

Research Article

Nonlocal Integrodifferential Boundary Value Problem for Nonlinear Fractional Differential Equations on an Unbounded Domain

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This paper investigates the existence of nonnegative solutions for nonlinear fractional differential equations with nonlocal fractional integrodifferential boundary conditions on an unbounded domain by means of Leray-Schauder nonlinear alternative theorem. An example is discussed for the illustration of the main work.

1. Introduction

Recent studies on fractional differential equations, appeared in several special issues and books, reveal an extensive development of various aspects of the subject. One of the reasons for the popularity of fractional calculus is the nonlocal behavior of fractional-order operators in contrast to the classical integer-order operators. This characteristic has motivated many experts on modelling to introduce the concept of fractional modelling by taking into account the ideas of fractional calculus. Examples include various disciplines of science and engineering such as physics, chemistry, biomathematics, dynamical processes in porous media, dynamics of earthquakes, material viscoelastic theory, and control theory of dynamical systems. Furthermore, the outcome of certain experimentations indicate that integral and derivative operators of fractional order possess some characteristics related to complex systems having long memory in time. For details and examples, we refer the reader to the works in [1–7].

Boundary value problems of fractional-order differential equations have been extensively investigated during the last few years, and a variety of results on the topic have been

established. A great deal of the work on fractional boundary value problems involves local/nonlocal boundary conditions on bounded and unbounded domains; for example, see [8–26].

In this paper, we study a new class of problems on fractional differential equations with nonlocal boundary conditions on unbounded domains. Precisely, we consider the following problem:

$$\begin{aligned} D^\alpha u(t) + f(t, u(t)) &= 0, \quad 1 < \alpha \leq 2, \quad t \in J = [0, +\infty), \\ I^{2-\alpha} u(0) &= 0, \\ D^{\alpha-1} u(+\infty) &= \lambda I^{\alpha-1} u(\eta), \quad 0 < \lambda, \eta < \infty, \end{aligned} \quad (1)$$

where D^α denotes Riemann-Liouville fractional derivative of order α , $f \in C(J \times \mathbb{R}, \mathbb{R}^+)$, and $\mathbb{R}^+ = [0, +\infty)$.

2. Preliminaries

In this section, we present some useful definitions and related theorems.

Definition 1 (see [4]). The Riemann-Liouville fractional derivative of order δ for a continuous function f is defined by

$$D^\delta f(t) = \frac{1}{\Gamma(n-\delta)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\delta-1} f(s) ds, \tag{2}$$

$$n = [\delta] + 1,$$

Provided that the right hand side is pointwise defined on $(0, \infty)$ and $[\delta]$ is the integer part of δ .

Definition 2 (see [4]). The Riemann-Liouville fractional integral of order δ for a function f is defined as

$$I^\delta f(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} f(s) ds, \quad \delta > 0, \tag{3}$$

provided that such integral exists.

Theorem 3 (see [27] (Leray-Schauder nonlinear alternative)). *Let C be a convex subset of a Banach space, and let U be an open subset of C with $0 \in U$. Then every completely continuous map $N : \bar{U} \rightarrow C$ has at least one of the following two properties:*

- (1) N has a fixed point in \bar{U} ;
- (2) there is an $x \in \partial U$ and $\lambda \in (0, 1)$ with $x = \lambda Nx$.

Theorem 4 (see [28]). *Let $U \subset X$ be a bounded set. Then U is relatively compact in X if the following conditions hold:*

- (i) for any $u(t) \in U$, $u(t)/(1+t^{\alpha-1})$ is equicontinuous on any compact interval of J ;
- (ii) for any $\varepsilon > 0$, there exists a constant $T = T(\varepsilon) > 0$ such that

$$\left| \frac{u(t_1)}{1+t_1^{\alpha-1}} - \frac{u(t_2)}{1+t_2^{\alpha-1}} \right| < \varepsilon \tag{4}$$

for any $t_1, t_2 \geq T$ and $u \in U$.

Now we list the assumptions needed in the sequel.

- (H_1): $\Gamma(2\alpha - 1) > \lambda\eta^{2\alpha-2}$.
- (H_2): there exist nonnegative functions $a(t), b(t)$ defined on $[0, \infty)$ and a constant $\rho > 0$ such that

$$|f(t, u(t))| \leq a(t) + b(t) |u(t)|^\rho,$$

$$\int_0^{+\infty} a(t) dt = a^* < +\infty, \tag{5}$$

$$\int_0^{+\infty} b(t) (1+t^{\alpha-1})^\rho dt = b^* < +\infty.$$

3. Some Lemmas

This section contains some preliminary works that we need to establish the main result for problem (1).

Lemma 5. *Let $\sigma(t) \in C([0, +\infty))$ with $\int_0^\infty \sigma(s) ds < \infty$. For $\Gamma(2\alpha - 1) \neq \lambda\eta^{2\alpha-2}$, the associated linear fractional boundary value problem,*

$$D^\alpha u(t) + \sigma(t) = 0, \quad 1 < \alpha \leq 2,$$

$$I^{2-\alpha} u(0) = 0, \tag{6}$$

$$D^{\alpha-1} u(+\infty) = \lambda I^{\alpha-1} u(\eta),$$

has a unique solution given by

$$u(t) = \int_0^{+\infty} G(t, s) \sigma(s) ds, \tag{7}$$

where

$$G(t, s) = \begin{cases} -(\Gamma(2\alpha - 1) - \lambda\eta^{2\alpha-2})(t-s)^{\alpha-1} \\ \quad + (\Gamma(2\alpha - 1) - \lambda(\eta-s)^{2\alpha-2})t^{\alpha-1}, & s \leq t, s \leq \eta, \\ \frac{1}{\Delta} \left\{ \begin{aligned} &(\Gamma(2\alpha - 1) - \lambda(\eta-s)^{2\alpha-2})t^{\alpha-1}, & 0 \leq t \leq s \leq \eta, \\ &-(\Gamma(2\alpha - 1) - \lambda\eta^{2\alpha-2})(t-s)^{\alpha-1} \\ &\quad + \Gamma(2\alpha - 1)t^{\alpha-1}, & 0 \leq \eta \leq s \leq t, \\ &\Gamma(2\alpha - 1)t^{\alpha-1}, & s \geq t, s \geq \eta, \end{aligned} \right. \end{cases} \tag{8}$$

$$\Delta = \Gamma(\alpha) (\Gamma(2\alpha - 1) - \lambda\eta^{2\alpha-2}). \tag{9}$$

Proof. It is well known that the fractional equation in (6) is equivalent to the integral equation:

$$u(t) = -I^\alpha \sigma(t) + c_1 t^{\alpha-1} + c_0 t^{\alpha-2}, \tag{10}$$

where $c_0, c_1 \in \mathbb{R}$ are arbitrary constants. From (10), we have

$$D^{\alpha-1} u(t) = -I^1 \sigma(t) + c_1 \Gamma(\alpha) = c_1 \Gamma(\alpha) - \int_0^t \sigma(s) ds,$$

$$D^{\alpha-2} u(t) = -I^2 \sigma(t) + c_1 \Gamma(\alpha - 1)t + c_0 \Gamma(\alpha - 1). \tag{11}$$

Using the given boundary conditions in (10), we find that $c_0 = 0$ and

$$c_1 = \frac{1}{\Gamma(\alpha) - A} \int_0^\infty \sigma(s) ds - \frac{\lambda}{\Gamma(\alpha) - A}$$

$$\times \int_0^\eta \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left[\int_0^s \frac{(s-x)^{\alpha-1}}{\Gamma(\alpha)} \sigma(x) dx \right] ds \tag{12}$$

$$= \frac{1}{\Gamma(\alpha) - A} \int_0^\infty \sigma(s) ds - \frac{\lambda}{\Gamma(\alpha) - A} I^{2\alpha-1} \sigma(\eta),$$

where

$$A = \lambda \int_0^\eta \frac{s^{\alpha-1} (\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds = \frac{\lambda \Gamma(\alpha) \eta^{2\alpha-2}}{\Gamma(2\alpha-1)}. \tag{13}$$

Substituting the values of c_1, c_2 into (10) gives

$$\begin{aligned}
 u(t) &= - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds - \frac{\lambda t^{\alpha-1}}{\Gamma(\alpha) - A} I^{2\alpha-1} \sigma(\eta) \\
 &\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha) - A} \int_0^\infty \sigma(s) ds \\
 &= - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds - \frac{\Gamma(2\alpha-1) \lambda t^{\alpha-1}}{\Gamma(\alpha) (\Gamma(2\alpha-1) - \lambda \eta^{2\alpha-2})} \\
 &\quad \times \int_0^\eta \frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} \sigma(s) ds + \frac{\Gamma(2\alpha-1) t^{\alpha-1}}{\Gamma(\alpha) (\Gamma(2\alpha-1) - \lambda \eta^{2\alpha-2})} \\
 &\quad \times \int_0^\infty \sigma(s) ds \\
 &= \int_0^\infty G(t,s) \sigma(s) ds,
 \end{aligned} \tag{14}$$

where $G(t,s)$ is defined by (8). □

Remark 6. In view of the assumption (H_1) , Green's function $G(t,s)$ satisfies the properties:

(1) $G(t,s) \geq 0$,

(2)

$$\frac{G(t,s)}{1+t^{\alpha-1}} \leq \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha) [\Gamma(2\alpha-1) - \lambda \eta^{2\alpha-2}]} \triangleq L. \tag{15}$$

For the forthcoming analysis, we introduce a space

$$X = \left\{ u \in C(J, \mathbb{R}) : \sup_{t \in J} \frac{|u(t)|}{1+t^{\alpha-1}} < +\infty \right\}, \tag{16}$$

equipped with the norm

$$\|u\|_X = \sup_{t \in J} \frac{|u(t)|}{1+t^{\alpha-1}}. \tag{17}$$

Notice that X is a Banach space.

Define an operator $T : X \rightarrow X$ as follows:

$$Tu(t) = \int_0^\infty G(t,s) f(s, u(s)) ds. \tag{18}$$

Observe that problem (1) has a solution only if the operator T has a fixed point.

Lemma 7. *If $(H_1), (H_2)$ hold, then the operator $T : X \rightarrow X$ is completely continuous.*

Proof. We divide the proof into several steps.

(i) The operator $T : X \rightarrow X$ is uniformly bounded. Let Ω be any bounded subset of X ; then there exists a constant L_1 such that $\|u\|_X \leq L_1$. By (H_2) , we have

$$\begin{aligned}
 \|Tu\|_X &= \sup_{t \in J} \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} |f(s, u(s))| ds \\
 &\leq L \int_0^\infty \left[a(s) + b(s) (1+s^{\alpha-1})^\rho \frac{|u(s)|^\rho}{(1+s^{\alpha-1})^\rho} \right] ds \\
 &\leq L (a^* + b^* L_1^\rho) \\
 &< \infty.
 \end{aligned} \tag{19}$$

This shows that $T\Omega$ is uniformly bounded.

(ii) $T : X \rightarrow X$ is continuous. Take $u_n, u \in X$ such that $\|u_n\|_X < \infty, \|u\|_X < \infty$, and $u_n \rightarrow u$ as $n \rightarrow \infty$. Then, by (H_2) , we have

$$\begin{aligned}
 \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} f(s, u_n(s)) ds &\leq L \int_0^\infty [a(s) + b(s) |u_n(s)|^\rho] ds \\
 &\leq La^* + Lb^* \|u_n\|_X^\rho < \infty,
 \end{aligned} \tag{20}$$

where L is defined by (15).

By the Lebesgue dominated convergence theorem and continuity of f , we obtain

$$\lim_{n \rightarrow \infty} \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} f(s, u_n(s)) ds = \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} f(s, u(s)) ds. \tag{21}$$

Taking the limit $n \rightarrow \infty$, we get

$$\begin{aligned}
 \|Tu_n - Tu\|_X &= \sup_{t \in J} \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} \\
 &\quad \times |f(s, u_n(s)) - f(s, u(s))| ds \rightarrow 0.
 \end{aligned} \tag{22}$$

Therefore, T is continuous.

(iii) $T : X \rightarrow X$ is equicontinuous. We consider two cases.

(a) Let $I \subset J$ be any compact interval, and let $t_1, t_2 \in I$ be such that $t_1 < t_2$. Let Ω be any bounded subset of X ; then for any $u \in \Omega$, we have

$$\begin{aligned}
 &\left| \frac{Tu(t_2)}{1+t_2^{\alpha-1}} - \frac{Tu(t_1)}{1+t_1^{\alpha-1}} \right| \\
 &= \left| \int_0^\infty \left(\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}} \right) f(s, u(s)) ds \right| \\
 &\leq \int_0^\infty \left| \frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}} \right| (a(s) + b(s) (1+s^{\alpha-1})^\rho \|u\|_X^\rho) ds.
 \end{aligned} \tag{23}$$

Since $G(t,s)$ is continuous on $J \times J$, we have that $G(t,s)/(1+t^{\alpha-1})$ is a uniformly continuous function on the compact set

$I \times I$. Moreover, for $s \geq t$, we have that this function only depends on t ; in consequence it is uniformly continuous on $I \times (J \setminus I)$. So we have that for all $s \in J$ and $t_1, t_2 \in I$, the following property holds.

For all $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that if $|t_1 - t_2| < \delta$, then $|G(t_2, s)/(1 + t_2^{\alpha-1}) - G(t_1, s)/(1 + t_1^{\alpha-1})| < \varepsilon$. By this, together with (23), and the fact that

$$\int_0^\infty (a(s) + b(s)(1 + s^{\alpha-1})^\rho L_1) ds < \infty, \tag{24}$$

we can get that $T\Omega$ is equicontinuous on I .

(b) In fact, when $t \rightarrow \infty$, we have

$$\lim_{t \rightarrow \infty} \frac{G(t, s)}{1 + t^{\alpha-1}} = \frac{1}{\Delta} \begin{cases} \lambda \eta^{2\alpha-2} - \lambda(\eta - s)^{2\alpha-2}, & s \leq \eta, \\ \lambda \eta^{2\alpha-2}, & 0 \leq \eta \leq s. \end{cases} \tag{25}$$

From this, it is not difficult to verify that for any given $\varepsilon > 0$, there exists a constant $T' = T'(\varepsilon) > 0$ such that

$$\left| \frac{G(t_1, s)}{1 + t_1^{\alpha-1}} - \frac{G(t_2, s)}{1 + t_2^{\alpha-1}} \right| < \varepsilon \tag{26}$$

for any $t_1, t_2 \geq T'$ and $s \in J$. Hence, T is equiconvergent at ∞ .

Thus the conclusion of Theorem 4 applies that T is relatively compact on J . So, $T : X \rightarrow X$ is completely continuous. This completes the proof. \square

4. Main Results

Theorem 8. Assume that (H_1) and (H_2) with $\rho = 1$ hold. If there exists $r > 0$ such that

$$r(1 - Lb^*) > La^* \tag{27}$$

with L given by (15), then problem (1) has a solution $u(t)$ satisfying

$$0 \leq \frac{u(t)}{1 + t^{\alpha-1}} \leq r, \quad \text{for } t \in J. \tag{28}$$

Proof. Let $U = \{u \in X, \|u\|_X < r\}$. For $u \in \partial U$, if there exist $v \in (0, 1)$ such that $u = vTu$, then we have

$$\begin{aligned} \|u\|_X &= \sup_{t \in J} \left| \frac{v(Tu)(t)}{1 + t^{\alpha-1}} \right| \\ &\leq \sup_{t \in J} \int_0^\infty \frac{G(t, s)}{1 + t^{\alpha-1}} |f(s, u(s))| ds \\ &\leq L \int_0^\infty |a(s) + b(s)u(s)| ds \\ &\leq La^* + Lb^* \|u\|_X. \end{aligned} \tag{29}$$

This implies that

$$r(1 - Lb^*) \leq La^*, \tag{30}$$

which contradicts (27). By Lemma 7 and Theorem 3, we conclude that problem (1) has a solution $u(t)$ satisfying

$$0 \leq \frac{u(t)}{1 + t^{\alpha-1}} \leq r, \quad t \in J. \tag{31}$$

This completes the proof. \square

In the next, we formulate existence results for the cases $0 < \rho < 1$ and $\rho > 1$. We do not provide the proof of these results as it is similar to that of Theorem 8. For that, we denote (H_2) with $0 < \rho < 1$ and $\rho > 1$, respectively, by (H_3) and (H_4) .

Theorem 9. Let the assumptions (H_1) and (H_3) hold. Then problem (1) has a solution $u(t)$ satisfying

$$0 \leq \frac{u(t)}{1 + t^{\alpha-1}} \leq r, \quad t \in J, \tag{32}$$

where $r > \max\{2La^*, (2Lb^*)^{1/(1-\rho)}\}$ with L given by (15).

Theorem 10. Suppose that (H_1) and (H_4) hold and that there exists $2La^* \leq r \leq (2Lb^*)^{1/(1-\rho)}$ with L given by (15). Then problem (1) has a solution $u(t)$ such that

$$0 \leq \frac{u(t)}{1 + t^{\alpha-1}} \leq r, \quad t \in J. \tag{33}$$

5. Example

Example 1. With $\alpha = 3/2$, $\lambda = 1/2$, and $\eta = 1$, we consider the following boundary value problem:

$$\begin{aligned} D^{3/2}u(t) + \frac{|u(t)| + \sin u(t)}{8(1 + t^{1/2})(1 + t)^2} + \frac{4}{(t + 4)^2} &= 0, \quad t \in [0, +\infty), \\ I^{1/2}u(0) &= 0, \\ D^{1/2}u(+\infty) &= \frac{1}{2}I^{1/2}u(1). \end{aligned} \tag{34}$$

Clearly the condition (H_1) holds as $\Gamma(2\alpha - 1) = \Gamma(2) = 1$, $\lambda\eta^{2\alpha-2} = 1/2$. Letting $a(t) = 4/(t+4)^2$, $b(t) = 1/(4(1+t^{1/2})(1+t)^2)$, we find that

$$\begin{aligned} f(t, u(t)) &= \frac{|u(t)| + \sin u(t)}{8(1 + t^{1/2})(1 + t)^2} + \frac{4}{(t + 4)^2} \\ &\leq a(t) + b(t)|u(t)|, \\ \int_0^{+\infty} a(t) dt &= 1 < +\infty, \\ \int_0^{+\infty} (1 + t^{\alpha-1})b(t) dt &= \frac{1}{4} < +\infty. \end{aligned} \tag{35}$$

This shows that (H_2) holds true. Finally, fixing $r > 1/(\sqrt{\pi} - 1)$, it can easily be verified that the condition (27) is satisfied. Thus all the conditions of Theorem 8 are satisfied. Therefore, by Theorem 8, problem (27) has a solution $u(t)$ such that

$$0 \leq \frac{u(t)}{1 + t^{\alpha-1}} \leq r. \tag{36}$$

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