Research Article

The Problem of Image Recovery by the Metric Projections in Banach Spaces

Yasunori Kimura¹ and Kazuhide Nakajo²

¹ Department of Information Science, Toho University, Miyama, Funabashi, Chiba 274-8510, Japan
² Sundai Preparatory School, Kanda-Surugadai, Chiyoda-ku, Tokyo 101-8313, Japan

Correspondence should be addressed to Yasunori Kimura; yasunori@is.sci.toho-u.ac.jp

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We consider the problem of image recovery by the metric projections in a real Banach space. For a countable family of nonempty closed convex subsets, we generate an iterative sequence converging weakly to a point in the intersection of these subsets. Our convergence theorems extend the results proved by Bregman and Crombez.

1. Introduction

Let $C_1, C_2, \ldots, C_r$ be nonempty closed convex subsets of a real Hilbert space $H$ such that $\bigcap_{i=1}^{r} C_i \neq \emptyset$. Then, the problem of image recovery may be stated as follows: the original unknown image $z$ is known a priori to belong to the intersection of $\bigcap_{i=1}^{r} C_i$; given only the metric projections $P_{C_i}$ of $H$ onto $C_i$ for $i = 1, 2, \ldots, r$, recover $z$ by an iterative scheme. Such a problem is connected with the convex feasibility problem and has been investigated by a large number of researchers.

Bregman [1] considered a sequence $\{x_n\}$ generated by cyclic projections, that is, $x_0 = x \in H, x_1 = P_{C_1}x, x_2 = P_{C_2}x_1, x_3 = P_{C_3}x_2, \ldots, x_n = P_{C_n}x_{n-1}, x_{n+1} = P_{C_{n+1}}x_n, x_{n+2} = P_{C_2}x_{n+1}, \ldots$. It was proved that $\{x_n\}$ converges weakly to an element of $\bigcap_{i=1}^{r} C_i$ for an arbitrary initial point $x \in H$.

Crombez [2] proposed a sequence $\{y_n\}$ generated by $y_0 = y \in H, y_{n+1} = \alpha_0 y_n + \sum_{i=1}^{r} \alpha_i (y_n + \lambda_i (P_{C_i}y_n - y_n))$ for $n = 0, 1, 2, \ldots$, where $0 < \alpha_i < 1$ for all $i = 0, 1, 2, \ldots, r$ with $\sum_{i=0}^{r} \alpha_i = 1$ and $0 < \lambda_i < 2$ for every $i = 1, 2, \ldots, r$. It was proved that $\{y_n\}$ converges weakly to an element of $\bigcap_{i=1}^{r} C_i$ for an arbitrary initial point $y \in H$.

Later, Kitahara and Takahashi [3] and Takahashi and Tamura [4] dealt with the problem of image recovery by convex combinations of nonexpansive retractions in a uniformly convex Banach space $E$. Alber [5] took up the problem of image recovery by the products of generalized projections in a uniformly convex and uniformly smooth Banach space $E$ whose duality mapping is weakly sequentially continuous (see also [6, 7]).

On the other hand, using the hybrid projection method proposed by Haugazeau [8] (see also [9–11] and references therein) and the shrinking projection method proposed by Takahashi et al. [12] (see also [13]), Nakajo et al. [14] and Kimura et al. [15] considered this problem by the metric projections and proved convergence of the iterative sequence to a common point of countable nonempty closed convex subsets in a uniformly convex and smooth Banach space $E$ and in a strictly convex, smooth, and reflexive Banach space $E$ having the Kadec-Klee property, respectively. Kohsaka and Takahashi [16] took up this problem by the generalized projections and obtained the strong convergence to a common point of a countable family of nonempty closed convex subsets in a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable (see also [17, 18]). Although these results guarantee the strong convergence, they need to use metric or generalized projections onto different subsets for each step, which are not given in advance.

In this paper, we consider this problem by the metric projections, which are one of the most familiar projections to deal with. The advantage of our results is that we use projections onto the given family of subsets only, to generate
the iterative scheme. Our convergence theorems extend the results of [1, 2] to a Banach space $E$, and they deduce the weak convergence to a common point of a countable family of nonempty closed convex subsets of $E$.

There are a number of results dealing with the image recovery problem from the aspects of engineering using nonlinear functional analysis (see, e.g., [19]). Comparing with these researches, we may say that our approach is more abstract and theoretical; we adopt a general Banach space with several conditions for an underlying space, and therefore, the technique of the proofs can be applied to various mathematical results related to nonlinear problems defined on Banach spaces.

2. Preliminaries

Throughout this paper, let $\mathbb{N}$ be the set of all positive integers, $\mathbb{R}$ the set of all real numbers, $E$ a real Banach space with norm $\| \cdot \|$, and $E^*$ the dual of $E$. For $x \in E$ and $x^* \in E^*$, we denote by $\langle x, x^* \rangle$ the value of $x^*$ at $x$. We write $x_n \rightharpoonup x$ to indicate that a sequence $\{x_n\}$ converges weakly to $x$. Similarly, $x_n \rightharpoonup^* x$ and $x_n \rightharpoonup x$ will symbolize weak and weak* convergence, respectively. We define the modulus $\delta_E$ of convexity of $E$ as follows: $\delta_E$ is a function of $[0,2]$ into $[0,1]$ such that

$$\delta_E(\varepsilon) = \inf \left\{ 1 - \frac{\| x + y \|}{2} : x, y \in E, \| x \| = 1, \| y \| = 1, \| x - y \| \geq \varepsilon \right\}$$

for every $\varepsilon \in [0,2]$. $E$ is called uniformly convex if $\delta_E(\varepsilon) > 0$ for each $\varepsilon > 0$. Let $p > 1$. $E$ is said to be $p$-uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\varepsilon) \geq c\varepsilon$ for every $\varepsilon \in [0,2]$. It is obvious that a $p$-uniformly convex Banach space is uniformly convex. $E$ is said to be strictly convex if $\| x + y \|/2 < 1$ for all $x, y \in E$ with $\| x \| = \| y \| = 1$ and $x \neq y$. We know that a uniformly convex Banach space is strictly convex and reflexive. For every $p > 1$, the (generalized) duality mapping $J_p : E \to 2^{E^*}$ of $E$ is defined by

$$J_p x = \{ y^* \in E^* : \langle x, y^* \rangle = \| y^* \|^p, \| y^* \| = \| x \|^{p-1} \}$$

for all $x \in E$. When $p = 2$, $J_2$ is called the normalized duality mapping. We have that for $p, q > 1$, $\| x \|^p J_p x = \| x \|^q J_q x$ for all $x \in E$. $E$ is said to be smooth to the limit

$$\lim_{t \to 0} \frac{\| x + ty \| - \| x \|}{t}$$

exists for every $x, y \in E$ with $\| x \| = \| y \| = 1$. We know that the duality mapping $J_p$ of $E$ is single valued for each $p > 1$. Then $J_p x \cap J_q y = \emptyset$ for all $p > 1$. If $E$ is reflexive, then $J_p$ is surjective, and $J_2$ is identical to the duality mapping $J_2^* : E^* \to 2^E$ defined by

$$J_2^* y^* = \{ x \in E : \langle x, y^* \rangle = \| y^* \|^q, \| x \| = \| y^* \|^{q-1} \}$$

for every $y^* \in E^*$, where $q = p/(p+1)$ and $p > 2$. Theorem 1. Let $E$ be a smooth Banach space and $p > 1$. Then, $E$ is $p$-uniformly convex if and only if there exists a constant $c > 0$ such that $\| x + y \|^p \geq \| x \|^p + \| y \|^p + c\| y \|^p$ holds for every $x, y \in E$.

Remark 2. For a $p$-uniformly convex and smooth Banach space $E$, we have that the constant $c$ in the theorem above satisfies $c \leq 1$. Let

$$c_0 = \sup \{ c > 0 : \| x + y \|^p \geq \| x \|^p + \| y \|^p + c\| y \|^p \forall x, y \in E \}.$$
Lemma 3. Let $p > 1$ and $E$ be a $p$-uniformly convex and smooth Banach space. Then, for each $x, y \in E$,

$$\phi_p(x, y) \geq c_0\|x - y\|^p \tag{8}$$

holds, where $c_0$ is maximum in Remark 2.

Proof. Let $x, y \in E$. By Theorem 1, we have

$$\|x\|^p \geq \|y\|^p + p\langle x - y, J_p y \rangle + c_0\|x - y\|^p, \tag{9}$$

where $c_0$ is maximum in Remark 2. Hence, we get

$$\phi_p(x, y) = \|x\|^p - \|y\|^p - p\langle x - y, J_p y \rangle \geq c_0\|x - y\|^p, \tag{10}$$

which is the desired result. \qed

Let $C$ be a nonempty closed convex subset of a strictly convex and reflexive Banach space $E$, and let $x \in E$. Then, there exists a unique element $x_0 \in C$ such that $\|x_0 - x\| = \inf_{y \in C} \|y - x\|$. Putting $x_0 = P_Cx$, we call $P_C$ the metric projection onto $C$ (see [24]). We have the following result [25, p. 196] for the metric projection onto $C$.

Lemma 4. Let $C$ be a nonempty closed convex subset of a strictly convex, reflexive, and smooth Banach space $E$, and let $x \in E$. Then, $y = P_Cx$ if and only if $\langle y - z, J_p(x - y) \rangle \geq 0$ for all $z \in C$, where $P_C$ is the metric projection onto $C$.

Remark 5. For $p > 1$, it holds that $\|x\|/p \cdot x = \|x\|^{p-1}/p \cdot x$ for every $x \in E$. Therefore, under the same assumption as Lemma 4, we have that $y = P_Cx$ if and only if $\langle y - z, J_p(x - y) \rangle \geq 0$ for all $z \in C$.

3. Main Results

Firstly, we consider the iteration of Crombez's type and get the following result.

Theorem 6. Let $p, q > 1$ be such that $1/p + 1/q = 1$. Let $\{C_n\}_{n \in \mathbb{N}}$ be a family of nonempty closed convex subsets of a $p$-uniformly convex and smooth Banach space $E$ whose duality mapping $J_p$ is weakly sequentially continuous. Suppose that $\bigcap_{n \in \mathbb{N}} C_n \neq \emptyset$. Let $\lambda_{nk} \in [0, 1]$, $\alpha_{nk} \in (0, 1)$ for all $n \in \mathbb{N}$ and $k = 1, 2, \ldots, n$ with $\sum_{k=1}^{n} \alpha_{nk} = 1$ for every $n \in \mathbb{N}$, where $c_0$ is maximum in Remark 2. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$ and

$$x_{n+1} = J_p^*\left(\sum_{k=1}^{n} \alpha_{nk} (J_p x_n - \lambda_{nk} J_p (x_n - P_{C_k} x_n))\right) \tag{11}$$

for every $n \in \mathbb{N}$. If $0 < \liminf_{n \to \infty} \lambda_{nk} \leq \limsup_{n \to \infty} \lambda_{nk} < 1/(1/(p-1))^{p-1} c_0$ and $\liminf_{n \to \infty} \alpha_{nk} > 0$ for each $k \in \mathbb{N}$, then $\{x_n\}$ converges weakly to a point in $\bigcap_{n=1}^{\infty} C_n$.

Proof. Let $y_{nk} = J_p^*\left(J_p x_n - \lambda_{nk} J_p (x_n - P_{C_k} x_n)\right)$ for $n \in \mathbb{N}$ and $k = 1, 2, \ldots, n$. Then, for $z \in \bigcap_{n \in \mathbb{N}} C_n$, we obtain

$$\phi_p(z, y_{nk}) - \phi_p(z, x_n)$$

$$= -\phi_p(y_{nk}, x_n) + p\langle y_{nk} - z, J_p y_{nk} - J_p x_n \rangle$$

$$= -\phi_p(y_{nk}, x_n) - p\lambda_{nk}\langle y_{nk} - z, J_p (x_n - P_{C_k} x_n) \rangle$$

$$= -\phi_p(y_{nk}, x_n) - p\lambda_{nk}\langle y_{nk} - x_n, J_p (x_n - P_{C_k} x_n) \rangle$$

$$- p\lambda_{nk}\langle x_n - z, J_p (x_n - P_{C_k} x_n) \rangle \tag{12}$$

for all $n \in \mathbb{N}$ and $k = 1, 2, \ldots, n$. Using Remark 5 with that $z \in C_k$, we get

$$\langle x_n - z, J_p (x_n - P_{C_k} x_n) \rangle$$

$$= \langle x_n - P_{C_k} x_n, J_p (x_n - P_{C_k} x_n) \rangle$$

$$+ \langle P_{C_k} x_n - z, J_p (x_n - P_{C_k} x_n) \rangle$$

$$\geq \|x_n - P_{C_k} x_n\|^p \tag{13}$$

for every $n \in \mathbb{N}$ and $k = 1, 2, \ldots, n$. Thus, by Lemma 3 we have

$$\phi_p(z, y_{nk}) - \phi_p(z, x_n)$$

$$\leq -c_0\|y_{nk} - x_n\|^p$$

$$- p\lambda_{nk}\|y_{nk} - x_n\| - p\lambda_{nk}\|y_{nk} - x_n\|$$

$$\leq -c_0\|y_{nk} - x_n\|^p + p\lambda_{nk}\|y_{nk} - x_n\|\|x_n - P_{C_k} x_n\|^{p-1}$$

$$- p\lambda_{nk}\|x_n - P_{C_k} x_n\|^p \tag{14}$$

for each $n \in \mathbb{N}$ and $k = 1, 2, \ldots, n$. Since it holds that

$$st \leq \frac{1}{\beta^p} + \beta^{p-1}\frac{t^p}{q} \tag{15}$$

for $s, t \geq 0, p, q > 1$ with $1/p + 1/q = 1$, and $\beta > 0$, we have

$$\|y_{nk} - x_n\|\|x_n - P_{C_k} x_n\|^{p-1}$$

$$\leq \frac{1}{\beta_k^p}\|y_{nk} - x_n\|^p$$

$$+ \beta_k^{1/(p-1)}\frac{t^p}{p}\|x_n - P_{C_k} x_n\|^p \tag{16}$$

for every $k \in \mathbb{N}$, $\beta_k > 0$ and $n \geq k$. Therefore, it follows that

$$\phi_p(z, y_{nk}) - \phi_p(z, x_n)$$

$$\leq \left(\frac{\lambda_{nk}}{\beta_k} - c_0\right)\|y_{nk} - x_n\|^p$$

$$+ \lambda_{nk}\left((p - 1)\beta_k^{1/(p-1)} - p\right)\|x_n - P_{C_k} x_n\|^p \tag{17}$$
for every \( n \in \mathbb{N}, k = 1, 2, \ldots, n, \) and \( \beta_k > 0. \) Since
\[
\phi_p(z, x_{n+1}) - \phi_p(z, x_n) \\
\leq \sum_{k=1}^{n} \alpha_{n,k} \left( \frac{\lambda_{n,k}}{\beta_k} - c_0 \right) \| y_{n,k} - x_n \|^p \\
+ \sum_{k=1}^{n} \alpha_{n,k} \left( (p - 1) \beta_k^{1/(p-1)} - p \right) \left\| (p - 1) \beta_k^{1/(p-1)} - p \right\| \| x_n - P_{C_i} x_n \|^p
\]
for every \( n \in \mathbb{N}, \) we have
\[
\phi_p(z, x_{n+1}) - \phi_p(z, x_n) \\
= \sum_{k=1}^{n} \alpha_{n,k} \phi_p(z, y_{n,k})
\]
for every \( n \in \mathbb{N}, \) we have
\[
\mu_1 - \mu_2 = \lim_{i \to \infty} \left( \phi_p(u_i, x_n) - \phi_p(u_{i+1}, x_n) \right)
\]
and \( J_p \) is weakly sequentially continuous, we have
\[
\mu_1 - \mu_2 = \| u_1 \|^p - \| u_2 \|^p + p \left\langle u_2 - u_1, J_p x_n \right\rangle
\]
So, we have that \( \{x_n\} \) is bounded from Lemma 3. Let \( \{x_m\} \) be a subsequence of \( \{x_n\} \) such that \( x_m \to u \). For fixed \( j \in I \), there exists a strictly increasing sequence \( \{m_k\} \subset \mathbb{N} \) such that \( n_k \leq m_k \leq n_k + M_j - 1 \) and \( i(m_k) = j \) for every \( k \in \mathbb{N} \). It follows that

\[
\|x_{m_k} - x_m\| \leq \sum_{l=m_k}^{m_k+M_j-1} \|x_{l+1} - x_l\| \tag{29}
\]

for all \( k \in \mathbb{N} \) which implies that \( x_{m_k} \to u \). Since \( \lim_{k \to \infty} \|x_{m_k} - P_{C_j}x_{m_k}\| = 0, u \in C_j \) for every \( j \in I \). So, we get \( u \in \bigcap_{j \in I} C_j \). As in the proof of Theorem 6, using that \( J_p \) is weakly sequentially continuous, we get that \( \{x_n\} \) converges weakly to a point in \( \bigcap_{j \in I} C_j \).

Suppose that the index set \( I \) is a finite set \( \{0, 1, 2, \ldots, N - 1\} \). For the cyclic iteration, the index mapping \( i \) is defined by \( i(j) = j \mod N \) for each \( j \in I \). Clearly it satisfies the assumption in Theorem 7. In the case where the index set \( I \) is countably infinite, that is, \( I = \mathbb{N}, \) one of the simplest examples of \( i : \mathbb{N} \to \mathbb{N} \) can be defined as follows:

\[
i(n) =
\begin{aligned}
1 & \quad (n = 2m - 1 \text{ for some } m \in \mathbb{N}), \\
2 & \quad (n = 2(2m - 1) \text{ for some } m \in \mathbb{N}), \\
3 & \quad (n = 4(2m - 1) \text{ for some } m \in \mathbb{N}), \\
& \quad \ldots, \\
& \quad (n = 2^{k-1}(2m - 1) \text{ for some } m \in \mathbb{N}), \\
& \quad \ldots
\end{aligned}
\tag{30}
\]

Then, the assumption in Theorem 7 is satisfied by letting \( M_j = 2^j \) for each \( j \in I = \mathbb{N} \).

4. Deduced Results

Since a real Hilbert space \( H \) is 2-uniformly convex and the maximum \( c_0 \) in Remark 2 is equal to 1, we get the following results. At first, we have the following theorem which generalizes the results of [2] by Theorem 6.

**Theorem 8.** Let \( \{C_n\}_{n \in \mathbb{N}} \) be a family of nonempty closed convex subsets of \( H \) such that \( \bigcap_{n \in \mathbb{N}} C_n \neq \emptyset \). Let \( \lambda_{n,k} \in [0,2] \) and \( \alpha_{n,k} \in [0,1] \) for all \( n \in \mathbb{N} \) and \( k = 1, 2, \ldots, n \) with \( \sum_{k=1}^{n} \alpha_{n,k} = 1 \) for every \( n \in \mathbb{N} \). Let \( \{x_n\} \) be a sequence generated by \( x_1 = x \in H \) and

\[
x_{n+1} = \sum_{k=1}^{n} \alpha_{n,k} (x_n - \lambda_{n,k} (x_n - P_{C_j} x_n))
\tag{31}
\]

for every \( n \in \mathbb{N} \). If it holds that

\[
0 < \liminf_{n \to \infty} \lambda_{n,k} \leq \limsup_{n \to \infty} \lambda_{n,k} < 2 \quad \text{and} \quad \liminf_{n \to \infty} \alpha_{n,k} > 0
\]

for each \( k \in \mathbb{N} \), then \( \{x_n\} \) converges weakly to a point in \( \bigcap_{j \in I} C_j \).

Next, we have the following theorem which extends the result of [1] by Theorem 7.

**Theorem 9.** Let \( I \) be a countable set and \( \{C_j\}_{j \in I} \) a family of nonempty closed convex subsets of \( H \) such that \( \bigcap_{j \in I} C_j \neq \emptyset \). Let \( \lambda_n \in [0,2] \) for all \( n \in \mathbb{N} \), and let \( \{x_n\} \) be a sequence generated by

\[
x_{n+1} = x_n - \lambda_n \left( x_n - P_{C_{i(n)}} x_n \right)
\tag{32}
\]

for every \( n \in \mathbb{N} \), where the index mapping \( i : \mathbb{N} \to I \) satisfies that, for every \( j \in I \), there exists \( M_j \in \mathbb{N} \) such that \( j \in [i(n), i(n+M_j-1)] \) for each \( n \in \mathbb{N} \). If \( 0 < \liminf_{n \to \infty} \lambda_{n,k} \leq \limsup_{n \to \infty} \lambda_{n,k} < 2 \), then \( \{x_n\} \) converges weakly to a point in \( \bigcap_{j \in I} C_j \).

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**References**


