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Research Article

Explicit Spectral Decimation for a Class of Self-Similar Fractals

Sergio A. Hernández and Federico Menéndez-Conde

Centro de Investigación en Matemáticas, Universidad Autónoma del Estado de Hidalgo, 42184 Pachuca, HGO, Mexico

Correspondence should be addressed to Federico Menéndez-Conde; fmclara@uaeh.edu.mx

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The method of spectral decimation is applied to an infinite collection of self-similar fractals. The sets considered are a generalization of the Sierpinski Gasket to higher dimensions; they belong to the class of nested fractals and are thus very symmetric. An explicit construction is given to obtain formulas for the eigenvalues of the Laplace operator acting on these fractals.

1. Introduction

In 1989, J. Kigami [1] gave an analytic definition of a Laplace operator acting on the Sierpinski Gasket; a few years later, this definition was extended to include the Laplacians on a large class of self-similar fractal sets [2], known as *postcritically finite* sets (PCF sets). The method of *spectral decimation* introduced by Fukushima and Shima in the 1990s provides a way to evaluate the eigenvalues of Kigami's Laplacian. In general terms, this method consists in finding the eigenvalues of the self-similar fractal set by taking limits of eigenvalues of discrete Laplacians that act on some graphs that approximate the fractal. The spectral decimation method was applied in [3] to the Sierpinski Gasket, in order to give an explicit construction which allows one to obtain the set of eigenvalues. In [4], it was shown that it is possible to apply the spectral decimation method to a large collection of p.c.f. sets, including the family of fractals known as *nested fractals* that was introduced by T. Lindstrøm in [5]. In addition to the Sierpinski Gasket, the spectral decimation method has been applied in several specific cases of p.c.f. fractals (e.g., [6–10]); also, the method has been proved useful to study the spectrum of particular fractals that are not p.c.f. (e.g., [11, 12]) and of fractafolds modeled on the Sierpinski Gasket [13]. The spectral decimation method has also shown to be a very useful tool for the analysis of the structure of the spectra of the Laplacians of some fractals (e.g., [14–16]).

In the present work, we develop in an explicit way the spectral decimation method for an infinite collection of self-similar sets that we will denote by $\mathbf{P}_n$ ($n \geq 2$ a positive integer). The definition of these sets is given in Definition 1. For the cases $n = 2, 3$, they correspond, respectively, to the unit interval and the Sierpinski Gasket. For larger values of n they give a quite natural extension of the Sierpinski Gasket to higher dimensions. The spectral decimation method for the cases $n = 2, 3$ is presented with thorough detail in [17]. Our presentation follows this reference to some extent. However, some technical difficulties arise for $n \geq 4$. This is mainly due to the fact that—even though the fractals considered are very symmetric—the graphs approximating the fractal are not as homogeneous as the ones approximating the Sierpinski Gasket. For instance, if we consider the graph obtained by taking away the boundary points from Γ_1 (see Definition 2 and Figures 1 and 2), then it will be a complete graph only for $n \leq 3$. A consequence of this is the appearance of sets of two types of vertices that have to be dealt with separately and which we denote by $F_{r,s}$ and $G_{r,s}$; for $n \leq 3$, the sets $G_{r,s}$ are empty. We also make the observation that the approximating graphs Γ_k are nonplanar when $n > 3$.

In Section 2, we present general facts about self similar sets, for the sake of completeness and in order to establish notation. In Section 3, we introduce the sets $\mathbf{P}_n$ that are the subject of study in this work; at the end of the section, we find the Hausdorff dimension of these fractals when embedded in the Euclidean space. In Section 4, we define the graphs that approximate the self-similar sets $\mathbf{P}_n$ and fix more notations. Our main result is presented in Section 5 (Theorem 3); it is shown that the eigenvalues and eigenfunctions of the

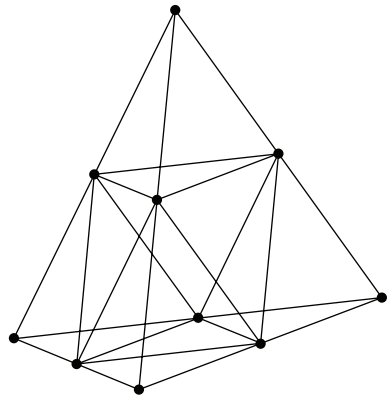

Figure 1: The first approximating graph Γ_1 for the fractal $\mathbf{P}_4$.

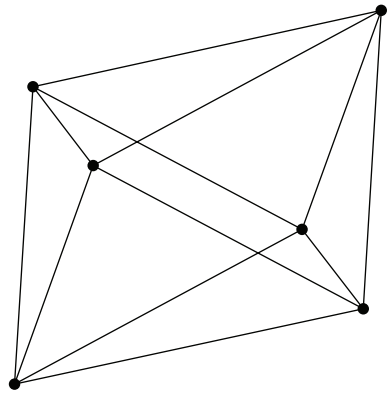

Figure 2: The graph Γ_1 for the fractal $\mathbf{P}_4$, minus the boundary points.

discrete Laplacians of the approximating graphs can be obtained recursively. Finally, in Section 6, it is shown that the eigenvalues of the Laplace operator in $\mathbf{P}_n$ can be recovered by taking limits of the discrete Laplacians; in order to do this, we solve the so-called *renormalization problem* for this case (see Theorem 5).

2. Notation and Preliminaries

We denote by $\mathbf{S}_n$ the shift space with n symbols. In this work, we will always consider these n symbols to be the numbers $0, 1, \ldots, n-1$. $\mathbf{S}_n$ is a compact space (see, e.g., [18]) when equipped with the metric

$$\delta\left(a_0a_1a_2a_3\ldots; b_0b_1b_2b_3\ldots\right) = r^k, \qquad 0 < r < 1, \tag{1}$$

where

$$k = \min\left\{j \geq 0 \mid a_j \neq b_j\right\}. \tag{2}$$

We will use the dot notation $\dot{a}$, meaning that the symbol a repeats to infinity.

Let $\mathbf{x} = x_0x_1x_2\ldots$ be an element of $\mathbf{S}_n$ and $a \in \{0, \ldots, n-1\}$. We denote by T_a the shift-operator given by

$$T_a(\mathbf{x}) = ax_0x_1x_2\ldots. \tag{3}$$

It is easy to see that

$$\delta\left(T(\mathbf{x}); T(\mathbf{y})\right) = r\delta(\mathbf{x}; \mathbf{y}), \tag{4}$$

so that T_a is a contraction (by factor r). The space $\mathbf{S}_n$ is a self-similar set, equal to n smaller copies of itself, with $\{T_0, \ldots, T_{n-1}\}$ the corresponding contractions. Even more so, it can be proved (Theorem 1.2.3. in [18]) that if K is any self-similar set, then it is homeomorphic to a quotient space of the form $\mathbf{S}_n/\sim$ for a suitable equivalence relation.

For $K = \mathbf{S}_n/\sim$ and a a word of length m

$$\mathbf{a} = (a_0a_1\cdots a_{m-1}), \tag{5}$$

denote by $T_{\mathbf{a}}$ the shift operator given by

$$T_{\mathbf{a}}(\mathbf{x}) = a_0\cdots a_{m-1}x_0x_1x_2\ldots. \tag{6}$$

The operator $T_{\mathbf{a}}$ is called an *m-contraction*, and the sets of the form $T_{\mathbf{a}}(K)$ are known as the *cells of level m* of the self-similar set K. We note that, for each choice of m, K is the union of the n^m cells of level m.

3. The Self-Similar Fractals $\mathbf{P}_n$

Here we will introduce the self-similar fractals $\mathbf{P}_n$ that are the subject of analysis in this work.

Definition 1. For $n \in \mathbb{N}$, define $\mathbf{P}_n$ as the quotient space $\mathbf{S}_n/\sim$, with the equivalence relation given by

$$a_0a_1a_2\cdots a_kb\dot{c} \sim a_0a_1a_2\cdots a_k\dot{cb}, \tag{7}$$

for any choice of symbols a_j, b, and c.

$\mathbf{P}_1$ is a trivial space with only one element, $\mathbf{P}_2$ is homeomorphic to a compact interval in $\mathbb{R}$, and $\mathbf{P}_3$ is homeomorphic to the well-known Sierpinski Gasket. For any value of n, $\mathbf{P}_n$ can be embedded in the Euclidean space; more precisely, there exists a (quite natural) homeomorphism between $\mathbf{P}_n$ and a compact self-similar set $K_n \subset R^{n-1}$. Below, we define the sets K_n; for these representations of $\mathbf{P}_n$, we will be able to find their Hausdorff dimensions.

Take n points $x_0, \ldots, x_{n-1} \in \mathbb{R}^{n-1}$ that do not lie in the same $(n-2)$-dimensional hyperplane; for $n = 3$, those points will be the "vertices" of the Sierpinski Gasket. For $n = 4$, the fractal K_4 will be some sort of a Sierpinski tetrahedron (see Figure 3), while the four points x_j will be the vertices of the tetrahedron.

Consider the contractions

$$f_i(x) = \frac{x + x_i}{2}, \quad i = 1, \ldots, n-1. \tag{8}$$

We note that f_i maps each $x \in \mathbb{R}^{n-1}$ to the midpoint of x and x_i (hence, leaving x_i fixed). Define K_n as the unique compact set such that

$$K_n = \bigcup_{i=0}^{n-1} f_i(K_n). \tag{9}$$

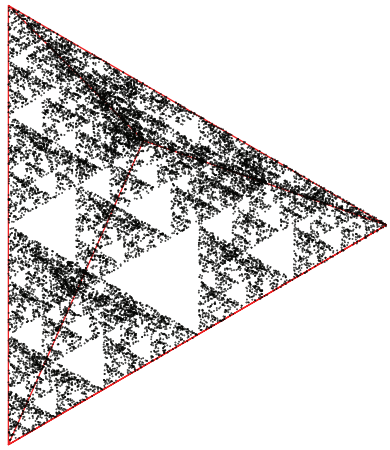

Figure 3: A representation of K_4, generated with Matlab.

We note that, for $i \neq j$, the sets $f_i(K_n)$ and $f_j(K_n)$ intersect at exactly one point: $f_i(x_j) = f_j(x_i)$. From this, it follows that the map $\pi : \mathbf{P}_n \to K_n$ given by

$$\pi(\omega_0\omega_1\omega_2\ldots) = \bigcap_{m\geq 0} f_{\omega_0} \circ f_{\omega_1} \circ \cdots \circ f_{\omega_m}(K_n) \tag{10}$$

is a well-defined homeomorphism; also, for every $k = 0, \ldots, n-1$, the following diagram commutes (cf. Theorem 1.2.3 in [18]):

$$\begin{array}{ccc} \mathbf{P}_n & \xrightarrow{T_k} & \mathbf{P}_n \\ \pi \downarrow & & \downarrow \pi \\ K_n & \xrightarrow[f_k]{} & K_n \end{array} \tag{11}$$

The sets K_n satisfy the Moran-Hutchinson *open set condition* namely, there exists a bounded nonempty open set $O \subset \mathbf{P}_n$ such that

$$\begin{gathered} f_i(O) \subset O, \quad \forall i \in \{0, \ldots, n-1\}, \\ f_i(O) \cap f_j(O), \quad \forall i \neq j. \end{gathered} \tag{12}$$

Just take $O = K_n \setminus \{x_0, \ldots, x_{n-1}\}$. From this and the fact that K_n is equal to n contractions of itself (by factor 1/2), it follows from Moran's theorem (Corollary 1.5.9 in [18]) that the Hausdorff dimension of K_n, with respect to theEuclidean metric, is equal to $\log n/\log 2$.

We end this section with two relevant notes.

For some values of n, it might be possible to embed K_n isometrically into the Euclidean space of a dimension m smaller than $n-1$. Of course, the dimension of the fractal gives a restriction to the minimal value of m.

The representations K_n are somehow useful to visualize the self-similar fractals $\mathbf{P}_n$. However, this representation and its metric do not play any role in the analysis carried out in the next sections; we will therefore focus on the more abstract definition of $\mathbf{P}_n$ given at the beginning of this section.

4. Graph Approximations of Self-Similar Sets

In this and the next sections, we consider the self-similar set $\mathbf{P}_n$ defined above, for an arbitrary but fixed value of $n \geq 2$.

Let V_0 be the set of points in $\mathbf{P}_n$ that have the form $\dot{k}$ with $k = 1, \ldots, n-1$. We call V_0 the *boundary* of $\mathbf{P}_n$. Likewise, for $m \in \mathbb{N}$, let V_m be the subset of $\mathbf{P}_n$ of points of the form $a_0 \cdots a_{m-1}\dot{k}$. In other words, $x \in V_m$ if and only if it belongs to the image of V_0 under some m-contraction.

Next, we define the graphs that will approximate $\mathbf{P}_n$.

Definition 2. Denote by Γ_0 the complete graph of n vertices, with V_0 being its set of vertices. For $m \in \mathbb{N}$, let Γ_m be the graph with the set of vertices V_m and edge relation established by requiring x to be connected with y if and only if there exists an m-contraction $T_{\mathbf{a}}$ such that both points x and y are in $T_{\mathbf{a}}(V_0)$.

We can see that an equivalent formulation is that two vertices x and y share an edge in Γ_m only when their first m symbols coincide. It is worth noting that even though $V_0 \subset V_1 \subset V_2 \ldots$, the edge relation is never preserved; this follows from the fact that if $x \neq y$ are connected in Γ_m, then their $(m+1)$-th symbols cannot be equal, so that they will not be connected in Γ_{m+1}.

For each $m \in \mathbb{N}$, let Δ_m be the graph Laplacian on Γ_m. We consider the Laplacian as acting on a space with boundary. More precisely, for a real-valued function u defined on V_m and x in $V_m \setminus V_0$,

$$\Delta_m u(x) = \sum_{y\sim x} (u(x) - u(y)), \tag{13}$$

with the sum over all vertices y that share an edge with x; the boundary values remain unchanged. Also, u is an eigenfunction of Δ_0 with eigenvalue λ, if

$$\Delta_m u(x) = \lambda u(x), \quad \forall x \in V_m \setminus V_0. \tag{14}$$

We denote by $E_m(\cdot,\cdot)$ the associated quadratic form (known as the *energy product* of the graph):

$$E_m(u,v) = (\Delta_m u, v) = \sum_{x\sim y} (u(x) - u(y))(v(x) - v(y)), \tag{15}$$

for u and v real-valued functions defined on V_m and the sum being taken over the pairs of vertices (x, y) that are connected to each other. Also, we use the abbreviation $E(u) = E(u,u)$.

5. Spectral Decimation

Let $m > 1$ and suppose u is an eigenfunction of Δ_{m-1}, with eigenvalue λ_{m-1}. We will show that it is always possible to extend this function to the domain V_m so that it will be an eigenfunction of Δ_m (with not the same eigenvalue).

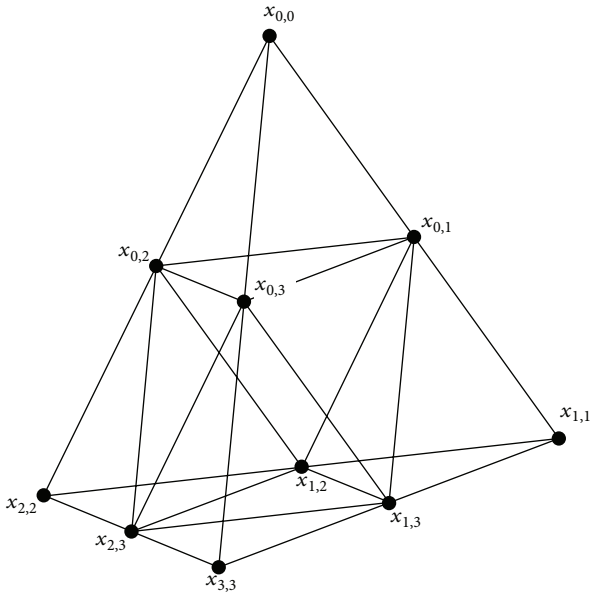

Figure 4: A cell of level $m-1$ of a graph Γ_m, approximating $\mathbf{P}_4$.

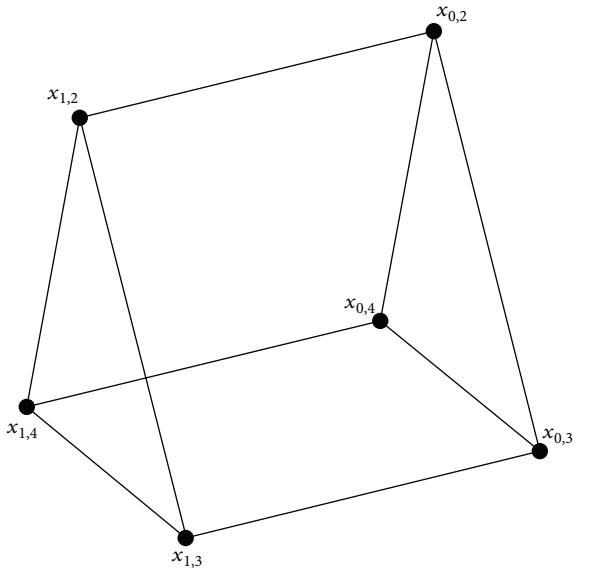

Figure 5: The graph determined by $F_{0,1}$ in a graph Γ_m, approximating $\mathbf{P}_5$.

In order to do this, we will derive necessary conditions for the extension to be an eigenfunction; in the process, it will become clear that those conditions are also sufficient.

Suppose that u is an eigenfunction of Δ_m with eigenvalue λ_m; we aim to write the values of u_m in $V_m \setminus V_{m-1}$ in terms of its values in V_{m-1}. Without loss of generality, we can restrict ourselves to the set $V_m \cap T_{\mathbf{a}}(\mathbf{P}_n)$ for a fixed $(m-1)$-contraction $T_{\mathbf{a}}$; this is because the vertices of Γ_m that belong to the set $(V_m \setminus V_{m-1}) \cap T_{\mathbf{a}}(\mathbf{P}_n)$ are not connected to any vertices outside the cell $T_{\mathbf{a}}(\mathbf{P}_n)$. Denote the elements of this set by

$$x_{b,c} = \mathbf{a}b\dot{c}, \quad b, c = 0, \ldots, n-1. \tag{16}$$

It is clear that $x_{b,c} = x_{c,b}$ and also that $x_{b,c} \in V_{m-1}$ if and only if $b = c$. This is shown in Figure 4.

For each point $x_{r,s} \in V_m \cap T_{\mathbf{a}}(\mathbf{P}_n)$, define the sets of vertices

$$\begin{aligned} F_{r,s} &= \left\{x_{i,j} \mid i \neq j, \{r,s\} \cap \{i,j\} \neq \emptyset\right\}, \\ G_{r,s} &= \left\{x_{i,j} \mid \ i \neq j, \{r,s\} \cap \{i,j\} = \emptyset\right\}. \end{aligned} \tag{17}$$

In other words, $F_{r,s}$ is the set of vertices (not in Γ_{m-1}) that are connected to the vertex $x_{r,s}$ in Γ_m, and $G_{r,s}$ is the set of vertices (not in Γ_{m-1}, either) that are not connected to it.

In the case $n = 4$, the graph $\Gamma_m \setminus \Gamma_{m-1}$ is an octahedron; hence, for each pair $\{r,s\}$, the subgraph determined by the vertices in $F_{r,s}$ is a 4-cycle one, while $G_{r,s}$ consists of a single vertex (the one opposite to $x_{r,s}$ in the octahedron). For general $\mathbf{P}_n$, we can see that the following.

The graph $\Gamma_m \setminus \Gamma_{m-1}$ has $n(n-1)$ vertices, all of them with degree $2(n-2)$. Each of these vertices is connected to another 2 vertices in Γ_{m-1}.

The subgraph determined by $F_{r,s}$ consists of two complete graphs, each one with $n-2$ vertices. The two complete graphs are joined to each other pairwise, thus forming a "prism," (a true prism only in the case $n = 5$, where the base is a 3-cycle one, as shown in Figure 5).

The subgraph determined by $G_{r,s}$ has $(n-2)(n-3)/2$ vertices, each one of them with degree $2(n-4)$.

In Γ_m, each vertex that belongs to $G_{r,s}$ is connected to exactly four vertices in $F_{r,s}$. On the other hand, each vertex that belongs to $F_{r,s}$ is connected to $n-2$ vertices in $G_{r,s}$.

Now, having noted all that, we proceed with the calculations. For every $r \neq s$ we have

$$\begin{aligned} \left(2(n-1) - \lambda_m\right) u\left(x_{r,s}\right) &= u\left(x_{r,r}\right) + u\left(x_{s,s}\right) \\ &\quad + \sum_{F_{r,s}} u\left(x_{i,j}\right). \end{aligned} \tag{18}$$

Adding this up over all the possible values of r and s and rearranging terms yields

$$\left(2 - \lambda_m\right) \sum_{r \neq s} u\left(x_{r,s}\right) = (n-1) \sum_{j=0}^{n-1} u\left(x_{j,j}\right), \tag{19}$$

which for any fixed $a \neq b$ can also be written in the form

$$\begin{aligned} &\left(2 - \lambda_m\right) \left(u\left(x_{a,b}\right) + \sum_{F_{a,b}} u\left(x_{i,j}\right) + \sum_{G_{a,b}} u\left(x_{i,j}\right) \right) \\ &= (n-1) \sum_{j=0}^{n-1} u\left(x_{j,j}\right). \end{aligned} \tag{20}$$

This, together with (18), allows us to express the sum of the values in $G_{a,b}$ in terms of $u(x_{a,b})$ and the values at points in V_{m-1}; namely, provided $\lambda_m \neq 2$, we have that

$$\sum_{G_{a,b}} u\left(x_{i,j}\right) = u\left(x_{a,a}\right) + u\left(x_{b,b}\right) - \left(2n - 1 - \lambda_m\right) u\left(x_{a,b}\right) + \frac{(n-1)}{2-\lambda_m} \sum_{j=0}^{n-1} u\left(x_{j,j}\right). \tag{21}$$

Next, we will take the sum of the same terms, but only over the $x_{i,j}$ inside the set $F_{a,b}$ for fixed values $a \neq b$. Since $F_{a,b}$ contains two complete graphs with $n-1$ vertices and these complete graphs are pairwise connected to each other, it is clear that each $x_{i,j} \in F_{a,b}$ is connected to other $n-1$ vertices in $F_{a,b}$. Also, recall that each $x_{i,j} \in G_{a,b}$ is connected to exactly four vertices in $F_{a,b}$. For the vertices in V_{m-1}, we note that $x_{j,j}$ is connected to $n-2$ vertices in $x_{i,j} \in F_{a,b}$ if $j = a, b$ and to only two vertices otherwise.

From the preceding discussion, it follows the equality

$$\begin{aligned}\left(n - \lambda_m\right) \sum_{F_{a,b}} u\left(x_{i,j}\right) &= 4 \sum_{G_{a,b}} u\left(x_{i,j}\right) \\ &+ (n-2)\left(u\left(x_{a,a}\right) + u\left(x_{b,b}\right)\right) \\ &+ 2(n-2)\, u\left(x_{a,b}\right) + 2 \sum_{j \neq a,b} u\left(x_{j,j}\right).\end{aligned} \tag{22}$$

Consider the expression given by (18) for $\{a, b\} = \{r, s\}$, multiply it by $n - \lambda_m$, and substitute equality (22) into it; this gives after arranging terms

$$\begin{aligned}\left(\lambda_m^2 - (3n-2)\lambda_m + 2\left(n^2 - 2n + 2\right)\right) u\left(x_{a,b}\right) &= 4 \sum_{G_{a,b}} u\left(x_{i,j}\right) \\ &+ 2 \sum_{j \neq a,b} u\left(x_{j,j}\right) + \left(2(n-1) - \lambda_m\right)\left(u\left(x_{a,a}\right) + u\left(x_{b,b}\right)\right).\end{aligned} \tag{23}$$

We want to get rid of the terms corresponding to $G_{a,b}$, so we replace it by (21). After straightforward computations, we can see that for $\lambda_m \neq 2$

$$\begin{aligned}&\left(\lambda_m^2 - (3n+2)\lambda_m + 2n(n+2)\right) u\left(x_{a,b}\right) \\ &\quad = \left(\lambda_m^2 - 2(n+2) + 8n\right)\left(u\left(x_{a,a}\right) + u\left(x_{b,b}\right)\right) \\ &\qquad + 2\left(2n - \lambda_m\right) \sum_{j \neq a,b} u\left(x_{j,j}\right)\left(2 - \lambda_m\right)^{-1}.\end{aligned} \tag{24}$$

The quadratic equation for λ_m in the left-hand side has roots $n+2$ and $2n$. The one in the right-hand side has roots 4 and $2n$. This gives us the following expression for $u(x_{a,b})$ in terms of the values of u in V_{m-1}:

$$u\left(x_{r,s}\right) = \frac{\left(4 - \lambda_m\right)\left(u\left(x_{r,r}\right) + u\left(x_{s,s}\right)\right) + 2\sum_{j \neq r,s} u\left(x_{j,j}\right)}{\left(2 - \lambda_m\right)\left((n+2) - \lambda_m\right)}, \tag{25}$$

valid for any eigenvalue $\lambda_m \neq 2,\ n+2,\ 2n$.

For $\lambda_m = 0$, this reduces to

$$u\left(x_{r,s}\right) = \frac{2}{n+2}\left(u\left(x_{r,r}\right) + u\left(x_{s,s}\right)\right) + \frac{1}{n+2} \sum_{j \neq r,s} u\left(x_{j,j}\right). \tag{26}$$

It is clear from the construction that if $u(x_{a,b})$ is defined by (25) and λ_m is given by (35), then we have that

$$\Delta_m u\left(x_{a,b}\right) = \lambda_m u\left(x_{a,b}\right) \quad (a \neq b). \tag{27}$$

It remains to verify that this is valid as well in V_{m-1}. Of course, this cannot be true for arbitrary values of λ_m but only at most for specific values depending on λ_{m-1}; we will find those values in what follows.

Take a point in V_{m-1}, say

$$x_{p,p} = \mathbf{a}\dot{p}, \quad \mathbf{a} = a_0 \cdots a_{m-1}. \tag{28}$$

Suppose that $a_k = q$ is the last symbol in $\mathbf{a}$ that is different from p; we can assume that such symbol exists, since otherwise $x_{p,p}$ would be in the boundary V_0. With this, the point $x_{p,p}$ can also be written in the form

$$x_{p,p} = \mathbf{a}'\dot{q}, \quad \mathbf{a}' = a_0 \cdots a_{k-1} p q \cdots q, \tag{29}$$

with the necessary number of q's to make $\mathbf{a}'$ a word of length $m-1$. Hence, $x_{p,p}$ is in exactly two different $(m-1)$-cells: $T_{\mathbf{a}}(\mathbf{P}_n)$ and $T_{\mathbf{a}'}(\mathbf{P}_n)$, corresponding to each one of its two representations.

Denote by $x'_{r,s}$ the points in $T_{\mathbf{a}'}(\mathbf{P}_n) \cap V_m$, defined as in (16) for the points in $T_{\mathbf{a}}(\mathbf{P}_n) \cap V_m$; in particular, $x_{p,p} = x'_{q,q}$ (see Figure 6). The value of u in the points $x'_{r,s}$ is given by the analogue of (25). The vertex $x_{p,p}$ is connected in Γ_m to the $2(n-1)$ points of the form $x_{p,j}$ and $x'_{q,j}$, from which it follows that u is an eigenfunction of Δ_m with eigenvalue λ_m, if and only if (25) holds for all $x_{r,s} \in V_m \setminus V_{m-1}$ and the following equality holds for all $x_{p,p} \in V_{m-1}$:

$$\left(2(n-1) - \lambda_m\right) u\left(x_{p,p}\right) = \sum_{i \neq p,\, j \neq q} \left(u\left(x_{p,j}\right) + u\left(x'_{q,j}\right)\right). \tag{30}$$

On the other hand, since we know that u is an eigenfunction of Δ_{m-1} with eigenvalue λ_{m-1}, we also have that

$$\left(2(n-1) - \lambda_{m-1}\right) u\left(x_{p,p}\right) = \sum_{i \neq p,\, j \neq q} \left(u\left(x_{i,i}\right) + u\left(x'_{j,j}\right)\right). \tag{31}$$

Replacing each term in the right-hand side of (30) by its expression given by (25), we can see that

$$\begin{aligned}&\left(2(n-1) - \lambda_m\right) u\left(x_{p,p}\right) \\ &\quad = 2(n-1)\left(4 - \lambda_m\right) u\left(x_{p,p}\right) + \left(2n - \lambda_m\right)\left(2 - \lambda_m\right)^{-1} \\ &\qquad \times \left((n+2) - \lambda_m\right)^{-1},\end{aligned} \tag{32}$$

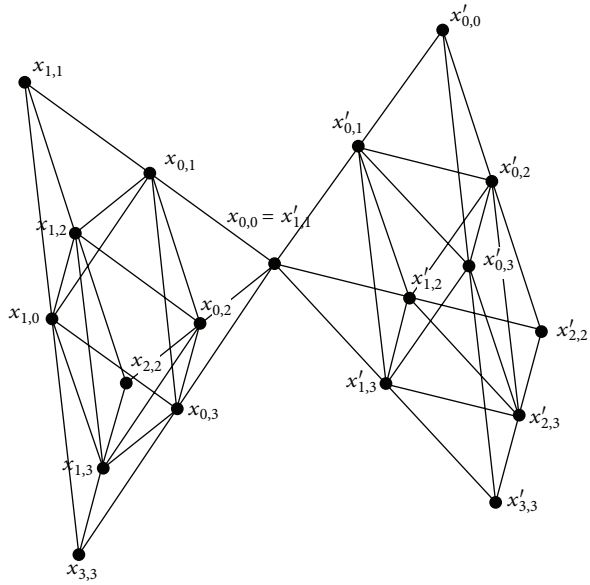

FIGURE 6: Two cells of level $m-1$ intersecting in a vertex of Γ_m.

and, using (31), this gives

$$\begin{aligned} u\left(x_{p,p}\right) &= \left[2(n-1)(4-\lambda_m) + (2n-\lambda_m)(2(n-1)-\lambda_m)\right] \\ &\quad \times u\left(x_{p,p}\right)(2(n-1)-\lambda_m)^{-1}(2-\lambda_m)^{-1} \\ &\quad \times ((n+2)-\lambda_m)^{-1}. \end{aligned} \tag{33}$$

Taking $u(x_{p,p}) \neq 0$ and cancelling out, after computations, the above equality reduces to the quadratic

$$\lambda_m^2 - (n+2)\lambda_m + \lambda_{m-1} = 0, \tag{34}$$

which in turn gives the following recursive characterization of the eigenvalues:

$$\lambda_m = \frac{(n+2) \pm \sqrt{(n+2)^2 - 4\lambda_{m-1}}}{2}. \tag{35}$$

Since this procedure can be reversed, we have proved the following result.

Theorem 3. *Let $\lambda_m \neq 2,\ n+2$, and $2n$, and let λ_{m-1} be given by* (34). *Suppose u is an eigenfunction of Δ_{m-1} with eigenvalue λ_{m-1}. Extend u to V_m by* (25). *Then u is an eigenfunction of Δ_m with eigenvalue λ_m. Conversely, if u is an eigenfunction of Δ_m with eigenvalue $\lambda_m \neq 2,\ n+2$, and $2n$, then its restriction to V_{m-1} is an eigenfunction of Δ_{m-1} with eigenvalue λ_{m-1}.*

6. The Laplacian on the Self-Similar Fractals

In order to define the Laplace operator of a p.c.f. fractal by means of graph approximations, it is required to solve the so-called *renormalization problem* for the fractal (e.g., [17], Chapter 4); roughly, this consists in normalizing the graph energies in Γ_m in order to obtain a self-similar energy in the fractal by taking the limit. This can be achieved if the energies are such that they remain constant for each harmonic extension from Γ_m to Γ_{m+1}. Below, we do this for the $\mathbf{P}_n$ sets.

Definition 4. For a given function u with domain V_{m-1}, we call the extension of u to V_m given by (26) its harmonic extension.

The next result gives the explicit solution of the renormalization problem for $\mathbf{P}_n$.

Theorem 5. *Let $u : V_{m-1} \to \mathbb{R}$ be arbitrary, and let $u' : V_m \to \mathbb{R}$ be its harmonic extension. Then*

$$E_m\left(u'\right) = \frac{n}{n+2} E_{m-1}(u). \tag{36}$$

Proof. Note that the energy at level k of a given function equals the sum of the energies at all the k'-cells for any $k' \leq k$, since different cells share no edges. This allows us to restrict ourselves to one fixed $m-1$-cell both while considering $E_m(u')$ and $E_{m-1}(u)$. We use the notation of Section 4 for the vertices of Γ_m in this cell and write $\widetilde{E}$ for the energy restricted to this cell. We can readily see that

$$\begin{aligned} \widetilde{E}_{m-1}(u) &= \sum_{i \neq j} \left(u\left(x_{i,i}\right) - u\left(x_{j,j}\right)\right)^2. \\ &= (n-1)\sum_{i=0}^{n-1} u^2\left(x_{i,i}\right) - 2\sum_{i \neq j} u\left(x_{i,i} u\left(x_i, j\right)\right). \end{aligned} \tag{37}$$

In order to evaluate the energy $\widetilde{E}_m$, we consider first the edges joining vertices in V_{m-1} with vertices in $V_m \setminus V_{m-1}$: the edge joining the vertex $x_{a,a}$ with the vertex $x_{a,k}$ contributes to the energy by

$$\begin{aligned} &\left(u\left(x_{a,a}\right) - u\left(x_{a,k}\right)\right)^2 \\ &\quad = \frac{1}{(n+2)^2}\left(nu\left(x_{a,a}\right) - 2u\left(x_{k,k}\right) - \sum_{j \neq a,k} u\left(x_{j,j}\right)\right)^2. \end{aligned} \tag{38}$$

When adding up this over all possible pairs $a \neq k$, each $x_{r,r}$ will appear $n-1$ times as the $x_{a,a}$, another $n-1$ times as the $x_{k,k}$, and $(n-1)(n-2)$ as one of the $x_{j,j}$'s. Each pair double product $2x_{r,r}x_{s,s}$ will appear twice for $\{r,s\} = \{a,k\}$, $2(n-2)$ times for $\{r,s\} = \{a,j\}$ for some j, also $2(n-2)$ times for $\{r,s\} = \{k,j\}$ for some j, and finally $(n-2)(n-3)$ times when both r and s are one of the j's. All this implies

that the contribution to the energy from these edges is, after simplification,

$$\begin{aligned}&\sum_{a\neq k}\left(u\left(x_{a,a}\right)-u\left(x_{a,k}\right)\right)^2\\&=\frac{(n-1)\left(n^2+n+2\right)}{(n+2)^2}\sum_{i=0}^{n-1}u^2\left(x_{i,i}\right)\\&\quad-\frac{2\left(n^2+n+2\right)}{(n+2)^2}\sum_{i\neq j}x_{i,i}x_{j,j}.\end{aligned}\tag{39}$$

On the other hand, the contribution from the edge that joins the vertices $x_{a,b}$ and $x_{a,c}$ (in $V_m\setminus V_{m-1}$) equals

$$\left(u\left(x_{a,b}\right)-u\left(x_{a,c}\right)\right)^2=\frac{1}{(n+2)^2}\left(u\left(x_{b,b}\right)-u\left(x_{c,c}\right)\right)^2.\tag{40}$$

Taking the sum over all of the vertices in $V_m\setminus V_{m-1}$ yields

$$\begin{aligned}&\sum_{a\neq b\neq c}\left(u\left(x_{a,b}\right)-u\left(x_{a,c}\right)\right)^2\\&=\frac{(n-1)(n-2)}{(n+2)^2}\sum_{i=0}^{n-1}u^2\left(x_{i,i}\right)-\frac{2(n-2)}{(n+2)^2}\sum_{i\neq j}u\left(x_{i,i}\right)u\left(x_{j,j}\right).\end{aligned}\tag{41}$$

From (39) and (41), it follows that the total energy of the cell is

$$\widetilde{E}_m\left(u'\right)=\frac{n(n-1)}{n+2}\sum_{i=0}^{n-1}u^2\left(x_{i,i}\right)-2n(n+2).\tag{42}$$

This, together with (37), gives

$$\widetilde{E}_m\left(u'\right)=\frac{n}{n+2}\widetilde{E}_{m-1}(u).\tag{43}$$

Taking this result for all the $m-1$-cells concludes the proof. □

Definition 6. The energy in $\mathbf{P}_n$ is given by

$$E(u)=\lim_{m\to\infty}\left(\frac{n+2}{n}\right)^m E_m(u).\tag{44}$$

The domain of $E(\cdot)$ being the space $D^{(n)}$ of functions such that the energy is finite. Write $D_0^{(n)}$ for the subspace of $D^{(n)}$ of functions that vanish on the boundary. The energy product $E(u,v)$ can be recovered by the polarization identity.

Let μ be a self-similar measure in $\mathbf{P}_n$; the Laplacian Δ_μ is given by the following:

Definition 7 (Kigami's Laplacian). With μ and Δ_μ as above, one says that u is in the domain of Δ_μ if there exists a continuous function f such that

$$E(u,v)=-\int_{P_n}fv\,d\mu,\quad\forall v\in D_0^{(n)}.\tag{45}$$

In such case, we define $\Delta_\mu u=f$.

Aside from the above weak representation, a pointwise formula can be obtained for $\Delta_m u$, proceeding exactly in the same way as in [17] (Theorem 2.2.1). In the case where μ is the standard measure in P_n (i.e., the only Borel regular measure such that the measure of every m-cell is equal to n^{-m}), the pointwise formula is

$$\Delta_\mu u(x)=\frac{n}{2}\lim_{m\to\infty}(n+2)^m\Delta_m u(x).\tag{46}$$

This leads to the following: if a sequence $\{\lambda_m\}$ is defined recursively by (35) (assuming that λ_m is never equal to n, $n+2$, or $2n$) and u_m is given by relation (25), then

$$\lambda=\frac{n}{2}\lim_{m\to\infty}(n+2)^m\lambda_m\tag{47}$$

is an eigenvalue of Δ_μ with eigenfunction u given by the limit $u_m\to u$. The limit above exists provided that the sign in relation (35) is chosen to be "+" for at most a finite number of times.

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