right]\right]. \quad (18)$$

Combining with the homotopy perturbation method, one has

$$\begin{aligned} &\sum_{n=0}^{\infty} p^n u_n(x,t) \\ &\quad = u_0(x,t) - p\left[L^{-1}\left[\frac{1}{s}L\left[\frac{1}{2}\frac{\partial\left(\sum_{n=0}^{\infty} p^n \mathcal{H}_n(u)\right)}{\partial x} \right.\right.\right. \\ &\qquad \left.\left.\left. - \sum_{n=0}^{\infty} p^n u_n + \sum_{n=0}^{\infty} p^n \mathcal{H}_n(u)\right]\right]\right], \end{aligned} \quad (19)$$

where $\mathcal{H}_n(u)$ is He's polynomials that represent nonlinear term u^2; we have a few terms of the He's polynomials for u^2 which are given by

$$\begin{aligned} \mathcal{H}_0(u) &= u_0^2, \\ \mathcal{H}_1(u) &= 2u_0u_1, \\ \mathcal{H}_2(u) &= u_1^2 + 2u_0u_2, \\ &\vdots \end{aligned} \quad (20)$$

Comparing the coefficient with identical powers in p,

$$\begin{aligned} u_0(x,t) &= 1 - e^{t-x}, \\ u_1 &= -L^{-1}\left[\frac{1}{s}\left[L\left[\frac{1}{2}\frac{\partial u_0^2}{\partial x} - u_0 + u_0^2\right]\right]\right] = 0, \\ u_2 &= -L^{-1}\left[\frac{1}{s}\left[L\left[\frac{1}{2}\frac{\partial(2u_0u_1)}{\partial x} - u_1 + 2u_0u_1\right]\right]\right] \\ &= e^{-x}\frac{t^{2\alpha}}{\Gamma[1+2\alpha]} = 0, \\ &\vdots \end{aligned} \quad (21)$$

and so on; in this manner the rest of component of the solution can be obtained. The solution of (14) and (15) in series form is given by

$$u(x,t) = 1 - e^{t-x}, \quad (22)$$

which is the exact solution. For this equation, the first-order approximate solution is justly the exact solution, and this proposed new method provides the solution in a rapid convergent. Furthermore, the new modified method can be easily extended to FDEs and this is the main purpose of our work.

3.2. Fractional Differential Equations. Let us consider the time fractional equation as follows:

$$ {}_0^C D_t^\alpha u(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), \quad (23) $$

$$ u^k(x,0^+) = a_k, \quad (24) $$

where $k = 0,\ldots,m-1$, $m = [\alpha]+1$, $g(x,t)$ is the source term, N represents the general nonlinear differential operator, and R is the linear differential operator. And the Caputo timefractional derivative operator of order $\alpha > 0$ is defined as

$$ \begin{aligned} {}_0^C D_t^\alpha u(x,t) &= \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m} d\tau, \\ & m = [\alpha]+1,\ m \in N, \end{aligned} \quad (25) $$

where $\Gamma(\cdot)$ denotes the Gamma function.

Now, we consider the application of the modified VIM [21, 22]. The following Laplace transform of the term ${}_0^C D_t^\alpha u(x,t)$ holds:

$$ \begin{aligned} L\left[{}_0^C D_t^\alpha u(x,t)\right] &= s^\alpha U(x,s) \\ &\quad - \sum_{k=0}^{m-1} u^k(x,0^+) s^{\alpha-1-k}, \\ & m-1 < \alpha \le m, \end{aligned} \quad (26) $$

where $U(x,s) = L[u(x,t)] = \int_0^\infty e^{-st} u(x,t)dt$. The detailed properties of fractional calculus and Laplace transform can be found in [1, 2]; we have chosen to the Caputo fractional derivative because it allows traditional initial and boundary conditions to be included in the formulation of the problem. Taking the above Laplace transform to both sides of (23) and (24), the iteration formula of (23) can be constructed as

$$ \begin{aligned} U_{n+1}(x,s) &= U_n(x,s) + \lambda(s) \\ &\quad \times \left[s^\alpha U_n(x,s) - \sum_{k=0}^{m-1} u^k(x,0^+) s^{\alpha-1-k} \right. \\ &\quad \left. + L\left[R\left[u_n(x,t)\right] + N\left[u_n(x,t)\right] - g(x,t)\right] \right]. \end{aligned} \quad (27) $$

Considering $L[R[u_n(x,t)] + N[u_n(x,t)]]$ as restricted terms, one can derive a Lagrange multiplier as

$$ \lambda = \frac{-1}{s^\alpha}. \quad (28) $$

With (28) and the inverse-Laplace transform L^{-1}, the iteration formula (27) can be explicitly given as

$$ \begin{aligned} u_{n+1}(x,t) &= u_n(x,t) - L^{-1} \\ &\quad \times \left[\frac{1}{s^\alpha} \left[s^\alpha U_n(x,s) - \sum_{k=0}^{m-1} u^k(x,0^+) s^{\alpha-1-k} \right. \right. \\ &\quad + L\left[R\left[u_n(x,t)\right]\right. \\ &\quad \left.\left.\left. + N\left[u_n(x,t)\right] - g(x,t)\right]\right]\right] \\ &= u_0(x,t) - L^{-1} \\ &\quad \times \left[\frac{1}{s^\alpha} \left[L\left[R\left[u_n(x,t)\right] + N\left[u_n(x,t)\right]\right]\right]\right]; \end{aligned} \quad (29) $$

$u_0(x,t)$ is an initial approximation of (23), and

$$ \begin{aligned} u_0(x,t) &= L^{-1}\left(\sum_{k=0}^{m-1} u^k(x,0^+) s^{\alpha-1-k} \right) \\ &\quad + L^{-1}\left[\frac{1}{s^\alpha} L\left[g(x,t)\right] \right] \\ &= u(x,0) + u'(x,0)t + \cdots + \frac{u^{m-1}(x,0) t^{m-1}}{(m-1)!} \\ &\quad + L^{-1}\left[\frac{1}{s^\alpha} L\left[g(x,t)\right] \right]. \end{aligned} \quad (30) $$

In the homotopy method, the basic assumption is that the solutions can be written as a power series in p:

$$ \begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} p^n u_n(x,t) \\ &= u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \cdots, \end{aligned} \quad (31) $$

and the nonlinear term can be decomposed as

$$ Nu(x,t) = \sum_{n=0}^{\infty} p^n \mathscr{H}_n(u), \quad (32) $$

where $p \in [0,1]$ is an embedding parameter. $\mathscr{H}_n(u)$ is He's polynomials [16, 23] that can be generated by

$$ \begin{aligned} \mathscr{H}_n(u_0,\ldots,u_n) &= \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[N\left(\sum_{i=0}^{n} p^i u_i \right) \right]_{p=0}, \quad n = 0,1,2,\ldots. \end{aligned} \quad (33) $$

The variational homotopy perturbation method is obtained by the elegant coupling of correction function (29) of variational iteration method with He's polynomials and is given by

$$\sum_{n=0}^{\infty}p^{n}u_{n}(x,t)=u_{0}(x,t)-p\left(L^{-1}\left[\frac{1}{s^{\alpha}}L\left[R\sum_{n=0}^{\infty}p^{n}u_{n}(x,t)\right]+\frac{1}{s^{\alpha}}L\left[\sum_{n=0}^{\infty}p^{n}\mathscr{H}_{n}(u)\right]\right]\right), \tag{34}$$

$u_0(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Equating the terms with identical powers in p, we obtain the following approximations:

$$\begin{aligned}
p^{0}&:u_{0}(x,t)=u(x,0)+u'(x,0)t+\cdots+\frac{u^{m-1}(x,0)t^{m-1}}{(m-1)!}+L^{-1}\left[\frac{1}{s^{\alpha}}L\left[g(x,t)\right]\right],\\
p^{1}&:u_{1}(x,t)=-L^{-1}\left[\frac{1}{s^{\alpha}}L\left[Ru_{0}(x,t)\right]+\frac{1}{s^{\alpha}}L\left[\mathscr{H}_{0}(u)\right]\right],\\
p^{2}&:u_{2}(x,t)=-L^{-1}\left[\frac{1}{s^{\alpha}}L\left[Ru_{1}(x,t)\right]+\frac{1}{s^{\alpha}}L\left[\mathscr{H}_{1}(u)\right]\right],\\
&\vdots
\end{aligned} \tag{35}$$

The best approximations for the solution are $u(x,t)=\sum_{n=0}^{\infty}u_n$. Let us apply the above method to solve fractional nonlinear equations of Caputo type.

Example 2. Consider the following nonlinear space time fractional equation [25]:

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}+u\frac{\partial^{\beta}u(x,t)}{\partial x^{\beta}}=x+xt^{2}, \tag{36}$$

$$u(x,0)=0, \tag{37}$$

where $0<\alpha,\ \beta\leqslant 1$, and the time-space fractional derivatives defined here are in Caputo sense. The Caputo space-fractional derivative operator of order $\beta>0$ is defined as

$${}_{0}^{C}D_{x}^{\beta}u(x,t)=\frac{1}{\Gamma(m-\beta)}\int_{0}^{x}(x-\xi)^{m-\beta-1}\frac{\partial^{m}u(\xi,t)}{\partial\xi^{m}}d\xi,\quad m=[\beta]+1,\ m\in N. \tag{38}$$

After taking the Laplace transform on both sides of (36) and (37), we get the following iteration formula:

$$U_{n+1}=U_{n}+\lambda(s)\left[s^{\alpha}U_{n}(x,s)-s^{\alpha-1}u(x,0)+L\left[u_{n}\frac{\partial^{\beta}u_{n}(x,t)}{\partial x^{\beta}}-\left(x+xt^{2}\right)\right]\right]. \tag{39}$$

As a result, after the identification of a Lagrange multiplier $\lambda(s)=-1/s^{\alpha}$, and with the inverse-Laplace transform, one can derive

$$u_{n+1}(x,y,t)=u_{0}(x,y,t)-L\left[u_{n}\frac{\partial^{\beta}u_{n}(x,t)}{\partial x^{\beta}}\right] \tag{40}$$

$u_0(x,y,t)$ is an initial approximation of (36), and

$$u_{0}(x,t)=L^{-1}\left[\frac{1}{s^{\alpha}}\left[L\left[x+xt^{2}\right]\right]\right]. \tag{41}$$

Applying the variational homotopy perturbation method, one has

$$\sum_{n=0}^{\infty}p^{n}u_{n}(x,t)=u_{0}(x,t)-p\left[L^{-1}\left[\frac{1}{s^{\alpha}}\left[L\left[\sum_{n=0}^{\infty}p^{n}\mathscr{H}_{n}(u)\right]\right]\right]\right], \tag{42}$$

where $\mathscr{H}_n(u)$ is He's polynomials that represent nonlinear term $u(\partial^{\beta}u(x,t)/\partial x^{\beta})$; we have a few terms of the He's polynomials for $u(\partial^{\beta}u(x,t)/\partial x^{\beta})$ which are given by

$$\begin{aligned}
\mathscr{H}_{0}(u)&=u_{0}\frac{\partial^{\beta}u_{0}}{\partial x^{\beta}},\\
\mathscr{H}_{1}(u)&=u_{0}\frac{\partial^{\beta}u_{1}}{\partial x^{\beta}}+u_{1}\frac{\partial^{\beta}u_{0}}{\partial x^{\beta}},\\
\mathscr{H}_{2}(u)&=u_{0}\frac{\partial^{\beta}u_{2}}{\partial x^{\beta}}+u_{1}\frac{\partial^{\beta}u_{1}}{\partial x^{\beta}}+u_{2}\frac{\partial^{\beta}u_{0}}{\partial x^{\beta}},\\
&\vdots
\end{aligned} \tag{43}$$

Comparing the coefficient with identical powers in p, one has

$$\begin{aligned}
u_0(x,t) &= \frac{xt^{\alpha}}{\Gamma(1+\alpha)} + \frac{2xt^{\alpha+2}}{\Gamma(3+\alpha)},\\
u_1 &= -L^{-1}\left[\frac{1}{s^{\alpha}}\left[L\left[u_0\frac{\partial^{\beta}u_0}{\partial x^{\beta}}\right]\right]\right]\\
&= -\frac{t^{3\alpha}x^{2-\beta}\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)\Gamma(1+3\alpha)\Gamma(2-\beta)}\\
&\quad - \frac{4t^{4+3\alpha}x^{2-\beta}\Gamma(5+2\alpha)}{\Gamma^2(3+\alpha)\Gamma(5+3\alpha)\Gamma(2-\beta)}\\
&\quad - \frac{4t^{2+3\alpha}x^{2-\beta}\Gamma(3+2\alpha)}{\Gamma(1+\alpha)\Gamma(3+\alpha)\Gamma(3+3\alpha)\Gamma(2-\beta)},\\
u_2 &= -L^{-1}\left[\frac{1}{s^{\alpha}}\left[L\left[u_0\frac{\partial^{\beta}u_1}{\partial x^{\beta}} + u_1\frac{\partial^{\beta}u_0}{\partial x^{\beta}}\right]\right]\right]\\
&= \frac{t^{5\alpha}x^{3-2\beta}\Gamma(1+2\alpha)\Gamma(1+4\alpha)}{\Gamma^3(1+\alpha)\Gamma(1+3\alpha)\Gamma(1+5\alpha)\Gamma^2(2-\beta)}\\
&\quad + \frac{2t^{2+5\alpha}x^{3-2\beta}\Gamma(1+2\alpha)\Gamma(3+4\alpha)}{\Gamma^2(1+\alpha)\Gamma(3+\alpha)\Gamma(1+3\alpha)\Gamma(3+5\alpha)\Gamma^2(2-\beta)}\\
&\quad + \frac{4t^{2+5\alpha}x^{3-2\beta}\Gamma(3+2\alpha)\Gamma(3+4\alpha)}{\Gamma^2(1+\alpha)\Gamma(3+\alpha)\Gamma(3+3\alpha)\Gamma(3+5\alpha)\Gamma^2(2-\beta)}\\
&\quad + \frac{8t^{4+5\alpha}x^{3-2\beta}\Gamma(3+2\alpha)\Gamma(5+4\alpha)}{\Gamma(1+\alpha)\Gamma^2(3+\alpha)\Gamma(3+3\alpha)\Gamma(5+5\alpha)\Gamma^2(2-\beta)}\\
&\quad + \frac{4t^{4+5\alpha}x^{3-2\beta}\Gamma(5+2\alpha)\Gamma(5+4\alpha)}{\Gamma(1+\alpha)\Gamma^2(3+\alpha)\Gamma(5+3\alpha)\Gamma(5+5\alpha)\Gamma^2(2-\beta)}\\
&\quad + \frac{8t^{6+5\alpha}x^{3-2\beta}\Gamma(5+2\alpha)\Gamma(7+4\alpha)}{\Gamma^3(3+\alpha)\Gamma(5+3\alpha)\Gamma(7+5\alpha)\Gamma^2(2-\beta)}\\
&\quad + \left(4t^{2+5\alpha}x^{3-2\beta}\Gamma(3+2\alpha)\Gamma(3+4\alpha)\Gamma(3-\beta)\right)\\
&\quad \times\left(\Gamma^2(1+\alpha)\Gamma(3+\alpha)\Gamma(3+3\alpha)\right.\\
&\qquad \left.\times\Gamma(3+5\alpha)\Gamma(3-2\beta)\Gamma(2-\beta)\right)^{-1}\\
&\quad + \left(8t^{4+5\alpha}x^{3-2\beta}\Gamma(3+2\alpha)\Gamma(5+4\alpha)\Gamma(3-\beta)\right)\\
&\quad \times\left(\Gamma(1+\alpha)\Gamma^2(3+\alpha)\Gamma(3+3\alpha)\right.\\
&\qquad \left.\times\Gamma(5+5\alpha)\Gamma(3-2\alpha)\Gamma(2-\beta)\right)^{-1}\\
&\quad + \left(4t^{4+5\alpha}x^{3-2\beta}\Gamma(5+2\alpha)\Gamma(5+4\alpha)\Gamma(3-\beta)\right)\\
&\quad \times\left(\Gamma(1+\alpha)\Gamma^2(3+\alpha)\Gamma(5+3\alpha)\right.\\
&\qquad \left.\times\Gamma(5+5\alpha)\Gamma(3-2\beta)\Gamma(2-\beta)\right)^{-1}\\
&\quad + \left(8t^{6+5\alpha}x^{3-2\beta}\Gamma(5+2\alpha)\Gamma(7+4\alpha)\Gamma(3-\beta)\right)\\
&\quad \times\left(\Gamma^3(1+\alpha)\Gamma(5+3\alpha)\Gamma(7+5\alpha)\right.\\
&\qquad \left.\times\Gamma(3-2\alpha)\Gamma(2-\beta)\right)^{-1},\\
&\ \ \vdots
\end{aligned} \tag{44}$$

The solution of (36) and (37) is given as $u(x,t) = u_0 + u_1 + u_2 + \cdots$. If we take $\alpha = \beta = 1$, one has

$$\begin{aligned}
u_0 &= xt + \frac{t^3x}{3},\\
u_1 &= -\frac{t^3x}{3} - \frac{2t^5x}{15} - \frac{t^7x}{63},\\
u_2 &= \frac{2t^5x}{15} + \frac{22t^7x}{315} + \frac{38t^9x}{2835} + \frac{2t^{11}x}{2079},\\
&\ \ \vdots
\end{aligned} \tag{45}$$

The noise terms $-(t^3x/3)$ between the components u_0 and u_1 can be canceled and the remaining term of u_0 still satisfies the equation. For this special case, the exact solution is therefore $u(x,t) = tx$ which was given in [25].

Example 3. Consider the following timefractional nonlinear system arising in thermoelasticity [26]:

$$\begin{aligned}
&\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} - a(u_x,\theta)u_{xx} + b(u_x,\theta)\theta_x = f(x,t),\\
&c(u_x,\theta)\frac{\partial^{\beta}v(x,t)}{\partial t^{\beta}} + b(u_x,\theta)u_{xt} - d(u_x,\theta)\theta_{xx} = g(x,t),
\end{aligned} \tag{46}$$

where $t > 0$, $x \in R^1$, $1 < \alpha \leq 2$, $0 < \beta \leq 1$, and the time fractional derivatives defined here are in Caputo sense. a, b, c, and d are defined by

$$\begin{aligned}
a(u_x,\theta) &= 2 - u_x\theta, \qquad b(u_x,\theta) = 2 + u_x\theta,\\
c(u_x,\theta) &= 1, \qquad d(u_x,\theta) = \theta,
\end{aligned} \tag{47}$$

and the right-hand side of (46) is replaced by

$$\begin{aligned}
f(x,t) &= \frac{2}{1+x^2} - \frac{2(1+t^2)(3x^2-1)}{(1+x^2)^3}a(w,v)\\
&\quad - \frac{2x(1+t)}{(1+x^2)^2}b(w,v),\\
g(x,t) &= \frac{1}{1+x^2}c(w,v) - \frac{4xt}{(1+x^2)^2b(w,v)}\\
&\quad - \frac{2(1+t)(3x^2-1)}{(1+x^2)^3}d(w,v),
\end{aligned} \tag{48}$$

where a, b, c, and d are defined above and

$$w \equiv w(x,t) = \frac{2x\left(1+t^2\right)}{\left(1+x^2\right)^2}, \qquad w \equiv w(x,t) = \frac{1+t}{1+x^2}, \tag{49}$$

with the initial conditions

$$u(x,0) = \frac{1}{1+x^2}, \qquad u_t(x,0) = 0, \qquad v(x,0) = \frac{1}{1+x^2}; \tag{50}$$

thus the exact solution of system is $u(x,t) = (1+t^2)/(1+x^2)$, $\theta = (1+t)/(1+x^2)$. After taking the Laplace transform to both sides of (46) and (50), we get the following iteration formula:

$$\begin{aligned} U_{n+1}(x,s) &= U_n(x,s) + \lambda_1(s) \\ &\quad \times \left[s^{\alpha}U_n(x,s) - s^{\alpha-1}u(x,0) - s^{\alpha-2}u_t(x,0)\right. \\ &\quad - L\left[2u_{nxx} - 2\theta_{nx}\right] \\ &\quad \left.- L\left[u_{nx}\theta_n u_{nxx} + u_{nx}\theta_n\theta_{nx}\right]\right], \\ \Theta_{n+1}(x,s) &= \Theta_n(x,s) + \lambda_2(s) \\ &\quad \times \left[s^{\beta}U_n(x,s) - s^{\beta-1}u(x,0)\right. \\ &\quad + L\left[-2u_{nxt}\right] \\ &\quad \left.- L\left[u_{nx}\theta_n u_{nxt} - \theta_n\theta_{nxx}\right]\right], \end{aligned} \tag{51}$$

where $\Theta(x,s) = L[\theta(x,t)] = \int_0^{\infty} e^{-st}\theta(x,t)dt$. As a result, after the identification of a Lagrange multiplier $\lambda_1(s) = -1/s^{\alpha}$, $\lambda_2(s) = -1/s^{\beta}$ and with the inverse-Laplace transform, one can derive the following iteration formula:

$$\begin{aligned} u_{n+1} &= u_0 + L^{-1}\left[\frac{1}{s^{\alpha}}\left[L\left[2u_{nxx} - 2\theta_{nx}\right]\right.\right. \\ &\quad \left.\left. - L\left[u_{nx}\theta_n u_{nxx} + u_{nx}\theta_n\theta_{nx}\right]\right]\right], \\ \theta_{n+1} &= \theta_0 + L^{-1}\left[\frac{1}{s^{\beta}}\left[L\left[-2u_{nxt}\right]\right.\right. \\ &\quad \left.\left. - L\left[u_{nx}\theta_n u_{nxt} - \theta_n\theta_{nxx}\right]\right]\right], \end{aligned} \tag{52}$$

$u_0(x,t), v_0(x,t)$ is an initial approximation of (46), and

$$\begin{aligned} u_0(x,t) &= u(x,0) + L^{-1}\left[\frac{1}{s^{\alpha}}L\left[f(x,t)\right]\right], \\ \theta_0(x,t) &= \theta(x,0) + L^{-1}\left[\frac{1}{s^{\beta}}L\left[g(x,t)\right]\right]. \end{aligned} \tag{53}$$

Applying the variational homotopy perturbation method, one has

$$\begin{aligned} \sum_{n=0}^{\infty} p^n u_n &= u_0 + p \\ &\quad \times \left[L^{-1}\left[\frac{1}{s^{\alpha}}\left[L\left[2\sum_{n=0}^{\infty} p^n u_{nxx} - 2\sum_{n=0}^{\infty} p^n \theta_{nx}\right]\right.\right.\right. \\ &\quad - L\left[\sum_{n=0}^{\infty} p^n \mathscr{H}_{1n}(u,\theta)\right. \\ &\quad \left.\left.\left.\left. + \sum_{n=0}^{\infty} p^n \mathscr{H}_{2n}(u,\theta)\right]\right]\right]\right], \\ \sum_{n=0}^{\infty} p^n \theta_n &= \theta_0 + p\left[L^{-1}\left[\frac{1}{s^{\beta}}\left[L\left[-2\sum_{n=0}^{\infty} p^n u_{nxt}\right]\right.\right.\right. \\ &\quad - L\left[\sum_{n=0}^{\infty} p^n \mathscr{H}_{3n}(u,\theta)\right. \\ &\quad \left.\left.\left.\left. - \sum_{n=0}^{\infty} p^n \mathscr{H}_{4n}(u,\theta)\right]\right]\right]\right], \end{aligned} \tag{54}$$

where $\mathscr{H}_{in}(u,\theta)$, $i = 1,2,3,4$, is He's polynomials that represent nonlinear terms $u_x\theta u_{xx}, u_x\theta\theta_x, u_x\theta u_{xt}, \theta\theta_{xx}$, respectively; we have a few terms of the He's polynomials for these nonlinear terms which are given by

$$\begin{aligned} \mathscr{H}_{10}(u,\theta) &= u_{0x}\theta_0 u_{0xx}, \\ \mathscr{H}_{11}(u,\theta) &= u_{0x}\theta_0 u_{1xx} + u_{0x}\theta_1 u_{0xx} + u_{1x}\theta_0 u_{0xx}, \\ &\vdots \\ \mathscr{H}_{20}(u,\theta) &= u_{0x}\theta_0\theta_{0x}, \\ \mathscr{H}_{21}(u,\theta) &= u_{0x}\theta_1\theta_{0x} + u_{0x}\theta_0\theta_{1x} + u_{1x}\theta_0 u_{0x}, \\ &\vdots \\ \mathscr{H}_{30}(u,\theta) &= u_{0x}\theta_0 u_{0xt}, \\ \mathscr{H}_{31}(u,\theta) &= u_{0x}\theta_0 u_{1xt} + u_{0x}\theta_1 u_{0xt} + u_{1x}\theta_0 u_{0xt}, \\ &\vdots \\ \mathscr{H}_{40}(u,\theta) &= \theta_0\theta_{0xx}, \\ \mathscr{H}_{41}(u,\theta) &= \theta_0\theta_{1xx} + u_{0x}\theta_1 u_{0xx}, \\ &\vdots \end{aligned} \tag{55}$$

Comparing the coefficient with identical powers in p, one has

$$\begin{aligned}
u_0(x,t) &= \frac{1}{1+x^2} \\
&+ \left(\frac{4x-12x^3}{(1+x^2)^6} + \frac{4x^2}{(1+x^2)^5} + \frac{4-12x^2}{(1+x^2)^3} + \frac{4x}{(1+x^2)^2} + \frac{2}{1+x^2}\right)\frac{t^{\alpha}}{\Gamma(\alpha)} \\
&+ \left(\frac{4x-12x^3}{(1+x^2)^6} + \frac{8x^2}{(1+x^2)^5} - \frac{4}{(1+x^2)^5}\right) \\
&\times \frac{t^{1+\alpha}}{\Gamma(2+\alpha)} + \left(\frac{480x-1440x^3}{(1+x^2)^6}\right)\frac{t^{5+\alpha}}{\Gamma(6+\alpha)} \\
&+ \left(\frac{16x-48x^3}{(1+x^2)^6} + \frac{16x^2}{(1+x^2)^5} + \frac{8+24x^2}{(1+x^2)^3}\right) \\
&\times \frac{t^{2+\alpha}}{\Gamma(3+\alpha)} \\
&+ \left(\frac{48-144x^3}{(1+x^2)^6} + \frac{48x^2}{(1+x^2)^5}\right)\frac{t^{3+\alpha}}{\Gamma(4+\alpha)} \\
&+ \left(\frac{96x-288x^3}{(1+x^2)^6} + \frac{96x^2}{(1+x^2)^5}\right)\frac{t^{4+\alpha}}{\Gamma(5+\alpha)}, \\
\theta_0(x,t) &= \frac{1}{1+x^2} + \left(\frac{2-6x^2}{(1+x^2)^4} + \frac{1}{1+x^2}\right)\frac{t^{\beta}}{\Gamma(1+\beta)} \\
&+ \left(\frac{8x^2}{(1+x^2)^5} + \frac{4-12x^2}{(1+x^2)^4} - \frac{8x}{(1+x^2)^2}\right) \\
&\times \frac{t^{1+\beta}}{\Gamma(2+\beta)} \\
&+ \left(\frac{16x^2}{(1+x^2)^5} + \frac{4-12x^2}{(1+x^2)^4}\right)\frac{t^{2+\beta}}{\Gamma(3+\beta)} \\
&\times \frac{48x^2}{(1+x^2)^5}\frac{t^{3+\beta}}{\Gamma(4+\beta)} \\
&+ \frac{192x^2}{(1+x^2)^5}\frac{t^{4+\beta}}{\Gamma(5+\beta)}, \\
u_1(x,t) &= L^{-1}\left[\frac{1}{s^{\alpha}}\left[L\left[2u_{0xx} - 2\theta_{0x}\right]\right.\right. \\
&\qquad \left.\left. - L\left[u_{0x}\theta_0 u_{0xt} + u_{0x}\theta_0\theta_{0x}\right]\right]\right], \\
\theta_1(x,t) &= L^{-1}\left[\frac{1}{s^{\beta}}\left[L\left[-2u_{0xt}\right] - L\left[u_{0x}\theta_0 u_{0xt} - \theta_0\theta_{0xx}\right]\right]\right], \\
u_2 &= L^{-1}\left[\frac{1}{s^{\alpha}}\left[L\left[2u_{1xx} - 2\theta_{1x}\right]\right.\right. \\
&\qquad - L\left[u_{0x}\theta_0 u_{1xx} + u_{0x}\theta_1 u_{0xx}\right. \\
&\qquad + u_{1x}\theta_0 u_{0xx} + u_{0x}\theta_1\theta_{0x} \\
&\qquad \left.\left.\left. + u_{0x}\theta_0\theta_{1x} + u_{1x}\theta_0 u_{0x}\right]\right]\right], \\
\theta_2 &= L^{-1}\left[\frac{1}{s^{\beta}}\left[L\left[-2u_{1xt}\right]\right.\right. \\
&\qquad - L\left[u_{0x}\theta_0 u_{1xt} + u_{0x}\theta_1 u_{0xt}\right. \\
&\qquad + u_{1x}\theta_0 u_{0xt} - \theta_0\theta_{1xx} \\
&\qquad \left.\left.\left. + u_{0x}\theta_1 u_{0xx}\right]\right]\right], \\
&\vdots
\end{aligned} \tag{56}$$

and so on; in this manner the rest of components of the solution can be obtained using the Mathematica symbolic computation software for purpose of simlification, the approximate solutions are not listed here.

4. Conclusion

In this paper, a new modification of variational iteration method is considered, which is based on Laplace transform and homotopy perturbation method. The fractional lagrange multiplier is accurately determined by the Laplace transform and the nonlinear one can be easily handled by the use of He's polynomials. Several fractional nonlinear nonhomogeneous equations are analytically solved as examples and the methodology is demonstrated. Examples 1, 2, and 3 have been successfully solved. And the results show that this method is a powerful and reliable method for finding the solution of the fractional nonlinear equations.

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