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Research Article

On the Hybrid Mean Value Involving Kloosterman Sums and Sums Analogous to Dedekind Sums

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The main purpose of this paper is using the properties of Gauss sums and the mean value theorem of Dirichlet L-functions to study one kind of hybrid mean value problems involving Kloosterman sums and sums analogous to Dedekind sums and give two exact computational formulae for them.

1. Introduction

Let c be a natural number and let d be an integer prime to c. The classical Dedekind sums

$$S(d,c)=\sum_{j=1}^{c}\left(\left(\frac{j}{c}\right)\right)\left(\left(\frac{dj}{c}\right)\right), \tag{1}$$

where

$$((x))=\begin{cases}x-[x]-\dfrac{1}{2}, & \text{if } x \text{ is not an integer;}\\ 0, & \text{if } x \text{ is an integer,}\end{cases} \tag{2}$$

describe the behaviour of the logarithm of the eta-function (see [1, 2]) under modular transformations. Gandhi [3] also introduced another sum analogous to Dedekind sums $S(h,k)$ as follows:

$$S_2(h,k)=\sum_{j=1}^{k}(-1)^j\left(\left(\frac{j}{k}\right)\right)\left(\left(\frac{hj}{k}\right)\right), \tag{3}$$

where k denotes any positive even number and h denotes any integer with $(h,k)=1$.

About the arithmetical properties of $S_2(h,k)$ and related sums, many authors had studied them and obtained a series of interesting results; see [1–9]. For example, the second author [7] proved the following conclusion.

Let $k=2^\beta M$ be a positive integer with $\beta\ge 1$ and $(M,2)=1$. Then we have the asymptotic formula

$$\begin{aligned}&\sum_{h=1}^{k}{}' \left|S_2(h,k)\right|^2\\ &\quad=\frac{5}{112}k\phi(k)\left(\frac{3}{5}-\frac{2}{2^{2\beta}}\right)\\ &\qquad\times\prod_{p^\alpha\|M}\frac{\left[(1+(1/p))^2-\left(1/p^{3\alpha+1}\right)\right]}{(1+(1/p)+(1/p^2))}\\ &\qquad+O\left(k\cdot\exp\left(\frac{4\ln\ln k}{\ln k}\right)\right),\end{aligned} \tag{4}$$

where $\sum_{h=1}^{\prime k}$ denotes the summation over all integers $1\le h\le k$ such that $(h,k)=1$, $\prod_{p^\alpha\|M}$ denotes the product over all prime divisors of M such that $p^\alpha\mid M$ and $p^{\alpha+1}\dagger M$, $\phi(k)$ is the Euler function, and $\exp(y)=e^y$.

The sum $S_2(h,k)$ is important, because it has close relations with the classical Dedekind sums $S(h,k)$. But unfortunately, so far, we knew that all results of $S_2(h,k)$ are the properties of their own, or the relationships between $S_2(h,k)$ and $S(h,k)$, and had nothing to do with the other arithmetic functions. If we can find some relations between $S_2(h,k)$ and other arithmetic function, that will be very useful for further study of the properties of $S_2(h,k)$.

On the other hand, we introduce the classical Kloosterman sums $K(n,q)$, which are defined as follows. For any positive integer $q>1$ and integer n,

$$K(n,q)={\sum_{b=1}^{q}}' e\left(\frac{nb+\overline{b}}{q}\right), \tag{5}$$

where $\overline{b}$ denotes the solution of the congruence $x\cdot b\equiv 1 \bmod q$ and $e(x)=e^{2\pi ix}$.

Some elementary properties of $K(n,q)$ can be found in [10, 11].

The main purpose of this paper is using the properties of the Gauss sums and the mean square value theorem of Dirichlet L-functions to study a hybrid mean value problem involving $S_2(h,k)$ and Kloosterman sums and give two exact computational formulae for them. That is, we will prove the following.

Theorem 1. *Let p be an odd prime. Then one has the identity*

$$\sum_{\substack{m=1\\(2m-1,p)=1}}^{p}\sum_{\substack{n=1\\(2n-1,p)=1}}^{p} K(2m-1,p)\cdot K(2n-1,p)\cdot S_2\left((2m-1)\cdot\overline{2n-1},2p\right)=-\frac{p(p-1)}{4}, \tag{6}$$

where $(2n-1)\cdot\overline{(2n-1)}\equiv 1 \bmod 2p$.

Theorem 2. *Let p be an odd prime; then one has the identity*

$$\sum_{\substack{m=1\\(2m-1,p)=1}}^{p}\sum_{\substack{n=1\\(2n-1,p)=1}}^{p} \left|K(2m-1,p)\right|^2\cdot\left|K(2n-1,p)\right|^2\cdot S_2\left((2m-1)\cdot\overline{(2n-1)},p\right)=\begin{cases}-\frac{1}{4}p^2(p-1)+3\cdot p^2\cdot h_p^2, & \text{if } p\equiv 3 \bmod 8;\\ -\frac{1}{4}p^2(p-1)-p^2\cdot h_p^2, & \text{if } p\equiv 7 \bmod 8;\\ -\frac{1}{4}p^2(p-1), & \text{if } p\equiv 1 \bmod 4,\end{cases} \tag{7}$$

where h_p denotes the class number of the quadratic field $\mathbf{Q}(\sqrt{-p})$.

2. Several Lemmas

In this section, we will give several lemmas, which are necessary in the proof of our theorems. Hereinafter, we will use many properties of Gauss sums, all of which can be found in [12], so they will not be repeated here. First we have the following.

Lemma 3. *Let p be an odd prime; then one has the identity*

$$\sum_{n=1}^{p}\chi(2n-1)\cdot\left|K(2n-1,p)\right|^2=\overline{\chi}(-1)\cdot\frac{\tau^3(\chi)\cdot\tau\left(\overline{\chi}^2\right)}{\tau(\overline{\chi})}. \tag{8}$$

Proof. It is clear that if n pass through a complete residue system $\bmod p$, then $2n-1$ also pass through a complete residue system mod$p$. So for any nonprincipal character $\chi \bmod p$, from the properties of Gauss sums $\tau(\chi)$ (see Theorem 8.9 of [12])

$$\chi(a)=\frac{1}{\tau(\overline{\chi})}\sum_{b=1}^{p-1}\overline{\chi}(b)\,e\left(\frac{ba}{p}\right), \tag{9}$$

we have the identity

$$\begin{aligned}
&\sum_{n=1}^{p}\chi(2n-1)\cdot\left|K(2n-1,p)\right|^2\\
&=\sum_{n=1}^{p}\chi(n)\left|K(n,p)\right|^2\\
&=\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{n=1}^{p-1}\chi(n)\,e\left(\frac{n(a-b)+\left(\overline{a}-\overline{b}\right)}{p}\right)\\
&=\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{n=1}^{p-1}\chi(n)\,e\left(\frac{nb(a-1)+\overline{b}(\overline{a}-1)}{p}\right)\\
&=\tau(\chi)\cdot\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\overline{\chi}(b(a-1))\,e\left(\frac{\overline{b}(\overline{a}-1)}{p}\right)\\
&=\tau^2(\chi)\cdot\sum_{a=1}^{p-1}\overline{\chi}(a-1)\,\overline{\chi}(\overline{a}-1)\\
&=\tau^2(\chi)\cdot\sum_{a=1}^{p-1}\chi(a)\,\overline{\chi}\left(-(a-1)^2\right)\\
&=\overline{\chi}(-1)\cdot\tau^2(\chi)\cdot\sum_{a=1}^{p-2}\chi(a+1)\,\overline{\chi}\left(a^2\right)\\
&=\overline{\chi}(-1)\cdot\tau^2(\chi)\cdot\sum_{a=1}^{p-2}\chi\left(\overline{a}+\overline{a}^2\right)\\
&=\overline{\chi}(-1)\cdot\tau^2(\chi)\cdot\sum_{a=1}^{p-1}\chi\left(a^2+a\right)\\
&=\overline{\chi}(-1)\cdot\tau^2(\chi)\cdot\frac{1}{\tau(\overline{\chi})}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\overline{\chi}(b)\,e\left(\frac{b\left(a^2+a\right)}{p}\right)\\
&=\overline{\chi}(-1)\cdot\tau^2(\chi)\cdot\frac{1}{\tau(\overline{\chi})}\cdot\sum_{b=1}^{p-1}\overline{\chi}(b)\,e\left(\frac{b}{p}\right)\sum_{a=1}^{p-1}\chi(a)\,e\left(\frac{ba}{p}\right)
\end{aligned}$$

$$= \overline{\chi}(-1) \cdot \tau^3(\chi) \cdot \frac{1}{\tau(\overline{\chi})} \cdot \sum_{b=1}^{p-1} \overline{\chi}^2(b) e\left(\frac{b}{p}\right)$$
$$= \overline{\chi}(-1) \cdot \frac{\tau^3(\chi) \cdot \tau(\overline{\chi}^2)}{\tau(\overline{\chi})}. \tag{10}$$

This proves Lemma 3. □

Lemma 4. *Let $q > 2$ be an integer; then for any integer a with $(a, q) = 1$, one has the identity*

$$S(a,q) = \frac{1}{\pi^2 q} \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(a) \left|L(1,\chi)\right|^2, \tag{11}$$

where $L(1,\chi)$ denotes the Dirichlet L-function corresponding to character χ mod d.

Proof. See Lemma 2 of [8]. □

Lemma 5. *Let p be an odd prime. Then for any odd number h with $(h, p) = 1$, one has the identity*

$$S_2(h,2p) = -S(h,p) + S(2h,p) + S(\overline{2}h,p), \tag{12}$$

where $\overline{2}$ satisfies the congruence $2 \cdot \overline{2} \equiv 1 \bmod p$.

Proof. Note that the divisors of $2p$ are $1, 2, p$, and $2p$. So from Lemma 4 and the definition of $S_2(h, 2p)$ and $S(h, k)$ we have

$$S_2(h,2p) = \sum_{j=1}^{2p} (-1)^j \left(\left(\frac{j}{2p}\right)\right)\left(\left(\frac{hj}{2p}\right)\right)$$
$$= 2\sum_{j=1}^{p} \left(\left(\frac{j}{p}\right)\right)\left(\left(\frac{hj}{p}\right)\right) - \sum_{j=1}^{2p} \left(\left(\frac{j}{2p}\right)\right)\left(\left(\frac{hj}{2p}\right)\right)$$
$$= 2S(h,p) - S(h,2p)$$
$$= 2S(h,p) - \frac{1}{2\pi^2 p} \sum_{d|2p} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(h) \left|L(1,\chi)\right|^2$$
$$= 2S(h,p) - \frac{p}{2\pi^2 (p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(h) \left|L(1,\chi)\right|^2 \tag{13}$$
$$- \frac{2p}{\pi^2 (p-1)} \sum_{\substack{\chi \bmod 2p \\ \chi(-1)=-1}} \chi(h) \left|L(1,\chi)\right|^2$$
$$= 2S(h,p) - \frac{p}{2\pi^2 (p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(h) \left|L(1,\chi)\right|^2 \tag{14}$$
$$- \frac{2p}{\pi^2 (p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(h)\lambda(h) \left|L(1,\chi\lambda)\right|^2,$$

where λ denotes the principal character mod 2.

From the Euler infinite product formula (see Theorem 11.6 of [12]) we have,

$$\left|L(1,\chi\lambda)\right|^2 = \prod_{p_1} \left|1 - \frac{\chi(p_1)\lambda(p_1)}{p_1}\right|^{-2} = \prod_{p_1>2} \left|1 - \frac{\chi(p_1)}{p_1}\right|^{-2}$$
$$= \left|1 - \frac{\chi(2)}{2}\right|^2 \cdot \prod_{p_1} \left|1 - \frac{\chi(p_1)}{p_1}\right|^{-2}$$
$$= \left(\frac{5}{4} - \frac{\chi(2)}{2} - \frac{\overline{\chi}(2)}{2}\right) \cdot \left|L(1,\chi)\right|^2, \tag{15}$$

where $\prod_p$ denotes the product over all primes p.

From Lemma 4 we also have the identity

$$S(h,p) = \frac{1}{\pi^2} \cdot \frac{p}{p-1} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(h) \left|L(1,\chi)\right|^2. \tag{16}$$

Note that h is an odd number; combining (14), (15), and (16) we have the identity

$$S_2(h,2p) = 2S(h,p) - \frac{1}{2}S(h,p) - \frac{2p}{\pi^2 (p-1)} \cdot \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(h) \left|L(1,\chi\lambda)\right|^2$$
$$= \frac{3}{2}S(h,p) - \frac{2p}{\pi^2 (p-1)} \cdot \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(h) \left(\frac{5}{4} - \frac{\chi(2)}{2} - \frac{\overline{\chi}(2)}{2}\right) \cdot \left|L(1,\chi)\right|^2$$
$$= -S(h,p) + S(2h,p) + S(\overline{2}h,p). \tag{17}$$

This proves Lemma 5. □

Lemma 6. *Let p be an odd prime. Then one has the identities*

(A)

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \left|L(1,\chi)\right|^2 = \frac{\pi^2}{12} \cdot \frac{(p-1)^2 \cdot (p-2)}{p^2}; \tag{18}$$

(B)

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2) \cdot \left|L(1,\chi)\right|^2 = \frac{\pi^2}{24} \cdot \frac{(p-1)^2 \cdot (p-5)}{p^2}. \tag{19}$$

Proof. From the definition of Dedekind sums we have

$$S(1,c) = \sum_{a=1}^{c-1} \left(\frac{a}{c} - \frac{1}{2}\right)^2 = \frac{(c-1)(c-2)}{12c}. \tag{20}$$

If $p \equiv 1 \bmod c$, then, from (20) and noting that the reciprocity theorem of Dedekind sums (see [5]), we have the computational formula

$$\begin{aligned} S(c,p) &= \frac{p^2+c^2+1}{12pc} - \frac{1}{4} - S(p,c) \\ &= \frac{p^2+c^2+1}{12pc} - \frac{1}{4} - S(1,c) \\ &= \frac{p^2+c^2+1}{12pc} - \frac{1}{4} - \frac{(c-1)(c-2)}{12c} \\ &= \frac{(p-1)(p-1-c^2)}{12pc}. \end{aligned} \tag{21}$$

Now taking $c = 1$ in (21), from (16) we may immediately deduce the identity

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1,\chi)|^2 = \frac{\pi^2}{12} \cdot \frac{(p-1)^2 \cdot (p-2)}{p^2}. \tag{22}$$

Taking $c = 2$ in (21), from (16) we can also deduce the identity

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2) \cdot |L(1,\chi)|^2 = \frac{\pi^2}{24} \cdot \frac{(p-1)^2 \cdot (p-5)}{p^2}. \tag{23}$$

Now Lemma 6 follows from (22) and (23). □

3. Proof of the Theorems

In this section, we will complete the proof of our theorems. First we prove Theorem 1. Note that if χ is a nonprincipal character mod p, then $|\tau(\chi)| = \sqrt{p}$ and

$$\begin{aligned} &\left| \sum_{m=1}^{p} \chi(2m-1) K(2m-1,p) \right| \\ &\quad = \left| \sum_{a=1}^{p-1} \sum_{m=1}^{p} \chi(m) e\left(\frac{ma+\bar{a}}{p} \right) \right| = \left| \tau^2(\chi) \right| = p. \end{aligned} \tag{24}$$

From (24) and Lemmas 4, 5, and 6 we have

$$\begin{aligned} &\sum_{\substack{m=1 \\ (2m-1,p)=1}}^{p} \sum_{\substack{n=1 \\ (2n-1,p)=1}}^{p} K(2m-1,p) \cdot K(2n-1,p) \\ &\qquad \cdot S_2\left((2m-1) \cdot \overline{2n-1}, 2p\right) \\ &= -\frac{p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \left| \sum_{n=1}^{p} \chi(2n-1) \cdot K(2n-1,p) \right|^2 \\ &\qquad \cdot |L(1,\chi)|^2 + \frac{p}{\pi^2(p-1)} \\ &\quad \times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2) \left| \sum_{n=1}^{p} \chi(2n-1) \cdot K(2n-1,p) \right|^2 \\ &\qquad \cdot |L(1,\chi)|^2 + \frac{p}{\pi^2(p-1)} \\ &\quad \times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \overline{\chi}(2) \left| \sum_{n=1}^{p} \chi(2n-1) \cdot K(2n-1,p) \right|^2 \\ &\qquad \cdot |L(1,\chi)|^2 \\ &= -\frac{p^3}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1,\chi)|^2 + \frac{p^3}{\pi^2(p-1)} \\ &\quad \times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2) |L(1,\chi)|^2 + \frac{p^3}{\pi^2(p-1)} \\ &\quad \times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \overline{\chi}(2) \cdot |L(1,\chi)|^2 \\ &= -\frac{p(p-1)(p-2)}{12} + \frac{p(p-1)(p-5)}{12} \\ &= -\frac{p(p-1)}{4}. \end{aligned} \tag{25}$$

This proves Theorem 1.

Now we prove Theorem 2. If $p \equiv 1 \bmod 4$, then from Lemmas 3, 5, and 6 we have

$$\begin{aligned} &\sum_{\substack{m=1 \\ (2m-1,p)=1}}^{p} \sum_{\substack{n=1 \\ (2n-1,p)=1}}^{p} |K(2m-1,p)|^2 \cdot |K(2n-1,p)|^2 \\ &\qquad \cdot S_2\left((2m-1) \cdot \overline{(2n-1)}, p\right) \\ &= -\frac{p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \left| \sum_{n=1}^{p} \chi(2n-1) \cdot |K(2n-1,p)|^2 \right|^2 \\ &\qquad \cdot |L(1,\chi)|^2 + \frac{p}{\pi^2(p-1)} \\ &\quad \times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2) \left| \sum_{n=1}^{p} \chi(2n-1) \cdot |K(2n-1,p)|^2 \right|^2 \\ &\qquad \cdot |L(1,\chi)|^2 + \frac{p}{\pi^2(p-1)} \end{aligned}$$

$$\times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \overline{\chi}(2) \left| \sum_{n=1}^{p} \chi(2n-1) \cdot |K(2n-1,p)|^2 \right|^2 \cdot |L(1,\chi)|^2$$
$$= -\frac{p^4}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1,\chi)|^2 + \frac{p^4}{\pi^2(p-1)} \times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2) |L(1,\chi)|^2 + \frac{p^4}{\pi^2(p-1)} \times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \overline{\chi}(2) \cdot |L(1,\chi)|^2$$
$$= -\frac{1}{12}p^2(p-1)(p-2) + \frac{1}{12}p^2(p-1)(p-5)$$
$$= -\frac{1}{4}p^2(p-1). \tag{26}$$

If $p \equiv 3 \bmod 4$, then note that the Legendre symbol $(-1/p) = \chi_2(-1) = -1$, $L(1,\chi_2) = \pi \cdot h_p/\sqrt{p}$ (see Dirichlet's class number formula, Chapter 6 of [13]), and

$$\tau(\chi_2^2) = \sum_{a=1}^{p-1} \left(\frac{a}{p}\right)^2 e\left(\frac{a}{p}\right) = \sum_{a=1}^{p-1} e\left(\frac{a}{p}\right) = -1, \tag{27}$$

so from Lemmas 3, 5, and 6 we have

$$\sum_{\substack{m=1 \\ (2m-1,p)=1}}^{p} \sum_{\substack{n=1 \\ (2n-1,p)=1}}^{p} |K(2m-1,p)|^2 \cdot |K(2n-1,p)|^2 \cdot S_2\left((2m-1)\cdot\overline{(2n-1)}, p\right)$$
$$= -\frac{p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \left| \sum_{n=1}^{p} \chi(2n-1) \cdot |K(2n-1,p)|^2 \right|^2 \cdot |L(1,\chi)|^2 + \frac{p}{\pi^2(p-1)}$$
$$\times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2) \left| \sum_{n=1}^{p} \chi(2n-1) \cdot |K(2n-1,p)|^2 \right|^2 \cdot |L(1,\chi)|^2 + \frac{p}{\pi^2(p-1)}$$
$$\times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \overline{\chi}(2) \left| \sum_{n=1}^{p} \chi(2n-1) \cdot |K(2n-1,p)|^2 \right|^2 \cdot |L(1,\chi)|^2$$
$$= -\frac{p^4}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1,\chi)|^2 + \frac{p^4}{\pi^2(p-1)} \times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2) |L(1,\chi)|^2 + \frac{p^4}{\pi^2(p-1)}$$
$$\times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \overline{\chi}(2) \cdot |L(1,\chi)|^2 + \frac{p^3}{\pi^2} \cdot |L(1,\chi_2)|^2 - \frac{p^3}{\pi^2} \cdot \left(\frac{2}{p}\right) \cdot |L(1,\chi_2)|^2 - \frac{p^3}{\pi^2} \cdot \left(\frac{2}{p}\right) \cdot |L(1,\chi_2)|^2$$
$$= -\frac{1}{4}p^2(p-1) + p^2 \cdot h_p^2 - 2\left(\frac{2}{p}\right) \cdot p^2 \cdot h_p^2. \tag{28}$$

Note that $(2/p) = (-1)^{(p^2-1)/8} = -1$ if $p \equiv 3 \bmod 8$; and $(2/p) = 1$ if $p \equiv 7 \bmod 8$, from (28) we may immediately deduce

$$\sum_{\substack{m=1 \\ (2m-1,p)=1}}^{p} \sum_{\substack{n=1 \\ (2n-1,p)=1}}^{p} |K(2m-1,p)|^2 \cdot |K(2n-1,p)|^2 \cdot S_2\left((2m-1)\cdot\overline{(2n-1)}, p\right)$$
$$= \begin{cases} -\frac{1}{4}p^2(p-1) + 3 \cdot p^2 \cdot h_p^2, & \text{if } p \equiv 3 \bmod 8; \\ -\frac{1}{4}p^2(p-1) - p^2 \cdot h_p^2, & \text{if } p \equiv 7 \bmod 8. \end{cases} \tag{29}$$

Now Theorem 2 follows from (26) and (29).

This completes the proofs of all results.

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