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Research Article

Global Attractors for the Three-Dimensional Viscous Primitive Equations of Large-Scale Atmosphere in Log-Pressure Coordinate

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We will first prove the existence of (V, V)- and (V, H^2)-absorbing sets for the three-dimensional primitive equations of large-scale atmosphere in log-pressure coordinate and then prove the existence of (V, H^3)-absorbing set by the use of the elliptic regularity theory. Finally, we obtain the existence of (V, V)- and (V, H^2)-global attractors for the three-dimensional viscous primitive equations of large-scale atmosphere in log-pressure coordinate by using the Sobolev compactness embedding theory.

1. Introduction

The fundamental equations governing the motion of the atmosphere consist of the Navier-Stokes equations with Coriolis force and thermodynamics, which account for the buoyancy forces and stratification effects under the Boussinesq approximation. Moreover, due to the shallowness of the atmosphere, that is, the depth of the fluid layer is very small in comparison with the radius of the earth, the vertical large-scale motion in the atmosphere is much smaller than the horizontal one, which in turn leads to modeling the vertical motion by hydrostatic balance [1–3]. But for large-scale atmospheric flows, it is often necessary to take account of compressibility. This may be done most easily by a change of vertical coordinates to log-pressure coordinate rather than a function of geometric height. As a result, one can obtain the system (1)–(4), which is known as the primitive equations for large-scale atmosphere in log-pressure coordinate.

In recent years, the primitive equations of the atmosphere, the ocean, and the coupled atmosphere-ocean have been extensively studied from the mathematical point of view [1, 4–13]. The mathematical framework of the primitive equations of the ocean was formulated, and the existence of weak solutions was proved by Lions et al. in [11]. In [1], the authors have studied the primitive equations of large-scale atmosphere in pressure coordinate, where the continuity equation is changed into the incompressible equation, but the thermodynamical equation is very complicated. For the sake of simplicity, the authors have made some approximation of the thermodynamical equation, and in this case, the authors have proved the existence and the uniqueness of strong, local-in-time solutions. By assuming that the initial data is small enough, the authors have studied the existence of strong, global-in-time solutions to the three-dimensional primitive equations of large-scale ocean and obtained the existence of strong, local-in-time solutions to the equations for all initial data in [14]. In [5], the authors have proved the maximum principles of the temperature for the primitive equations of the atmosphere in pressure coordinate. The existence and the uniqueness of strong, global-in-time solutions to the primitive equations in thin domains for a large set of initial datum whose sizes depend inversely on the thickness were established in [9]. In [15], Temam and Ziane have considered the existence of strong, local-in-time solutions for the primitive equations of the atmosphere, the ocean, and the coupled atmosphere-ocean. Asymptotic analysis of the primitive equations under the small depth assumption was established in [10]. In [8], the authors have proved the existence of weak solutions and a trajectory attractors for the moist atmospheric equations in geophysics. The existence and the uniqueness

of strong, global-in-time solutions for the three-dimensional viscous primitive equations of large-scale ocean and atmosphere dynamics were established by the authors in [4, 16]. In [6], the authors have considered the long-time dynamics of the primitive equations of large-scale atmosphere and obtained a weakly compact global attractor $\mathscr{A}$ which captures all the trajectories with respect to the weak topology of V. The existence and the uniqueness of strong solutions, the existence of smooth solutions, and the existence of a compact global attractor in V for the primitive equations with more regular initial data than that in [6] were established in [7]. In [12], the authors have proved the existence and the uniqueness of z-weak, global-in-time solutions when the initial conditions satisfy some regularity. The existence of global attractor in V for the primitive equations with initial datum in V and the heat source $Q \in L^2(\Omega)$ was proved by use of the Aubin-Lions compactness lemma in [17]. In [18], the authors have proved the existence of the global attractor in $(H^2(\Omega))^3$ for the three-dimensional viscous primitive equations of large-scale ocean and atmosphere dynamics. In this paper, we will consider the primitive equations of large-scale atmosphere in log-pressure coordinate, where the continuity equation is transformed into the incompressible equation with the weighted function e^{-z/H_s}, which is more complicated than that in pressure coordinate, but we do not need to make any approximation of the thermodynamical equation such that the accuracy of the temperature is higher than that in pressure coordinate which will reveal the real world better. Moreover, the appearance of the weighted function e^{-z/H_s} makes the calculations more difficult. In order to make sure of the existence of the absorbing set, we need to add some additional conditions (10) on h, which means that the thickness of Ω is very small. In [18], there are not any restrictions on the thickness of Ω to ensure the existence of the absorbing sets, and the $H^1(\Omega)$ estimate of T can be obtained directly by taking the inner product (15) with $-\Delta_3 T$. But in this paper, thanks to the appearance of the weighted function e^{-z/H_s}, we have to first estimate T_z in L^2 and then $\nabla_2 T$ in L^2 to obtain a priori estimates on T in H^1. In this paper, we will prove the existence of (V, H^2)-global attractor for the three-dimensional primitive equations of large-scale atmosphere in log-pressure coordinate under the assumptions (10) on h.

In this paper, we will consider the following three-dimensional viscous primitive equations of large-scale atmosphere in log-pressure coordinate [3]:

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v + w \frac{\partial v}{\partial z} + f_0 \vec{k} \times v + \nabla \Phi + L_1 v = 0, \tag{1}$$

$$\frac{\partial \Phi}{\partial z} = \frac{RT}{H_s}, \tag{2}$$

$$\nabla \cdot \left(e^{-z/H_s} v \right) + \frac{\partial}{\partial z} \left(e^{-z/H_s} w \right) = 0, \tag{3}$$

$$\frac{\partial T}{\partial t} + v \cdot \nabla T + w \frac{\partial T}{\partial z} + L_2 T = Q, \tag{4}$$

in the domain

$$\Omega = M \times (-h, 0) \subset \mathbb{R}^3, \tag{5}$$

where M is a bounded domain in $\mathbb{R}^2$ with smooth boundary. Here, $v = (v_1, v_2)$, (v_1, v_2, w) is the velocity field, T is the temperature, Φ is the geopotential, $f_0 = 2\Omega_1 \sin \nu_0$ is the Coriolis parameter, $\vec{k}$ is the vertical unit vector, R is a positive constant, and Q is the heat source. The operators L_1 and L_2 are given by

$$\begin{aligned} L_1 &= -\frac{1}{\mathrm{Re}_1} \Delta - \frac{1}{\mathrm{Re}_2} \frac{\partial^2}{\partial z^2}, \\ L_2 &= -\frac{1}{Rt_1} \Delta - \frac{1}{Rt_2} \frac{\partial^2}{\partial z^2}, \end{aligned} \tag{6}$$

where Re_1 and Re_2 are positive constants representing the horizontal and the vertical Reynolds numbers, respectively, and Rt_1 and Rt_2 are positive constants which stand for the horizontal and the vertical heat diffusivities, respectively. And $z = -H_s \log(p/p_*)$ is used as a vertical coordinate, where H_s is a constant "scale height" and p_* is a constant reference pressure. For the sake of simplicity, let $\nabla = (\partial_x, \partial_y)$ be the horizontal gradient operator and $\Delta = \partial_x^2 + \partial_y^2$ the horizontal Laplacian.

We denote the different parts of the boundary $\partial\Omega$ by

$$\begin{aligned} \Gamma_u &= \left\{ (x, y, z) \in \overline{\Omega} : z = 0 \right\}, \\ \Gamma_b &= \left\{ (x, y, z) \in \overline{\Omega} : z = -h \right\}, \\ \Gamma_l &= \left\{ (x, y, z) \in \overline{\Omega} : (x, y) \in \partial M, -h \le z \le 0 \right\}. \end{aligned} \tag{7}$$

We equip (1)–(4) with the following boundary conditions, with nonslip and nonflux on the side walls and bottom [4]:

$$\begin{gathered} \left. \frac{\partial v}{\partial z} \right|_{\Gamma_u} = 0, \qquad w|_{\Gamma_u} = 0, \qquad \left. \left(\frac{1}{Rt_2} \frac{\partial T}{\partial z} + \alpha T \right) \right|_{\Gamma_u} = 0, \\ \left. \frac{\partial v}{\partial z} \right|_{\Gamma_b} = 0, \qquad w|_{\Gamma_b} = 0, \qquad \left. \frac{\partial T}{\partial z} \right|_{\Gamma_b} = 0, \\ v \cdot \vec{n}|_{\Gamma_l} = 0, \qquad \left. \frac{\partial v}{\partial \vec{n}} \times \vec{n} \right|_{\Gamma_l} = 0, \qquad \left. \frac{\partial T}{\partial \vec{n}} \right|_{\Gamma_l} = 0, \end{gathered} \tag{8}$$

where α is a positive constant.

In addition, (1)–(8) are supplemented with the following initial conditions:

$$\begin{aligned} v(x, y, z, 0) &= v_0(x, y, z), \\ T(x, y, z, 0) &= T_0(x, y, z). \end{aligned} \tag{9}$$

Additionally, assume that h satisfy

$$\begin{gathered} \frac{9h^2}{4H_s^2} + \frac{4h}{\alpha H_s^2 Rt_2} < 1, \\ H_s^2 \mathrm{Re}_2 \lambda > 1. \end{gathered} \tag{10}$$

This paper is organized as follows. Section 2 is devoted to introduce the mathematical framework and the definition of solutions for (1)–(10) and give some auxiliary lemmas used in the sequel. In Section 3, we will give the existence and the uniqueness of solutions for (1)–(10). Next, some a priori estimates of solutions corresponding to (1)–(10) will be established, which imply the existence of (V, V)- and (V, H^2)-absorbing sets for (1)–(10). In Section 4, we first obtain the existence of the (V, V)-global attractor for (1)–(10) by $H^2 \subset V$ compactly, and then get the existence of (V, H^3)-absorbing set of (1)–(10) by the elliptic regularity theory. Finally, we prove the existence of the (V, H^2)-global attractor for the three-dimensional viscous primitive equations of large-scale atmosphere in log-pressure coordinate by the Sobolev compactness embedding theory.

Throughout this paper, let C be positive constants independent of t which may be different from line to line.

2. Mathematical Setting of (1)–(10)

2.1. Reformulation of (1)–(9). Integrating the continuity equation (3) and taking the boundary conditions for w into account, we obtain

$$w(x,y,z,t) = -\int_{-h}^{z} e^{(z-\zeta)/H_s} \nabla \cdot v(x,y,\zeta,t)\, d\zeta,$$
$$\int_{-h}^{0} e^{-\zeta/H_s} \nabla \cdot v(x,y,\zeta,t)\, d\zeta = 0. \tag{11}$$

And then, we consider the vertical momentum equation, that is, the second equation (2). First, we define an unknown function on $z = 0$ (i.e., $p = p_*$), where p_* is a constant reference pressure, say

$$\Phi_s : M \longrightarrow \mathbb{R}, \tag{12}$$

which is the geopotential of the atmosphere on $z = 0$.

Then,

$$\Phi(x,y,z,t) = \Phi_s(x,y,t) + \frac{R}{H_s}\int_0^z T(x,y,\zeta,t)\, d\zeta. \tag{13}$$

Therefore, (1)–(4) can be rewritten as follows:

$$\frac{\partial v}{\partial t} + (v\cdot\nabla)v - \left(\int_{-h}^{z} e^{(z-\zeta)/H_s}\, \nabla\cdot v(x,y,\zeta,t)\, d\zeta\right)\frac{\partial v}{\partial z} + \nabla\Phi_s(x,y,t) + f_0\vec{k}\times v + \frac{R}{H_s}\int_0^z \nabla T(x,y,\zeta,t)\, d\zeta + L_1 v = 0, \tag{14}$$

$$\frac{\partial T}{\partial t} + v\cdot\nabla T - \left(\int_{-h}^{z} e^{(z-\zeta)/H_s}\nabla\cdot v(x,y,\zeta,t)\, d\zeta\right)\frac{\partial T}{\partial z} + L_2 T = Q, \tag{15}$$

with the following boundary conditions:

$$\left.\frac{\partial v}{\partial z}\right|_{\Gamma_b} = 0, \qquad \left.\frac{\partial v}{\partial z}\right|_{\Gamma_u} = 0,$$
$$v\cdot\vec{n}|_{\Gamma_l} = 0, \qquad \left.\frac{\partial v}{\partial \vec{n}}\times\vec{n}\right|_{\Gamma_l} = 0, \tag{16}$$

$$\left.\frac{\partial T}{\partial z}\right|_{\Gamma_b} = 0, \qquad \left.\left(\frac{1}{Rt_2}\frac{\partial T}{\partial z} + \alpha T\right)\right|_{\Gamma_u} = 0, \qquad \left.\frac{\partial T}{\partial \vec{n}}\right|_{\Gamma_l} = 0, \tag{17}$$

and the initial data

$$v(x,y,z,0) = v_0(x,y,z),$$
$$T(x,y,z,0) = T_0(x,y,z). \tag{18}$$

At last, we will divide (14) into two systems with respect to $\overline{v}$ and $\widetilde{v}$ [4], where $\overline{v}$ and $\widetilde{v}$ are defined by

$$\overline{v} = \frac{1}{A}\int_{-h}^{0} e^{-z/H_s}\, v(x,y,z)\, dz,$$
$$\widetilde{v} = v - \overline{v}, \tag{19}$$

where

$$A = \int_{-h}^{0} e^{-z/H_s} dz = H_s\left(e^{h/H_s} - 1\right), \tag{20}$$

and we have $\overline{\widetilde{v}} = 0$.

Therefore, $\overline{v}$ satisfies the following systems:

$$\frac{\partial \overline{v}}{\partial t} + (\overline{v}\cdot\nabla)\overline{v} - \frac{1}{\mathrm{Re}_1}\Delta\overline{v} + \nabla\Phi_s(x,y,t) + \overline{(\nabla\cdot\overline{v})\overline{v} + (\overline{v}\cdot\nabla)\overline{v}} + f_0\vec{k}\times\overline{v} + \overline{\frac{R}{H_s}\int_0^z \nabla T(x,y,\zeta,t)\, d\zeta - \frac{1}{H_s\mathrm{Re}_2}\frac{\partial\overline{\widetilde{v}}}{\partial z}} = 0, \tag{21}$$

with the boundary conditions

$$\nabla\cdot\overline{v} = 0, \qquad \overline{v}\cdot\vec{n}|_{\Gamma_l} = 0, \qquad \left.\frac{\partial\overline{v}}{\partial\vec{n}}\times\vec{n}\right|_{\Gamma_l} = 0, \tag{22}$$

and $\widetilde{v}$ satisfies the following systems

$$\frac{\partial\widetilde{v}}{\partial t} + (\widetilde{v}\cdot\nabla)\widetilde{v} - \left(\int_{-h}^{z} e^{(z-\zeta)/H_s}\nabla\cdot\widetilde{v}(x,y,\zeta,t)\, d\zeta\right)\frac{\partial\widetilde{v}}{\partial z} + \frac{R}{H_s}\int_0^z \nabla T(x,y,\zeta,t)\, d\zeta + f_0\vec{k}\times\widetilde{v} + L_1\widetilde{v} + (\widetilde{v}\cdot\nabla)\overline{v} + (\overline{v}\cdot\nabla)\widetilde{v} - \overline{(\nabla\cdot\widetilde{v})\widetilde{v} + (\widetilde{v}\cdot\nabla)\widetilde{v} + \frac{R}{H_s}\int_0^z \nabla T(x,y,\zeta,t)\, d\zeta - \frac{1}{H_s\mathrm{Re}_2}\frac{\partial\widetilde{v}}{\partial z}} = 0, \tag{23}$$

with the boundary conditions

$$\left.\frac{\partial\widetilde{v}}{\partial z}\right|_{\Gamma_u} = 0, \qquad \left.\frac{\partial\widetilde{v}}{\partial z}\right|_{\Gamma_b} = 0, \qquad \widetilde{v}\cdot\vec{n}|_{\Gamma_l} = 0,$$
$$\left.\frac{\partial\widetilde{v}}{\partial\vec{n}}\times\vec{n}\right|_{\Gamma_l} = 0. \tag{24}$$

2.2. Some Function Spaces. In this subsection, we will quote briefly some notations and function spaces used in this paper.

First of all, we introduce the notations for some function spaces on Ω as follows:

$$\begin{aligned}\mathscr{V}_1 = \Bigg\{ & v \in \left(C^\infty\left(\overline{\Omega}\right)\right)^2 : \left.\frac{\partial v}{\partial z}\right|_{\Gamma_u} = 0, \left.\frac{\partial v}{\partial z}\right|_{\Gamma_b} = 0, v\cdot\vec{n}|_{\Gamma_l} = 0, \\ & \left.\frac{\partial v}{\partial \vec{n}}\times\vec{n}\right|_{\Gamma_l} = 0, \int_{-h}^{0} e^{-z/H_s}\nabla\cdot v\left(x,y,z\right)dz = 0\Bigg\}, \\ \mathscr{V}_2 = \Bigg\{ & T \in C^\infty\left(\overline{\Omega}\right) : \left.\frac{\partial T}{\partial z}\right|_{\Gamma_b} = 0, \\ & \left.\left(\frac{1}{Rt_2}\frac{\partial T}{\partial z} + \alpha T\right)\right|_{\Gamma_u} = 0, \left.\frac{\partial T}{\partial \vec{n}}\right|_{\Gamma_l} = 0\Bigg\}.\end{aligned} \tag{25}$$

Let L^p (or H^s) be the weighted Hilbert spaces of L^p(or H^s) functions or L^p(or H^s)-vector valued functions on Ω and denote the usual $L^p(\Omega)$-norms by $\|\cdot\|_{L^p(\Omega)}$, where $1 \le p \le \infty$(or $s \ge 0$). Denote the inner product in L^2 by $(\cdot,\cdot)$ and the norm in L^p by $\|\cdot\|_p$, respectively, given by

$$\begin{aligned}\left(h_1,h_2\right) &= \int_\Omega e^{-z/H_s}h_1\cdot h_2 dx\,dy\,dz, \\ \|h\|_p^p &= \int_\Omega e^{-z/H_s}|h|^p dx\,dy\,dz,\end{aligned} \tag{26}$$

for any $h_1, h_2 \in L^2$ and $h \in L^p$ $(1 \le p \le \infty)$.

Similarly, define

$$\begin{aligned}\left(\left(v_1,v_2\right)\right) &= \frac{1}{\mathrm{Re}_1}\int_\Omega e^{-z/H_s}\nabla v_1\cdot\nabla v_2 dx\,dy\,dz \\ &\quad + \frac{1}{\mathrm{Re}_2}\int_\Omega e^{-z/H_s}\frac{\partial v_1}{\partial z}\cdot\frac{\partial v_2}{\partial z}dx\,dy\,dz, \\ \left(\left(T_1,T_2\right)\right) &= \frac{1}{Rt_1}\int_\Omega e^{-z/H_s}\nabla T_1\cdot\nabla T_2 dx\,dy\,dz \\ &\quad + \frac{1}{Rt_2}\int_\Omega e^{-z/H_s}\frac{\partial T_1}{\partial z}\frac{\partial T_2}{\partial z}dx\,dy\,dz \\ &\quad + \alpha\int_{\Gamma_u} T_1T_2 dx\,dy,\end{aligned} \tag{27}$$

for any $v_1, v_2 \in \mathscr{V}_1$ and $T_1, T_2 \in \mathscr{V}_2$, respectively.

Let V_1, V_2 be the closure of $\mathscr{V}_1, \mathscr{V}_2$ with respect to the following norms:

$$\|v\|^2 = \left((v,v)\right), \qquad \|T\|^2 = \left((T,T)\right), \tag{28}$$

for any $v \in \mathscr{V}_1$ and $T \in \mathscr{V}_2$, respectively, and let $V = V_1 \times V_2$.

Next, we recall some results used in the sequel.

Lemma 1 (see [19]). (i) *for $u \in H^1(M)$,*

$$\begin{aligned}\|u\|_{L^4(M)} &\le C\|u\|_{L^2(M)}^{1/2}\|u\|_{H^1(M)}^{1/2}, \\ \|u\|_{L^6(M)} &\le C\|u\|_{L^2(M)}^{2/3}\|u\|_{H^1(M)}^{1/3}, \\ \|u\|_{L^8(M)} &\le C\|u\|_{L^4(M)}^{1/2}\|u\|_{H^1(M)}^{1/2}.\end{aligned} \tag{29}$$

(ii) *for $u \in V_i$ $(i = 1, 2)$,*

$$\|u\|_4 \le C\|u\|_2^{1/4}\|u\|^{3/4}. \tag{30}$$

Proposition 2. *There exists a positive constant λ such that*

$$\|v\|^2 \ge \lambda\|v\|_2^2, \tag{31}$$

for any $v \in V_1$.

Lemma 3 (see [4, 6]). *Let $v, v_1 \in V_1$, $T \in V_2$, and $p \ge 1$. Then, we have*

(1) $\int_\Omega[(v\cdot\nabla)v_1 - (\int_{-h}^{z} e^{(z-\zeta)/H_s}\nabla\cdot v(x,y,\zeta)d\zeta)(\partial v_1/\partial z)]\cdot(e^{-z/H_s}|v_1|^{p-1}v_1) = 0$,

(2) $\int_\Omega[v\cdot\nabla T - (\int_{-h}^{z} e^{(z-\zeta)/H_s}\nabla\cdot v(x,y,\zeta)d\zeta)(\partial T/\partial z)]e^{-z/H_s}|T|^{p-1}T = 0$,

(3) $\int_\Omega e^{-z/H_s}\nabla\Phi_s(x,y)\cdot v(x,y,z) = 0$.

Lemma 4 (see [4], (Minkowski inequality)). *Let (X,μ), (Y,ν) be two measure spaces and $f(x,y)$ a measurable function about $\mu\times\nu$ on $X\times Y$, and $\rho(y)$ is a measurable positive function. If $f(\cdot,y) \in L^p(X,\mu)$ $(1 \le p \le \infty)$ for a.e.$y \in Y$ and $\int_Y \rho(y)\|f(\cdot,y)\|_{L^p(X,\mu)}d\nu(y) < \infty$, then*

$$\begin{aligned}&\left\|\int_Y \rho\left(y\right) f\left(\cdot,y\right)d\nu\left(y\right)\right\|_{L^p(X,\mu)} \\ &\quad\le \int_Y \rho\left(y\right)\left\|f\left(\cdot,y\right)\right\|_{L^p(X,\mu)}d\nu\left(y\right).\end{aligned} \tag{32}$$

Finally, we will recall the definition of strong solutions and state it as follows.

Definition 5. Assume that $(v_0,T_0) \in V$ and $Q \in H^1$ and let τ be any fixed positive time. (v,T) is called a strong solution of (14)–(18) on $[0,\tau]$, if

$$\begin{aligned}&(v,T) \in L^\infty\left(0,\tau;V\right)\cap L^2\left(0,\tau;H^2\right), \\ &\left(v_t,T_t\right) \in L^1\left(0,\tau;L^2\right),\end{aligned} \tag{33}$$

and they satisfy

$$\begin{aligned}
&(v(t),\phi)\\
&\quad+\int_{t_0}^{t}\int_{\Omega} e^{-z/H_s}\Bigg[(v\cdot\nabla)v-\Bigg(\int_{-h}^{z} e^{(z-\zeta)/H_s}\nabla\cdot v(x,y,\zeta,t)\,d\zeta\Bigg)\\
&\qquad\times\frac{\partial v}{\partial z}+f_0\vec{k}\times v+\frac{R}{H_s}\int_0^z \nabla T(x,y,\zeta,t)\,d\zeta\Bigg]\\
&\quad\cdot\phi+\int_{t_0}^{t}((v,\phi))\\
&=\frac{1}{H_s\mathrm{Re}_2}\int_{t_0}^{t}(\partial_z v,\phi)+(v(t_0),\phi),\\
&(T(t),\psi)\\
&\quad+\int_{t_0}^{t}\int_{\Omega} e^{-z/H_s}\Bigg[v\cdot\nabla T-\Bigg(\int_{-h}^{z} e^{(z-\zeta)/H_s}\nabla\cdot v(x,y,\zeta,t)\,d\zeta\Bigg)\\
&\qquad\times\frac{\partial T}{\partial z}\Bigg]\psi+\int_{t_0}^{t}((T,\psi))\\
&=\int_{t_0}^{t}(Q,\psi)+\frac{1}{H_sRt_2}\int_{t_0}^{t}(\partial_z T,\psi)+(T(t_0),\psi),
\end{aligned}\tag{34}$$

for every $\phi\in\mathscr{V}_1$ and $\psi\in\mathscr{V}_2$, almost every $t_0, t\in[0,\tau]$.

3. Some a Priori Estimates for (1)–(10)

We start with the following general existence and uniqueness of solutions which can be obtained by the standard Fatou-Galerkin methods [20–22] and similar methods in [4]. Here, we only state it as follows.

Theorem 6. *Assume that $(v_0,T_0)\in V$ and $Q\in H^1$ and let τ be any fixed positive time. Then, there exists a unique strong solution (v,T) of (14)–(18) on $[0,\tau]$, which satisfies*

$$(v,T)\in L^{\infty}(0,\tau;V)\cap L^2\left(0,\tau;H^2\right)\tag{35}$$

and depends continuously on the initial data (v_0,T_0); that is, the mapping $(v_0,T_0)\to(v,T)$ is continuous in V.

By Theorem 6, we can define the operator semigroup $\{S(t)\}_{t\ge0}$ in V as

$$S(\cdot)(v_0,T_0):\mathbb{R}^+\times V\longrightarrow V,\tag{36}$$

which is (V,V)-continuous.

In this section, we will give some a priori estimates which imply the existence of the (X,Z)-absorbing set B_0 for (1)–(10). That is, for any bounded subset $B\subset X$, there exists $T=T(B)$ such that $S(t)B\subset B_0$ for any $t\ge T$, where $\{S(t)\}_{t\ge0}$ is a semigroup on Banach space X generated by (1)–(10).

3.1. L^2 Estimates of T. Taking the L^2 inner product of (15) with T, using the boundary conditions (17) and using Lemma 3, we deduce that

$$\frac{1}{2}\frac{d}{dt}\|T\|_2^2+\|T\|^2\le\|Q\|_2\|T\|_2+\frac{1}{H_sRt_2}\left\|\frac{\partial T}{\partial z}\right\|_2\|T\|_2.\tag{37}$$

Employing Hölder's inequality yields

$$\frac{d}{dt}\|T\|_2^2+\|T\|^2\le\frac{2}{H_s^2Rt_2}\|T\|_2^2+H_s^2Rt_2\|Q\|_2^2.\tag{38}$$

Thanks to

$$\|T\|_2^2\le2h\|T(z=0)\|_{L^2(M)}^2+h^2\left\|\frac{\partial T}{\partial z}\right\|_2^2+\frac{h^2}{4H_s^2}\|T\|_2^2,\tag{39}$$

we obtain

$$\begin{aligned}
&\frac{\|T\|_2^2}{h^2Rt_2+2h/\alpha}\\
&\quad\le\frac{4H_s^2}{4H_s^2-h^2}\left(\frac{1}{Rt_2}\left\|\frac{\partial T}{\partial z}\right\|_2^2+\alpha\|T(z=0)\|_{L^2(M)}^2\right).
\end{aligned}\tag{40}$$

Setting $\delta=1-8(Rt_2h^2+2h/\alpha)/Rt_2(4H_s^2-h^2)$ and $\delta_1=\delta(4H_s^2-h^2)/4H_s^2(Rt_2h^2+2h/\alpha)$, we find that

$$\begin{aligned}
\frac{d}{dt}\|T\|_2^2+\delta\|T\|^2&\le H_s^2Rt_2\|Q\|_2^2,\\
\frac{d}{dt}\|T\|_2^2+\delta_1\|T\|_2^2&\le H_s^2Rt_2\|Q\|_2^2,
\end{aligned}\tag{41}$$

which implies that

$$\|T(t)\|_2^2\le\|T_0\|_2^2e^{-\delta_1t}+\frac{H_s^2Rt_2}{\delta_1}\|Q\|_2^2\left(1-e^{-\delta_1t}\right).\tag{42}$$

Therefore, there exist three positive constants ρ_1,ρ_2,T_1 such that

$$\|T(t)\|_2^2\le\rho_1,\tag{43}$$

$$\int_t^{t+1}\|T\|^2\,d\tau\le\rho_2,\tag{44}$$

for any $t\ge T_1$. For brevity, we omit writing out explicitly these bounds here, and we also omit writing out other similar bounds in our future discussion for all other uniform a priori estimates.

3.2. L^2 Estimates of v. Multiplying (14) by v, integrating over Ω, and using Lemma 3, we deduce that

$$\frac{1}{2}\frac{d}{dt}\|v\|_2^2+\|v\|^2\le C\|T\|_2\|\nabla v\|_2+\frac{1}{H_s\mathrm{Re}_2}\left\|\frac{\partial v}{\partial z}\right\|_2\|v\|_2.\tag{45}$$

Setting $\delta_3 = 1 - 1/\mathrm{Re}_2\lambda H_s^2$ and $\delta_4 = \lambda - 1/\mathrm{Re}_2 H_s^2$, we obtain

$$\begin{aligned} &\frac{d}{dt}\|v\|_2^2 + \delta_3\|v\|^2 \le C\|T\|_2^2, \\ &\frac{d}{dt}\|v\|_2^2 + \delta_4\|v\|_2^2 \le C\|T\|_2^2. \end{aligned} \tag{46}$$

From the classical Gronwall inequality, we have

$$\|v(t)\|_2^2 \le \|v_0\|_2^2 e^{-\delta_4 t} + C\rho_1, \tag{47}$$

for all $t \ge T_1$, which implies that there exist three positive constants ρ_3,ρ_4,$T_2 \ge T_1$ such that

$$\|v(t)\|_2^2 \le \rho_3, \tag{48}$$

$$\int_t^{t+1} \|v\|^2 d\tau \le \rho_4, \tag{49}$$

for all $t \ge T_2$.

3.3. $L^6(\Omega)$ Estimates of T. Taking the $L^2(\Omega)$ inner product of (15) with $|T|^4T$ and using Lemma 3, we have

$$\frac{1}{6}\frac{d}{dt}\|T\|_6^6 + \frac{5}{9}\left\||T|^3\right\|^2 \le \left\||T|^3\right\|_{10/3}^{5/3}\|Q\|_2 + \frac{1}{H_s R t_2}\left\||T|^3\right\|_{10/3}^{5/3}\|T_z\|_2. \tag{50}$$

From the inequality $\|\varphi\|_{10/3} \le C\|\varphi\|_2^{2/5}\|\varphi\|^{3/5}$, we get

$$\frac{d}{dt}\|T\|_6^6 + 2\left\||T|^3\right\|^2 \le C\|T_z\|_2^2\|T\|_6^4 + C\|Q\|_2^2\|T\|_6^4, \tag{51}$$

which implies that

$$\frac{d}{dt}\|T\|_6^2 \le C\|T_z\|_2^2 + C\|Q\|_2^2. \tag{52}$$

Therefore, by virtue of the uniform Gronwall lemma, from (44) and $\|T\|_6 \le C\|T\|_{H^1(\Omega)}$, we deduce that

$$\|T\|_6^2 \le \rho_5, \tag{53}$$

for any $t \ge T_2 + 1$.

3.4. $(L^6(\Omega))^2$ Estimates of $\widetilde{v}$. Multiplying (23) by $|\widetilde{v}|^4\widetilde{v}$, integrating over Ω, and using Lemma 3, we deduce that

$$\begin{aligned} &\frac{1}{6}\frac{d}{dt}\|\widetilde{v}\|_6^6 + \frac{1}{\mathrm{Re}_1}\int_\Omega e^{-z/H_s}|\nabla\widetilde{v}|^2|\widetilde{v}|^4 dx\,dy\,dz \\ &\quad + \frac{1}{\mathrm{Re}_2}\int_\Omega e^{-z/H_s}\left|\partial_z\widetilde{v}\right|^2|\widetilde{v}|^4 dx\,dy\,dz + \frac{4}{9\mathrm{Re}_1} \\ &\quad \times \int_\Omega e^{-z/H_s}\left|\nabla|\widetilde{v}|^3\right|^2 dx\,dy\,dz \\ &\quad + \frac{4}{9\mathrm{Re}_2}\int_\Omega e^{-z/H_s}\left|\partial_z|\widetilde{v}|^3\right|^2 dx\,dy\,dz \\ &\le \frac{1}{\mathrm{Re}_2 H_s}\|\widetilde{v}_z\|_2\left\||\widetilde{v}|^3\right\|_{10/3}^{5/3} + C\|\widetilde{v}\|_4\|\widetilde{v}\|_6^{3/2}\left\||\widetilde{v}|^3\right\|^{1/2} \\ &\quad \times \left(\int_\Omega |\nabla\widetilde{v}|^2|\widetilde{v}|^4 dx\,dy\,dz\right)^{1/2} + C\|\nabla T\|_2\|\widetilde{v}\|_6^2\left\||\widetilde{v}|^3\right\| \\ &\quad + \int_\Omega e^{-z/H_s}\overline{(\widetilde{v}\cdot\nabla)\widetilde{v} + (\nabla\cdot\widetilde{v})\widetilde{v}}\cdot|\widetilde{v}|^4\widetilde{v} dx\,dy\,dz. \end{aligned} \tag{54}$$

It follows from the Sobolev interpolation inequality and Lemma 4 that

$$\begin{aligned} &\int_\Omega e^{-z/H_s}\overline{(\widetilde{v}\cdot\nabla)\widetilde{v} + (\nabla\cdot\widetilde{v})\widetilde{v}}\cdot|\widetilde{v}|^4\widetilde{v} dx\,dy\,dz \\ &\quad \le C\int_M\left(\int_{-h}^0 e^{-z/H_s}|\nabla\widetilde{v}|\,|\widetilde{v}|^4 dz\right) \\ &\qquad \times\left(\int_{-h}^0 e^{-z/H_s}|\widetilde{v}|^2\, dz\right) dx\,dy \\ &\quad \le C\int_M\left(\int_{-h}^0 e^{-z/H_s}|\nabla\widetilde{v}|^2|\widetilde{v}|^4 dz\right)^{1/2} \\ &\qquad \times\left(\int_{-h}^0 e^{-z/H_s}|\widetilde{v}|^4 dz\right) dx\,dy \\ &\quad \le C\left(\int_\Omega e^{-z/H_s}|\nabla\widetilde{v}|^2|\widetilde{v}|^4 dx\,dy\,dz\right)^{1/2} \\ &\qquad \times\left(\int_M\left(\int_{-h}^0 e^{-z/H_s}|\widetilde{v}|^4 dz\right)^2 dx\,dy\right)^{1/2}, \\ &\left(\int_M\left(\int_{-h}^0 e^{-z/H_s}|\widetilde{v}|^4 dz\right)^2 dx\,dy\right)^{1/2} \\ &\quad \le \int_{-h}^0 e^{-z/H_s}\left(\int_M|\widetilde{v}|^8\, dx\,dy\right)^{1/2} dz \\ &\quad \le C\int_{-h}^0 e^{-z/H_s}\|\widetilde{v}\|_{L^6(M)}^3\|\widetilde{v}\|_{H^1(M)}\, dz \\ &\quad \le C\|\widetilde{v}\|_6^3\,\|\widetilde{v}\|, \end{aligned} \tag{55}$$

which implies that

$$\begin{aligned}&\int_\Omega e^{-z/H_s}\overline{(\widetilde{v}\cdot\nabla)\,\widetilde{v}+(\nabla\cdot\widetilde{v})\,\widetilde{v}}\cdot|\widetilde{v}|^4\widetilde{v}\,dx\,dy\,dz\\&\quad\le C\Big(\int_\Omega e^{-z/H_s}|\nabla\widetilde{v}|^2|\widetilde{v}|^4\,dx\,dy\,dz\Big)^{1/2}\|\widetilde{v}\|_6^3\,\|\widetilde{v}\|,\end{aligned}\tag{56}$$

where we use the inequality $\|\widetilde{v}\|_{L^8(M)}^8\le C\|\widetilde{v}\|_{L^6(M)}^6\|\widetilde{v}\|_{H^1(M)}^2$.

We deduce from (54) and (56) that

$$\frac{d}{dt}\|\widetilde{v}\|_6^2\le C\left(\|\widetilde{v}\|^2+\|\overline{v}\|_4^4\right)\|\widetilde{v}\|_6^2+C\|\widetilde{v}_z\|_2^2+C\|\nabla T\|_2^2.\tag{57}$$

Therefore, it follows from (44)–(49) and the uniform Gronwall lemma that

$$\|\widetilde{v}\|_6^2\le\rho_6,\tag{58}$$

$$\int_t^{t+1}\int_\Omega e^{-z/H_s}|\nabla\widetilde{v}|^2|\widetilde{v}|^4dx\,dy\,dz\,d\tau\le\rho_7,\tag{59}$$

for any $t\ge T_2+2$.

3.5. $(H^1(M))^2$ Estimates of $\overline{v}$. Taking the $L^2(\Omega)$ inner product of (21) with $-\Delta\overline{v}$ and combining the boundary conditions (22), we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla\overline{v}\|_{L^2(M)}^2+\frac{1}{\mathrm{Re}_1}\int_M|\Delta\overline{v}|^2\,dx\,dy\\&\quad\le\frac{1}{\mathrm{Re}_2H_s}\|\widetilde{v}_z\|_2\|\Delta\overline{v}\|_2+C\|\overline{v}\|_4\|\nabla\overline{v}\|_4\|\Delta\overline{v}\|_2\\&\qquad+C\int_M\left(\int_{-h}^0e^{-z/H_s}\,|\nabla\widetilde{v}|\,|\widetilde{v}|\,dz\right)|\Delta\overline{v}|\,dx\,dy\\&\qquad+C\|\overline{v}\|_2\|\Delta\overline{v}\|_2+C\|\nabla T\|_2\|\Delta\overline{v}\|_2.\end{aligned}\tag{60}$$

By using the Hölder inequality, we get

$$\begin{aligned}&\int_M\left(\int_{-h}^0e^{-z/H_s}\,|\nabla\widetilde{v}|\,|\widetilde{v}|\,dz\right)|\Delta\overline{v}|\,dx\,dy\\&\le\int_M\left(\int_{-h}^0e^{-z/H_s}\,|\nabla\widetilde{v}|\,dz\right)^{1/2}\\&\quad\times\left(\int_{-h}^0e^{-z/H_s}\,|\nabla\widetilde{v}|\,|\widetilde{v}|^2dz\right)^{1/2}|\Delta\overline{v}|\,dx\,dy\\&\le\left(\int_M\left(\int_{-h}^0e^{-z/H_s}\,|\nabla\widetilde{v}|\,dz\right)^2dx\,dy\right)^{1/4}\\&\quad\times\left(\int_M\left(\int_{-h}^0e^{-z/H_s}\,|\nabla\widetilde{v}|\,|\widetilde{v}|^2dz\right)^2dx\,dy\right)^{1/4}\|\Delta\overline{v}\|_2\\&\le\left(\int_{-h}^0e^{-z/H_s}\Big(\int_M|\nabla\widetilde{v}|^2dx\,dy\Big)^{1/2}\,dz\right)^{1/2}\\&\quad\times\left(\int_{-h}^0e^{-z/H_s}\Big(\int_M|\nabla\widetilde{v}|^2|\widetilde{v}|^4\,dx\,dy\Big)^{1/2}\,dz\right)^{1/2}\|\Delta\overline{v}\|_2\\&\le C\|\nabla\overline{v}\|_2^{1/2}\Big(\int_\Omega e^{-z/H_s}|\nabla\widetilde{v}|^2|\widetilde{v}|^4dx\,dy\,dz\Big)^{1/4}\|\Delta\overline{v}\|_2.\end{aligned}\tag{61}$$

It follows from (60) and (61) that

$$\begin{aligned}&\frac{d}{dt}\|\nabla\overline{v}\|_{L^2(M)}^2+\frac{1}{\mathrm{Re}_1}\int_M|\Delta\overline{v}|^2dx\,dy\\&\le C\|\overline{v}\|_{L^2(M)}^2\|\nabla\overline{v}\|_{L^2(M)}^4+C\|\widetilde{v}_z\|_2^2+C\|\nabla T\|_2^2+C\|\nabla\overline{v}\|_2^2\\&\quad+C\left(\int_\Omega e^{-z/H_s}|\nabla\widetilde{v}|^2|\widetilde{v}|^4dx\,dy\,dz\right)+C\|\overline{v}\|_2^2.\end{aligned}\tag{62}$$

In view of (44)–(49) and (59) and the uniform Gronwall lemma, we obtain

$$\|\nabla\overline{v}\|_{L^2(M)}^2\le\rho_8,\tag{63}$$

for any $t\ge T_2+3$.

3.6. $(L^2(\Omega))^2$ Estimates of v_z. Denote that $u=v_z$. It is clear that u satisfies the following equation obtained by differentiating (14) with respect to z:

$$\begin{aligned}&\frac{\partial u}{\partial t}+L_1u+(v\cdot\nabla)\,u\\&\quad-\left(\int_{-h}^ze^{(z-\zeta)/H_s}\nabla\cdot v\,(x,y,\zeta,t)\,d\zeta\right)\frac{\partial u}{\partial z}\\&\quad-\frac{1}{H_s}\left(\int_{-h}^ze^{(z-\zeta)/H_s}\nabla\cdot v\,(x,y,\zeta,t)\,d\zeta\right)u\\&\quad+(u\cdot\nabla)\,v-(\nabla\cdot v)\,u+f_0\vec{k}\times u+\frac{R}{H_s}\nabla T=0,\end{aligned}\tag{64}$$

with the boundary conditions

$$\begin{aligned}&u|_{\Gamma_u}=0,\qquad u|_{\Gamma_b}=0,\\&u\cdot\vec{n}|_{\Gamma_l}=0,\qquad\frac{\partial u}{\partial\vec{n}}\times\vec{n}\Big|_{\Gamma_l}=0.\end{aligned}\tag{65}$$

Multiplying (64) by u, integrating over Ω, and using Lemma 3, we get

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \|u\|^2 \le \frac{1}{\mathrm{Re}_2 H_s}\|u_z\|_2\|u\|_2 + C\|v\|_6\|u\|_3\|\nabla u\|_2 + C\|T\|_2\|\nabla u\|_2. \tag{66}$$

It follows from the Young inequality and the Sobolev embedding theorem that

$$\frac{d}{dt}\|u\|_2^2 + \|u\|^2 \le C\left(1 + \|v\|_6^4\right)\|u\|_2^2 + C\|T\|_2^2. \tag{67}$$

Since

$$\|v\|_6 \le Ch^{-1/3}\|v\|_2 + Ch^{1/6}\|\nabla\overline{v}\|_2 + \|\widetilde{v}\|_6, \tag{68}$$

which implies that

$$\|v\|_6^2 \le \rho_9, \tag{69}$$

for any $t \ge T_2 + 3$.

By virtue of the uniform Gronwall lemma, from (43), (49), and (69), we deduce that

$$\|v_z\|_2^2 \le \rho_{10}, \tag{70}$$

$$\int_t^{t+1}\|v_z\|^2\,d\tau \le \rho_{11}, \tag{71}$$

for any $t \ge T_2 + 4$.

3.7. $(L^2(\Omega))^2$ Estimates of ∇v. Taking the $L^2(\Omega)$ inner product of (14) with $-\Delta v$ and combining the boundary conditions (16), we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla v\|_2^2 + \frac{1}{\mathrm{Re}_1}\int_\Omega e^{-z/H_s}|\Delta v|^2\,dx\,dy\,dz \\ &\quad + \frac{1}{\mathrm{Re}_2}\int_\Omega e^{-z/H_s}|\nabla v_z|^2\,dx\,dy\,dz \\ &\le \frac{1}{\mathrm{Re}_2 H_s}\|v_z\|_2\|\Delta v\|_2 + C\|v_z\|_2^{1/2}\|\nabla v_z\|_2^{1/2}\|\nabla v\|_2^{1/2}\|\Delta v\|_2^{3/2} \\ &\quad + C\|\nabla T\|_2\|\Delta v\|_2 + C\|v\|_6\|\nabla v\|_3\|\Delta v\|_2 + C\|v\|_2\|\Delta v\|_2.\end{aligned} \tag{72}$$

We derive from (72) and the Young inequality that

$$\begin{aligned}&\frac{d}{dt}\|\nabla v\|_2^2 + \frac{1}{\mathrm{Re}_1}\int_\Omega e^{-z/H_s}|\Delta v|^2 dx\,dy\,dz \\ &\quad + \frac{1}{\mathrm{Re}_2}\int_\Omega e^{-z/H_s}|\nabla v_z|^2 dx\,dy\,dz \\ &\le C\left(\|v\|_6^4 + \|v_z\|_2^2\|\nabla v_z\|_2^2\right)\|\nabla v\|_2^2 + C\|\nabla T\|_2^2 + C\|v\|_2^2 + C\|v_z\|_2^2.\end{aligned} \tag{73}$$

By virtue of (44)–(49), (69)–(71), and the uniform Gronwall lemma, we get

$$\|\nabla v\|_2^2 \le \rho_{12}, \tag{74}$$

$$\begin{aligned}&\frac{1}{\mathrm{Re}_1}\int_t^{t+1}\int_\Omega e^{-z/H_s}|\Delta v|^2 dx\,dy\,dz\,d\tau \\ &\quad + \frac{1}{\mathrm{Re}_2}\int_t^{t+1}\int_\Omega e^{-z/H_s}|\nabla v_z|^2 dx\,dy\,dz\,d\tau \le \rho_{13},\end{aligned} \tag{75}$$

for any $t \ge T_2 + 5$.

3.8. $L^2(\Omega)$ Estimates of T_z. Denote $\theta = T_z$. It is clear that θ satisfies the following equation obtained by differentiating (15) with respect to z:

$$\begin{aligned}&\frac{\partial\theta}{\partial t} + L_2\theta + v\cdot\nabla\theta - \left(\int_{-h}^{z} e^{(z-\zeta)/H_s}\nabla\cdot v(x,y,\zeta,t)\,d\zeta\right)\frac{\partial\theta}{\partial z} \\ &\quad - \frac{1}{H_s}\left(\int_{-h}^{z} e^{(z-\zeta)/H_s}\nabla\cdot v(x,y,\zeta,t)\,d\zeta\right)\theta \\ &\quad + v_z\cdot\nabla T - (\nabla\cdot v)\theta = \partial_z Q,\end{aligned} \tag{76}$$

with the boundary conditions

$$\left(\frac{1}{Rt_2}\theta + \alpha T\right)\Big|_{\Gamma_u} = 0, \qquad \theta|_{\Gamma_b} = 0, \qquad \frac{\partial\theta}{\partial\vec{n}}\Big|_{\Gamma_l} = 0. \tag{77}$$

Multiplying (76) by θ, integrating over Ω, and using Lemma 3, we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\theta\|_2^2 + \frac{1}{Rt_1}\int_\Omega e^{-z/H_s}|\nabla\theta|^2\,dx\,dy\,dz \\ &\quad + \frac{1}{Rt_2}\int_\Omega e^{-z/H_s}|\partial_z\theta|^2\,dx\,dy\,dz + \alpha\int_{\Gamma_u}\frac{\partial\theta}{\partial z}T\,dx\,dy \\ &\le \frac{1}{Rt_2 H_s}\|\theta_z\|_2\|\theta\|_2 + C\|v\|_6\|\theta\|_3\|\nabla\theta\|_2 + C\|Q_z\|_2\|\theta\|_2 \\ &\quad + C\|\nabla v_z\|_2\|T\|_6\|\theta\|_3 + C\|v_z\|_3\|T\|_6\|\nabla\theta\|_2.\end{aligned} \tag{78}$$

We deduce from the Young inequality that

$$\begin{aligned}
&\alpha \int_{\Gamma_u} \frac{\partial \theta}{\partial z} T \, dx\, dy \\
&\quad = \alpha Rt_2 \int_{\Gamma_u} \left(\frac{\partial T}{\partial t} + v \cdot \nabla T - \frac{1}{Rt_1} \Delta T - Q \right) T \, dx\, dy \\
&\quad = \frac{\alpha Rt_2}{2} \frac{d}{dt} \int_{\Gamma_u} |T|^2 \, dx\, dy + \alpha Rt_2 \int_{\Gamma_u} (v \cdot \nabla T)\, T \, dx\, dy \\
&\qquad + \frac{\alpha Rt_2}{Rt_1} \int_{\Gamma_u} |\nabla T|^2 \, dx\, dy - \alpha Rt_2 \int_{\Gamma_u} QT \, dx \, dy, \\
&\left| \alpha Rt_2 \int_{\Gamma_u} (v \cdot \nabla T)\, T \, dx\, dy - \alpha Rt_2 \int_{\Gamma_u} QT \, dx\, dy \right| \\
&\quad \le C \|v\|_{L^4(\Gamma_u)} \|\nabla T\|_{L^2(\Gamma_u)} \|T\|_{L^4(\Gamma_u)} + C \|Q\|_{L^2(\Gamma_u)} \|T\|_{L^2(\Gamma_u)} \\
&\quad \le C \|v\| \, \|\nabla T\|^{3/2}_{L^2(\Gamma_u)} \|T\|^{1/2}_{L^2(\Gamma_u)} + C \|Q\| \, \|T\|_{L^2(\Gamma_u)}.
\end{aligned} \tag{79}$$

It follows from (78) and (79) that

$$\begin{aligned}
&\frac{d}{dt} \left(\|\theta\|_2^2 + \alpha Rt_2 \|T\|^2_{L^2(\Gamma_u)} \right) \\
&\qquad + \frac{1}{Rt_1} \int_\Omega e^{-z/H_s} |\nabla \theta|^2 dx\, dy\, dz \\
&\qquad + \frac{1}{Rt_2} \int_\Omega e^{-z/H_s} |\theta_z|^2 dx\, dy\, dz + \frac{\alpha Rt_2}{Rt_1} \int_{\Gamma_u} |\nabla T|^2 dx\, dy \\
&\quad \le C \left(1 + \|v\|_6^4 + \|T\|_6^4 + \|v\|^4 \right) \left(\|\theta\|_2^2 + \alpha Rt_2 \|T\|^2_{L^2(\Gamma_u)} \right) \\
&\qquad + C \left(\|Q\|^2 + \|v_z\|^2 + \|v_z\|_2^2 \|T\|_6^4 \right).
\end{aligned} \tag{80}$$

As a result of (44), (53), (69)–(71), and (74), we get

$$\|T_z\|_2^2 + \alpha Rt_2 \|T\|^2_{L^2(\Gamma_u)} \le \rho_{14}, \tag{81}$$

$$\begin{aligned}
&\frac{1}{Rt_1} \int_t^{t+1} \int_\Omega e^{-z/H_s} |\nabla T_z|^2 dx\, dy\, dz\, d\tau \\
&\qquad + \frac{1}{Rt_2} \int_t^{t+1} \int_\Omega e^{-z/H_s} \left| \partial_z^2 T \right|^2 dx\, dy\, dz\, d\tau \\
&\qquad + \frac{\alpha Rt_2}{Rt_1} \int_t^{t+1} \int_{\Gamma_u} |\nabla T|^2 dx\, dy\, d\tau \le \rho_{15},
\end{aligned} \tag{82}$$

for any $t \ge T_2 + 6$.

3.9. $L^2(\Omega)$ Estimates of ∇T. Taking the $L^2(\Omega)$ inner product of (15) with $-\Delta T$ and combining the boundary conditions (17), we deduce that

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|\nabla T\|_2^2 + \frac{1}{Rt_1} \int_\Omega e^{-z/H_s} |\Delta T|^2 dx\, dy\, dz \\
&\qquad + \frac{1}{Rt_2} \int_\Omega |\nabla T_z|^2 dx\, dy\, dz + \alpha \int_{\Gamma_u} |\nabla T|^2 dx\, dy \\
&\quad \le \frac{1}{Rt_2 H_s} \|T_z\|_2 \|\Delta T\|_2 + C \|v\|_6 \|\nabla T\|_3 \|\Delta T\|_2 + C \|Q\|_2 \|\Delta T\|_2 \\
&\qquad + C \|\nabla v\|_2^{1/2} \|\Delta v\|_2^{1/2} \|T_z\|_2^{1/2} \|\nabla T_z\|_2^{1/2} \|\Delta T\|_2.
\end{aligned} \tag{83}$$

We derive from the Young inequality that

$$\begin{aligned}
&\frac{d}{dt} \|\nabla T\|_2^2 + \frac{1}{Rt_1} \int_\Omega e^{-z/H_s} |\Delta T|^2 dx\, dy\, dz \\
&\qquad + \frac{1}{Rt_2} \int_\Omega e^{-z/H_s} |\nabla T_z|^2 dx\, dy\, dz \\
&\qquad + \alpha \int_M |\nabla T\,(z=0)|^2 dx\, dy \\
&\quad \le C \|v\|_6^4 \|\nabla T\|_2^2 + C \|\nabla v\|_2^2 \|\Delta v\|_2^2 \|T_z\|_2^2 + C \|Q\|_2^2 + C \|T_z\|_2^2.
\end{aligned} \tag{84}$$

Using (44), (69), (74), (75), and (81), we get

$$\|\nabla T\|_2^2 \le \rho_{16}, \tag{85}$$

$$\begin{aligned}
&\frac{1}{Rt_1} \int_t^{t+1} \int_\Omega e^{-z/H_s} |\Delta T|^2 \, dx\, dy\, dz\, d\tau \\
&\qquad + \frac{1}{Rt_2} \int_t^{t+1} \int_\Omega e^{-z/H_s} |\nabla T_z|^2 \, dx\, dy\, dz\, d\tau \\
&\qquad + \alpha \int_t^{t+1} \int_M |\nabla T\,(z=0)|^2 \, dx\, dy\, d\tau \\
&\quad \le \rho_{17},
\end{aligned} \tag{86}$$

for any $t \ge T_2 + 7$.

3.10. $(L^6(\Omega))^2$ *Estimates of* $\partial_z v$. Taking the $L^2(\Omega)$ inner product of (64) with $|u|^4u$, combining the boundary conditions (65), and using Lemma 3, we deduce that

$$\begin{aligned}
&\frac{1}{6}\frac{d}{dt}\|u\|_6^6 + \frac{1}{\mathrm{Re}_1}\int_\Omega e^{-z/H_s}|\nabla u|^2|u|^4dx\,dy\,dz \\
&\quad + \frac{1}{\mathrm{Re}_2}\int_\Omega e^{-z/H_s}\left|\partial_z u\right|^2|u|^4dx\,dy\,dz \\
&\quad + \frac{4}{9\mathrm{Re}_1}\int_\Omega e^{-z/H_s}\left|\nabla|u|^3\right|^2dx\,dy\,dz \\
&\quad + \frac{4}{9\mathrm{Re}_2}\int_\Omega e^{-z/H_s}\left|\partial_z\,|u|^3\right|^2dx\,dy\,dz \\
&\le \frac{1}{\mathrm{Re}_2 H_s}\|u_z\|_2\left\||u|^3\right\|_{10/3}^{5/3} + C\|v\|_6\left\||u|^3\right\|_3\left\|\nabla|u|^3\right\|_2 \\
&\quad + C\|\nabla T\|_2\left\||u|^3\right\|_{10/3}^{5/3} + C\|v\|_6\left\||u|^3\right\|_3\left\|\nabla u|\,|u|^2\right\|_2.
\end{aligned} \tag{87}$$

From the Young inequality, we obtain

$$\frac{d}{dt}\|u\|_6^2 \le C\|v\|_6^4\|u\|_6^2 + C\|\nabla T\|_2^2 + C\|u_z\|_2^2. \tag{88}$$

Thanks to (44), (69), and (71), we have

$$\|v_z\|_6^2 \le \rho_{18}, \tag{89}$$

for any $t \ge T_2 + 8$.

3.11. $L^6(\Omega)$ *Estimates of* $\partial_z T$. Taking the $L^2(\Omega)$ inner product of (76) with $|\theta|^4\theta$, combining the boundary conditions (77), and using Lemma 3, we deduce that

$$\begin{aligned}
&\frac{1}{6}\frac{d}{dt}\|\theta\|_6^6 + \frac{5}{9Rt_1}\int_\Omega e^{-z/H_s}\left|\nabla|\theta|^3\right|^2dx\,dy\,dz \\
&\quad + \frac{5}{9Rt_2}\int_\Omega e^{-z/H_s}\left|\partial_z|\theta|^3\right|^2dx\,dy\,dz \\
&\quad + \alpha^5Rt_2^4\int_{\Gamma_u}\frac{\partial\theta}{\partial z}|T|^4T\,dx\,dy \\
&\le \frac{1}{Rt_2H_s}\|\theta_z\|_2\left\||\theta|^3\right\|_{10/3}^{5/3} + C\|v\|_6\left\||\theta|^3\right\|_3\left\|\nabla\,|\theta|^3\right\|_2 \\
&\qquad + C\|v\|_6\left\||\theta|^3\right\|_3\left\|\nabla\theta|\,|\theta|^2\right\|_2 + \|Q_z\|_2\left\||\theta|^3\right\|_{10/3}^{5/3} \\
&\quad + \left|-\int_\Omega e^{-z/H_s}\left(v_z\cdot\nabla T\right)|\theta|^4\theta dx\,dy\,dz\right|.
\end{aligned} \tag{90}$$

It follows from the Hölder inequality that

$$\begin{aligned}
&\left|-\int_\Omega e^{-z/H_s}\left(v_z\cdot\nabla T\right)|\theta|^4\theta dx\,dy\,dz\right| \\
&\quad \le C\int_\Omega e^{-z/H_s}\left|v_z\right|\left|\nabla T\right|\left|\theta\right|^5dx\,dy\,dz \\
&\quad \le C\|v_z\|_6\|\nabla T\|_3\left\||\theta|^3\right\|_{10/3}^{5/3} \\
&\quad \le C\|v_z\|_6\|\nabla T\|_2^{1/2}\|\nabla T\|_{H^1(\Omega)}^{1/2}\left\||\theta|^3\right\|_2^{2/3}\left\||\theta|^3\right\|_{H^1(\Omega)}, \\
&\alpha^5Rt_2^4\int_{\Gamma_u}\frac{\partial\theta}{\partial z}|T|^4Tdx\,dy \\
&\quad = \alpha^5Rt_2^5\int_{\Gamma_u}\left(\frac{\partial T}{\partial t} + v\cdot\nabla T - \frac{1}{Rt_1}\Delta T - Q\right)|T|^4Tdx\,dy \\
&\quad = \frac{\alpha^5Rt_2^5}{6}\frac{d}{dt}\int_{\Gamma_u}|T|^6dx\,dy + \alpha^5Rt_2^5\int_{\Gamma_u}\left(v\cdot\nabla T\right)|T|^4Tdx\,dy \\
&\qquad + \frac{5\alpha^5Rt_2^5}{9Rt_1}\int_{\Gamma_u}\left|\nabla|T|^3\right|^2dx\,dy - \alpha^5Rt_2^5\int_{\Gamma_u}Q|T|^4Tdx\,dy, \\
&\left|\alpha^5Rt_2^5\int_{\Gamma_u}\left(v\cdot\nabla T\right)|T|^4Tdx\,dy - \alpha^5Rt_2^5\int_{\Gamma_u}Q|T|^4Tdx\,dy\right| \\
&\le C\|v\|_{L^4(\Gamma_u)}\left\|\nabla|T|^3\right\|_{L^2(\Gamma_u)}\left\||T|^3\right\|_{L^4(\Gamma_u)} + C\|Q\|_{L^2(\Gamma_u)}\left\||T|^3\right\|_{L^{10/3}(\Gamma_u)}^{5/3} \\
&\le C\|v\|\left\|\nabla\,|T|^3\right\|_{L^2(\Gamma_u)}^{3/2}\|T\|_{L^6(\Gamma_u)}^{3/2} + C\|Q\|\left\||T|^3\right\|_{L^2(\Gamma_u)}\left\||T|^3\right\|_{H^1(\Gamma_u)}^{2/3}.
\end{aligned} \tag{91}$$

We derive from the Young inequality that

$$\begin{aligned}
&\frac{d}{dt}\left(\|\theta\|_6^2 + \alpha Rt_2\|T\|_{L^6(\Gamma_u)}^2\right) \\
&\quad \le C\left(1 + \|v\|_6^4 + \|v\|^4\right)\left(\|\theta\|_6^2 + \alpha Rt_2\|T\|_{L^6(\Gamma_u)}^2\right) \\
&\qquad + C\|\theta_z\|_2^2 + C\|Q\|^2 + C\|v_z\|_6^4 + C\|\nabla T\|_2^2\|\nabla T\|_{H^1(\Omega)}^2.
\end{aligned} \tag{92}$$

As a consequence of the uniform Gronwall lemma, we get

$$\|T_z\|_6^2 + \alpha Rt_2\|T\|_{L^6(\Gamma_u)}^2 \le \rho_{19}, \tag{93}$$

for any $t \ge T_2 + 9$.

3.12. H Estimates of $(\partial_t v, \partial_t T)$. Denote that $\pi = \partial_t v$, $\xi = \partial_t T$. It is clear that π, ξ satisfy the following equations obtained by differentiating (14) and (15) with respect to t, respectively,

$$\begin{aligned}
&\frac{\partial \pi}{\partial t} + L_1\pi + (v\cdot\nabla)\,\pi \\
&\quad - \left(\int_{-h}^{z} e^{(z-\zeta)/H_s}\nabla\cdot v\,(x,y,\zeta,t)\,d\zeta\right)\frac{\partial \pi}{\partial z} \\
&\quad + f_0\vec{k}\times\pi + \nabla\partial_t\Phi_s\,(x,y,t) + (\pi\cdot\nabla)\,v \\
&\quad - \left(\int_{-h}^{z} e^{(z-\zeta)/H_s}\nabla\cdot \pi\,(x,y,\zeta,t)\,d\zeta\right)\frac{\partial v}{\partial z} \\
&\quad + \frac{R}{H_s}\int_0^z \nabla\xi\,(x,y,\zeta,t)\,d\zeta = 0,
\end{aligned} \tag{94}$$

$$\begin{aligned}
&\frac{\partial \xi}{\partial t} + L_2\xi + v\cdot\nabla\xi - \left(\int_{-h}^{z} e^{(z-\zeta)/H_s}\nabla\cdot v\,(x,y,\zeta,t)\,d\zeta\right)\frac{\partial \xi}{\partial z} \\
&\quad + \pi\cdot\nabla T - \left(\int_{-h}^{z} e^{(z-\zeta)/H_s}\nabla\cdot \pi\,(x,y,\zeta,t)\,d\zeta\right)\frac{\partial T}{\partial z} = 0,
\end{aligned} \tag{95}$$

with the boundary conditions

$$\begin{aligned}
&\left.\frac{\partial \pi}{\partial z}\right|_{\Gamma_u} = 0, \qquad \left.\frac{\partial \pi}{\partial z}\right|_{\Gamma_b} = 0, \\
&\pi\cdot\vec{n}|_{\Gamma_l} = 0, \qquad \left.\frac{\partial \pi}{\partial \vec{n}}\times\vec{n}\right|_{\Gamma_l} = 0,
\end{aligned} \tag{96}$$

$$\left.\left(\frac{1}{Rt_2}\frac{\partial \xi}{\partial z} + \alpha\xi\right)\right|_{\Gamma_u} = 0, \qquad \left.\frac{\partial \xi}{\partial z}\right|_{\Gamma_b} = 0, \qquad \left.\frac{\partial \xi}{\partial \vec{n}}\right|_{\Gamma_l} = 0, \tag{97}$$

Multiplying (94) and (95), respectively, by π, ξ, integrating over Ω, and using Lemma 3, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\pi\|_2^2 + \|\pi\|^2 \\
&\quad \le \frac{1}{\mathrm{Re}_2 H_s}\|\pi_z\|_2\|\pi\|_2 + C\|v\|_6\|\pi\|_3\|\nabla\pi\|_2 \\
&\qquad + C\|v_z\|_6\|\pi\|_3\|\nabla\pi\|_2 + C\|\nabla\pi\|_2\|\xi\|_2, \\
&\frac{1}{2}\frac{d}{dt}\|\xi\|_2^2 + \|\xi\|^2 \\
&\quad \le \frac{1}{Rt_2 H_s}\|\xi_z\|_2\|\xi\|_2 + C\|T\|_6\|\pi\|_3\|\nabla\xi\|_2 \\
&\qquad + C\|T\|_6\|\xi\|_3\|\nabla\pi\|_2 + C\|T_z\|_6\|\xi\|_3\|\nabla\pi\|_2.
\end{aligned} \tag{98}$$

It follows from the Young inequality that

$$\begin{aligned}
&\frac{d}{dt}\left(\|\xi\|_2^2 + \|\pi\|_2^2\right) \\
&\quad \le C\left(1 + \|T\|_6^4 + \|v\|_6^4 + \|T_z\|_6^4 + \|v_z\|_6^4\right)\left(\|\xi\|_2^2 + \|\pi\|_2^2\right).
\end{aligned} \tag{99}$$

Multiplying (15) by ξ and integrating over Ω, we obtain

$$\begin{aligned}
\|\xi\|_2^2 &\le \|Q\|_2\|\xi\|_2 + C\|v\|_6\|\nabla T\|_3\|\xi\|_2 \\
&\quad + \|L_2T\|_2\|\xi\|_2 + C\|T_z\|_6\|\nabla v\|_3\|\xi\|_2.
\end{aligned} \tag{100}$$

Similarly, we have

$$\begin{aligned}
\|\pi\|_2^2 &\le C\|v\|_6\|\nabla v\|_3\|\pi\|_2 + C\|v_z\|_6\|\nabla v\|_3\|\pi\|_2 \\
&\quad + C\|v\|_2\|\pi\|_2 + C\|\nabla T\|_2\|\pi\|_2 + \|L_1v\|_2\|\pi\|_2.
\end{aligned} \tag{101}$$

We derive from the Young inequality that

$$\begin{aligned}
\|\pi\|_2^2 + \|\xi\|_2^2 &\le C\left(1 + \|v\|_6^4 + \|v_z\|_6^4 + \|T_z\|_6^4\right)\left(\|\nabla T\|_2^2 + \|\nabla v\|_2^2\right) \\
&\quad + C\|Q\|_2^2 + C\|L_2T\|_2^2 + C\|L_1v\|_2^2 + C\|v\|_2^2.
\end{aligned} \tag{102}$$

By use of the uniform Gronwall inequality, we have

$$\|v_t\|_2^2 + \|T_t\|_2^2 \le \rho_{20}, \tag{103}$$

for any $t \ge T_2 + 10$.

Moreover, we have

$$\int_t^{t+1}\|v_t\|^2 d\tau + \int_t^{t+1}\|T_t\|^2 d\tau \le \rho_{21}, \tag{104}$$

for any $t \ge T_2 + 10$.

3.13. H Estimates of (L_1v, L_2T). Multiplying (15) by L_2T and integrating over Ω, we obtain

$$\begin{aligned}
\|L_2T\|_2^2 &\le C\|v\|_6\|\nabla T\|_3\|L_2T\|_2 + C\|T_z\|_6\|\nabla v\|_3\|L_2T\|_2 \\
&\quad + \|Q\|_2\|L_2T\|_2 + \|\partial_t T\|_2\|L_2T\|_2.
\end{aligned} \tag{105}$$

Similarly, we have

$$\begin{aligned}
\|L_1v\|_2^2 &\le C\|v\|_6\|\nabla v\|_3\|L_1v\|_2 + C\|v_z\|_6\|\nabla v\|_3\|L_1v\|_2 \\
&\quad + C\|v\|_2\|L_1v\|_2 + C\|\nabla T\|_2\|L_1v\|_2 + \|v_t\|_2\|L_1v\|_2.
\end{aligned} \tag{106}$$

It follows from the Young inequality that

$$\begin{aligned}
&\|L_1v\|_2^2 + \|L_2T\|_2^2 \\
&\quad \le C\left(1 + \|v\|_6^4 + \|T_z\|_6^4 + \|v_z\|_6^4\right)\left(\|\nabla T\|_2^2 + \|\nabla v\|_2^2\right) \\
&\qquad + C\left(\|v_t\|_2^2 + \|T_t\|_2^2\right) + C\left(\|v\|_2^2 + \|Q\|_2^2\right).
\end{aligned} \tag{107}$$

By virtue of (48), (69), (74), (85), (89), (93), and (103), we have

$$\|L_1v\|_2^2 + \|L_2T\|_2^2 \le \rho_{22}, \tag{108}$$

for any $t \ge T_2 + 10$.

3.14. L^2 Estimates of $(\nabla v_t, \nabla T_t)$. Taking the H_1 inner product of (94) with $-\Delta\pi$ and combining the boundary conditions (96), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\pi\|_2^2 + \frac{1}{\mathrm{Re}_1}\int_\Omega e^{-z/H_s}|\Delta\pi|^2 dx\,dy\,dz \\
&\quad + \frac{1}{\mathrm{Re}_2}\int_\Omega e^{-z/H_s}\left|\nabla\pi_z\right|^2 dx\,dy\,dz \\
&\le \frac{1}{\mathrm{Re}_2 H_s}\left\|\pi_z\right\|_2\|\Delta\pi\|_2 + C\|v\|_6\|\nabla\pi\|_3\|\Delta\pi\|_2 \\
&\quad + C\|\nabla v\|_2^{1/2}\|\Delta v\|_2^{1/2}\left\|\pi_z\right\|_2^{1/2}\left\|\nabla\pi_z\right\|_2^{1/2}\|\Delta\pi\|_2 \\
&\quad + C\|\pi\|_2\|\Delta\pi\|_2 + C\|\pi\|_6\|\nabla v\|_3\|\Delta\pi\|_2 \\
&\quad + C\|\nabla\pi\|_2^{1/2}\left\|v_z\right\|_2^{1/2}\left\|\nabla v_z\right\|_2^{1/2}\|\Delta\pi\|_2^{3/2} + C\left\|\nabla\xi\right\|_2\|\Delta\pi\|_2.
\end{aligned} \tag{109}$$

Therefore, in view of the Young inequality, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|\nabla\pi\|_2^2 + \frac{1}{\mathrm{Re}_1}\int_\Omega e^{-z/H_s}|\Delta\pi|^2 dx\,dy\,dz \\
&\quad + \frac{1}{\mathrm{Re}_2}\int_\Omega e^{-z/H_s}\left|\nabla\pi_z\right|^2 dx\,dy\,dz \\
&\le C\left(\|v\|_6^4 + \left\|v_z\right\|_2^2\left\|\nabla v_z\right\|_2^2\right)\|\nabla\pi\|_2^2 + C\|\pi\|_2^2 + C\left\|\nabla\xi\right\|_2^2 \\
&\quad + C\left\|\pi_z\right\|_2^2 + C\|\nabla v\|_2^2\|\Delta v\|_2^2\left\|\pi_z\right\|_2^2 + C\|\pi\|^2\|\nabla v\|_3^2.
\end{aligned} \tag{110}$$

Employing the uniform Gronwall inequality, we get

$$\left\|\nabla v_t\right\|_2^2 \le \rho_{22}, \tag{111}$$

for any $t \ge T_2 + 11$.

Moreover, we have

$$\begin{aligned}
&\frac{1}{\mathrm{Re}_1}\int_t^{t+1}\int_\Omega e^{-z/H_s}\left|\Delta v_t\right|^2 dx\,dy\,dz\,d\tau \\
&\quad + \frac{1}{\mathrm{Re}_2}\int_t^{t+1}\int_\Omega e^{-z/H_s}\left|\nabla\partial_z v_t\right|^2 dx\,dy\,dz\,d\tau \le \rho_{23}.
\end{aligned} \tag{112}$$

Similarly, we can also obtain

$$\begin{aligned}
&\left\|\nabla T_t\right\|_2^2 \le \rho_{24}, \\
&\frac{1}{\mathrm{Re}_1}\int_t^{t+1}\int_\Omega e^{-z/H_s}\left|\Delta T_t\right|^2 dx\,dy\,dz\,d\tau \\
&\quad + \frac{1}{\mathrm{Re}_2}\int_t^{t+1}\int_\Omega e^{-z/H_s}\left|\nabla\partial_z T_t\right|^2 dx\,dy\,dz\,d\tau \le \rho_{25},
\end{aligned} \tag{113}$$

for any $t \ge T_2 + 12$.

3.15. L^2 Estimates of $(\partial_z v_t, \partial_z T_t)$. Denote that $\beta = \pi_z$, $\gamma = \xi_z$. It is clear that β, γ satisfy the following equations obtained by differentiating (94) and (95) with respect to z, respectively,

$$\begin{aligned}
&\frac{\partial\beta}{\partial t} + L_1\beta + \left(v_z\cdot\nabla\right)\pi + \left(v\cdot\nabla\right)\beta \\
&\quad - \left(\int_{-h}^{z} e^{(z-\zeta)/H_s}\nabla\cdot v\left(x,y,\zeta,t\right)d\zeta\right)\frac{\partial\beta}{\partial z} \\
&\quad - \frac{1}{H_s}\left(\int_{-h}^{z} e^{(z-\zeta)/H_s}\nabla\cdot v\left(x,y,\zeta,t\right)d\zeta\right)\beta \\
&\quad - \left(\nabla\cdot v\right)\beta + f_0\vec{k}\times\beta + \left(\beta\cdot\nabla\right)v + \left(\pi\cdot\nabla\right)v_z \\
&\quad - \left(\int_{-h}^{z} e^{(z-\zeta)/H_s}\nabla\cdot\pi\left(x,y,\zeta,t\right)d\zeta\right)\frac{\partial v_z}{\partial z} \\
&\quad - \frac{1}{H_s}\left(\int_{-h}^{z} e^{(z-\zeta)/H_s}\nabla\cdot\pi\left(x,y,\zeta,t\right)d\zeta\right)\frac{\partial v}{\partial z} \\
&\quad - \left(\nabla\cdot\pi\right)v_z + \frac{R}{H_s}\nabla\xi = 0,
\end{aligned} \tag{114}$$

$$\begin{aligned}
&\frac{\partial\gamma}{\partial t} + L_2\gamma + v_z\cdot\nabla\xi + v\cdot\nabla\gamma \\
&\quad - \left(\int_{-h}^{z} e^{(z-\zeta)/H_s}\nabla\cdot v\left(x,y,\zeta,t\right)d\zeta\right)\frac{\partial\gamma}{\partial z} \\
&\quad - \left(\nabla\cdot\pi\right)T_z - \frac{1}{H_s}\left(\int_{-h}^{z} e^{(z-\zeta)/H_s}\nabla\cdot v\left(x,y,\zeta,t\right)d\zeta\right)\gamma \\
&\quad - \left(\nabla\cdot v\right)\gamma + \beta\cdot\nabla T + \pi\cdot\nabla T_z \\
&\quad - \left(\int_{-h}^{z} e^{(z-\zeta)/H_s}\nabla\cdot\pi\left(x,y,\zeta,t\right)d\zeta\right)\frac{\partial T_z}{\partial z} \\
&\quad - \frac{1}{H_s}\left(\int_{-h}^{z} e^{(z-\zeta)/H_s}\nabla\cdot\pi\left(x,y,\zeta,t\right)d\zeta\right)\frac{\partial T}{\partial z} = 0,
\end{aligned} \tag{115}$$

with the boundary conditions

$$\begin{aligned}
&\beta\big|_{\Gamma_u} = 0, \qquad \beta\big|_{\Gamma_b} = 0, \qquad \beta\cdot\vec{n}\big|_{\Gamma_l} = 0, \\
&\frac{\partial\beta}{\partial\vec{n}}\times\vec{n}\big|_{\Gamma_l} = 0,
\end{aligned} \tag{116}$$

$$\left(\frac{1}{Rt_2}\gamma + \alpha\xi\right)\Big|_{\Gamma_u} = 0, \qquad \gamma\big|_{\Gamma_b} = 0, \qquad \frac{\partial\gamma}{\partial\vec{n}}\Big|_{\Gamma_l} = 0. \tag{117}$$

Taking the L^2 inner product of (114) with β and combining the boundary conditions (116), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\beta\|_2^2 + \|\beta\|^2 \\
&\quad \le \frac{1}{\mathrm{Re}_2 H_s}\left\|\beta_z\right\|_2\|\beta\|_2 + C\left\|v_z\right\|_6\|\nabla\pi\|_3\|\beta\|_2 \\
&\quad + C\|v\|_6\|\beta\|_3\|\nabla\beta\|_2 + C\|\pi\|_3\left\|\nabla v_z\right\|_2\|\beta\|_6 \\
&\quad + C\|\nabla\pi\|_3\left\|\partial_z v_z\right\|_2\|\beta\|_6 + C\left\|\xi\right\|_2\|\nabla\beta\|_2.
\end{aligned} \tag{118}$$

Therefore, in view of the Young inequality, we obtain

$$\begin{aligned}&\frac{d}{dt}\|\beta\|_2^2+\|\beta\|^2\\&\quad\le C\left(1+\|v\|_6^4\right)\|\beta\|_2^2+C\|v_z\|_6^2\|\nabla\pi\|_3^2\\&\qquad+C\|\pi\|_3^2\|\nabla v_z\|_2^2+C\|\nabla\pi\|_3^2\|\partial_z v_z\|_2^2+C\|\xi\|_2^2.\end{aligned}\tag{119}$$

Employing the uniform Gronwall inequality, we get

$$\|\partial_z v_t\|_2^2\le\rho_{26},\tag{120}$$

for any $t\ge T_2+13$.

Similarly, we can also obtain

$$\int_\Omega e^{-z/H_s}\left|\partial_z T_t\right|^2dx\,dy\,dz+\alpha Rt_2\int_{\Gamma_u}\left|T_t\right|^2dx\,dy\le\rho_{27},\tag{121}$$

for any $t\ge T_2+14$.

From (43), (48), (70), (74), (81), (85) and (108), we get the following results.

Theorem 7. *Assume that $Q\in H^1(\Omega)$ and $(v_0,T_0)\in V$. Let $\{S(t)\}_{t\ge0}$ be a semigroup generated by the initial boundary problem* (14)–(18). *Then, there exists an absorbing set in H^2. That is, for any bounded subset B of V, there exist a positive constant $\mathscr{R}_1$ and a positive time $\tau_1=\tau_{1,B}$ which depend on the norm of B, such that*

$$\begin{aligned}\|(v(t),T(t))\|_{H^2}^2&=\left\|S(t)\left(v_0,T_0\right)\right\|_{H^2}^2\\&=\|v(t)\|_{H^2}^2+\|T(t)\|_{H^2}^2\\&\le\mathscr{R}_1,\end{aligned}\tag{122}$$

for any $t\ge\tau_1$.

Corollary 8. *Assume that $Q\in H^1$ and $(v_0,T_0)\in V$. Let $\{S(t)\}_{t\ge0}$ be a semigroup generated by the initial boundary problem* (14)–(18). *Then, there exist an absorbing set in V. That is, for any bounded subset B of V, there exist a positive constant $\mathscr{R}_2$ and a positive time $\tau_2=\tau_{2,B}$, depending on the norm of B, such that*

$$\begin{aligned}\|(v(t),T(t))\|_V^2&=\left\|S(t)\left(v_0,T_0\right)\right\|_V^2\\&=\|v(t)\|_{V_1}^2+\|T(t)\|_{V_2}^2\\&\le\mathscr{R}_2,\end{aligned}\tag{123}$$

for any $t\ge\tau_2$.

4. The Existence of the Global Attractors

In this section, we will recall some definitions and lemmas about the global attractor and prove the existence of the global attractor in H^2.

Definition 9 (see [23]). Let X be a Banach space and $\{S(t)\}_{t\ge0}$ a family operators on X. We say that $\{S(t)\}_{t\ge0}$ is a norm-to-weak continuous semigroup on X, if $\{S(t)\}_{t\ge0}$ satisfies that

(i) $S(0)=\mathrm{Id}$ (the identity),

(ii) $S(t)S(s)=S(t+s)$, for all $t,s\ge0$,

(iii) $S(t_n)x_n\rightharpoonup S(t)x$ if $t_n\to t$ and $x_n\to x$ in X.

Lemma 10 (see [24]). *Let X, Y be two Banach spaces and X^*, Y^* be the dual spaces of X, Y, respectively. If X is dense in Y, the injection $i:X\to Y$ is continuous and its adjoint $i^*:Y^*\to X^*$ is dense. Let $\{S(t)\}_{t\ge0}$ be a semigroup on X and Y, respectively, and assume furthermore that $\{S(t)\}_{t\ge0}$ is continuous or weak continuous on Y. Then, $\{S(t)\}_{t\ge0}$ is a norm-to-weak continuous semigroup on X if and only if $\{S(t)\}_{t\ge0}$ maps compact subsets of $\mathbb{R}^+\times X$ into bounded sets of X.*

Definition 11 (see [20]). Let X be a Banach space and let $\{S(t)\}_{t\ge0}$ be a continuous semigroup on X. A subset $\mathscr{A}$ in X is called a global attractor if and only if

(1) $\mathscr{A}$ is invariant; that is, $S(t)\mathscr{A}=\mathscr{A}$ for all $t\ge0$,

(2) $\mathscr{A}$ is compact in X,

(3) $\mathscr{A}$ attracts each bounded subset B in X.

Definition 12 (see [20]). Let $\{S(t)\}_{t\ge0}$ be a semigroup on Banach space X. $\{S(t)\}_{t\ge0}$ is called (X,Z) uniformly compact, if for any bounded (in X) set $B\subset X$, there exists $t_0=t_0(B)$, such that $\cup_{t\ge t_0}S(t)B$ is relatively compact in Z.

Lemma 13 (see [20, 24]). *Let X be a Banach space and $\{S(t)\}_{t\ge0}$ a C^0 semigroup on X. Then $\{S(t)\}_{t\ge0}$ has a global attractor $\mathscr{A}$ in X provided that the following conditions hold true:*

(1) *$\{S(t)\}_{t\ge0}$ has a bounded absorbing set B_0 in X,*

(2) *$\{S(t)\}_{t\ge0}$ is uniformly compact.*

Lemma 14 (see [20, 24, 25]). *Let $\{S(t)\}_{t\ge0}$ be a norm-to-weak continuous semigroup such that $\{S(t)\}_{t\ge0}$ is uniformly compact. If there exists an absorbing set B, then $\{S(t)\}_{t\ge0}$ has a global attractor $\mathscr{A}$ and*

$$\mathscr{A}=\bigcap_{s\ge0}\overline{\bigcup_{t\ge s}S(t)B}.\tag{124}$$

By virtue of the Rellich-Kondrachov theorem, we obtain the following.

Corollary 15. *Assume that $Q\in H^1$, $(v_0,T_0)\in V$. Then, the semigroup $\{S(t)\}_{t\ge0}$ corresponding to* (14)–(18) *has a (V,V)-global attractor $\mathscr{A}_V$.*

To prove the existence of the (V,H^3)-absorbing set, we need the following lemma.

Lemma 16 (see [7]). *Let $v \in V_1, \Phi_s \in L^2(M)$ be a solution to the following Stokes problem:*

$$\begin{aligned} L_1 v + \nabla \Phi_s &= g, \\ \int_{-h}^{0} \nabla \cdot v(x,y,z)\, dz &= 0, \\ \left.\frac{\partial v}{\partial z}\right|_{z=-h} &= 0, \qquad \left.\frac{\partial v}{\partial z}\right|_{z=0} = 0. \end{aligned} \tag{125}$$

If $g \in (W^{m,\alpha}(\Omega))^2$, $1 < \alpha < \infty$, $m \geq 0$, then

$$v \in V_1 \cap \left(W^{m+2,\alpha}(\Omega)\right)^2, \qquad \Phi_s \in W^{m+1,\alpha}(M). \tag{126}$$

Next, we will prove the existence of absorbing set in H^3 by the use of Lemma 16, which implies the existence of (V, H^2)-global attractor of the initial boundary problem (14)–(18).

Theorem 17. *Assume that $Q \in H^1$ and $(v_0, T_0) \in V$. Let $\{S(t)\}_{t\geq 0}$ be a semigroup generated by the initial boundary problem* (14)–(18). *Then, there exists an absorbing set in H^3. That is, for any bounded subset B of V, there exists a positive constant $\mathscr{R}_3$ depending on the norm of B, such that*

$$\begin{aligned} \|(v(t), T(t))\|_{H^3}^2 &= \left\|S(t)(v_0, T_0)\right\|_{H^3}^2 \\ &= \|v(t)\|_{H^3}^2 + \|T(t)\|_{H^3}^2 \\ &\leq \mathscr{R}_3, \end{aligned} \tag{127}$$

for any $t \geq \tau_1$, where τ_1 is specified in Theorem 7.

Proof. From the results of a priori estimates in the previous section, we know that there exists a strong solution (v, T) to (14)–(18), which satisfies for any $\mathscr{T} > \tau_1$, $v_z \in L^\infty(\tau_1, \mathscr{T}; L^6)$, $T_z \in L^\infty(\tau_1, \mathscr{T}; L^6)$, $v_t \in L^\infty(\tau_1, \mathscr{T}; V_1)$, $T_t \in L^\infty(\tau_1, \mathscr{T}; V_2)$,$(v, T) \in L^\infty(\tau_1, \mathscr{T}; H^2)$, where τ_1 is specified in Theorem 7. Then,

$$L_1 v + \nabla \Phi_s = g, \tag{128}$$

where

$$\begin{aligned} -g = \frac{\partial v}{\partial t} + (v \cdot \nabla) v - \left(\int_{-h}^{z} e^{(z-\zeta)/H_s} \nabla \cdot v(x,y,\zeta,t)\, d\zeta\right) \frac{\partial v}{\partial z} \\ + \frac{R}{H_s} \int_0^z \nabla T(x,y,\zeta,t)\, d\zeta + f_0 \vec{k} \times v. \end{aligned} \tag{129}$$

Since $v \in L^\infty(\tau_1, \mathscr{T}; H^2)$ and $v_z \in L^\infty(\tau_1, \mathscr{T}; L^6)$, for any $u \in L^{3/2}$, we have

$$\begin{aligned} &\left|\int_\Omega e^{-z/H_s} \left(\int_{-h}^{z} e^{(z-\zeta)/H_s} \nabla \cdot v(x,y,\zeta,t)\, d\zeta\right) \frac{\partial v}{\partial z} \cdot u \, dx\, dy\, dz\right| \\ &\quad \leq C \|\nabla v\|_6 \|v_z\|_6 \|u\|_{3/2}, \end{aligned} \tag{130}$$

which implies that

$$\left(\int_{-h}^{z} e^{(z-\zeta)/H_s} \nabla \cdot v(x,y,\zeta,t)\, d\zeta\right) \frac{\partial v}{\partial z} \in L^\infty\left(\tau_1, \mathscr{T}; L^3\right). \tag{131}$$

Since (v, T) is a strong solution to (14)–(18) and $v_t \in L^\infty(\tau_1, \mathscr{T}; V_1)$, it is clear that $\partial v/\partial t + (v\cdot\nabla)v - (R/H_s)\int_0^z \nabla T(x, y, \zeta, t)d\zeta + f_0\vec{k} \times v \in L^\infty(\tau_1, \mathscr{T}; L^3)$. Thus,

$$g \in L^\infty\left(\tau_1, \mathscr{T}; L^3\right). \tag{132}$$

By the use of Lemma 16, we get

$$v \in L^\infty\left(\tau_1, \mathscr{T}; V_1 \cap \left(W^{2,3}(\Omega)\right)^2\right). \tag{133}$$

Using the Sobolev embedding theorem $(W^{2,3}(\Omega))^2 \subset (W^{1,12}(\Omega))^2$, we know that

$$v \in L^\infty\left(\tau_1, \mathscr{T}; V_1 \cap \left(W^{1,12}(\Omega)\right)^2\right). \tag{134}$$

Now, since $v \in L^\infty(\tau_1, \mathscr{T}; V_1 \cap (W^{1,12}(\Omega))^2)$ and $v_z \in L^\infty(\tau_1, \mathscr{T}; L^6)$, for any $u \in L^{4/3}$, we have

$$\begin{aligned} &\left|\int_\Omega e^{-z/H_s} \left(\int_{-h}^{z} e^{(z-\zeta)/H_s} \nabla \cdot v(x,y,\zeta,t)\, d\zeta\right) \frac{\partial v}{\partial z} \cdot u dx\, dy\, dz\right| \\ &\quad \leq C \|\nabla v\|_{12} \|v_z\|_6 \|u\|_{4/3}, \end{aligned} \tag{135}$$

which implies that

$$\left(\int_{-h}^{z} e^{(z-\zeta)/H_s} \nabla \cdot v(x,y,\zeta,t)\, d\zeta\right) \frac{\partial v}{\partial z} \in L^\infty\left(\tau_1, \mathscr{T}; L^4\right). \tag{136}$$

Therefore, we find that

$$g \in L^\infty\left(\tau_1, \mathscr{T}; L^4\right). \tag{137}$$

From Lemma 16, we know that

$$v \in L^\infty\left(\tau_1, \mathscr{T}; V_1 \cap \left(W^{2,4}(\Omega)\right)^2\right), \tag{138}$$

which implies that

$$\left(\int_{-h}^{z} e^{(z-\zeta)/H_s} \nabla \cdot v(x,y,\zeta,t)\, d\zeta\right) \frac{\partial v}{\partial z} \in L^\infty\left(\tau_1, \mathscr{T}; H^1\right). \tag{139}$$

Since $v_t \in L^\infty(\tau_1, \mathscr{T}; V_1)$ and $v \in L^\infty(\tau_1, \mathscr{T}; V_1 \cap (W^{2,4}(\Omega))^2)$, it is clear that $\partial v/\partial t + (v\cdot\nabla)v - (R/H_s)\int_0^z \nabla T(x, y, \zeta, t)d\zeta + f_0\vec{k} \times v \in L^\infty(\tau_1, \mathscr{T}; H^1)$. Thus,

$$g \in L^\infty\left(\tau_1, \mathscr{T}; H^1\right). \tag{140}$$

From Lemma 16, we deduce that

$$v \in L^\infty\left(\tau_1, \mathscr{T}; V_1 \cap H^3\right). \tag{141}$$

Employing the elliptic regularity theory, we obtain

$$T \in L^\infty\left(\tau_1, \mathscr{T}; V_2 \cap H^3\right). \tag{142}$$

This complete the proof of Theorem 17. □

Since $H^2 \subset V$ is compact, from Theorem 7, we know that the semigroup generated by (14)–(18) is compact in V. From Theorem 7 and Lemma 10, we deduce that the semigroup generated by (14)–(18) maps a compact set of H^2 to be a bounded set of H^2; that is, the semigroup generated by (14)–(18) is norm to weak continuous in H^2. Furthermore, $H^3 \subset H^2$ compactly, and from Theorem 7 and Lemma 13, we immediately get the following result.

Theorem 18. *Assume that* $Q \in H^1(\Omega)$, $(v_0, T_0) \in V$. *Then, the semigroup* $\{S(t)\}_{t\geq 0}$ *corresponding to* (14)–(18) *has a* (V, H^2)*-global attractor* $\mathscr{A}$.

By using the Rellich-Kondrachov theorem, we can obtain the existence of the $(V, H^{2+\delta})$-global attractor for $\delta \in [0, 1)$.

Appendix

Proof of Proposition 2

If this conclusion is not true, then for each natural number n, there exists $v_n \in V_1$ such that

$$\|v_n\|_2^2 = 1, \qquad \|v_n\|^2 \leq \frac{1}{n}. \tag{A.1}$$

Let

$$\|v_n\|_{H^1}^2 = \|v_n\|_2^2 + \|v_n\|^2. \tag{A.2}$$

Hence,

$$\|v_n\|_{H^1}^2 \leq 1 + \frac{1}{n} \leq 2, \tag{A.3}$$

which implies that there exists a subsequence of $\{v_n\}_{n\geq 1}$ (still denoted by $\{v_n\}_{n\geq 1}$) such that

$$\begin{aligned} v_n &\longrightarrow v, \quad \text{weakly in } H^1, \\ v_n &\longrightarrow v, \quad \text{strongly in } L^2. \end{aligned} \tag{A.4}$$

Therefore, we have

$$\|v\|_2^2 = 1, \qquad \|v\|_{H^1}^2 \leq \liminf_{n\to\infty} \|v_n\|_{H^1}^2 = 1, \tag{A.5}$$

which implies that

$$\|v\|^2 = 0. \tag{A.6}$$

Combining $\|v\|^2 = 0$ with $v \cdot \vec{n} = 0$, we obtain

$$v = 0, \tag{A.7}$$

this is a contradiction.

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