Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 750147, 8 pages
http://dx.doi.org/10.1155/2013/750147

Research Article

A Robust Weak Taylor Approximation Scheme for Solutions of Jump-Diffusion Stochastic Delay Differential Equations

Yanli Zhou,[1] Yonghong Wu,[1] Xiangyu Ge,[2] and B. Wiwatanapataphee[3]

[1] *Department of Maths and Statistics, Curtin University, Perth, WA 6845, Australia*
[2] *School of Statistics & Maths, Zhongnan University of Economics and Law, Wuhan 430073, China*
[3] *Department of Mathematics, Faculty of Science, Mahidol University, Bangkok 10400, Thailand*

Correspondence should be addressed to Yanli Zhou; ylzhou8507@gmail.com and B. Wiwatanapataphee; scbww@mahidol.ac.th

Received 1 December 2012; Revised 13 March 2013; Accepted 14 March 2013

Academic Editor: Xinguang Zhang

Stochastic delay differential equations with jumps have a wide range of applications, particularly, in mathematical finance. Solution of the underlying initial value problems is important for the understanding and control of many phenomena and systems in the real world. In this paper, we construct a robust Taylor approximation scheme and then examine the convergence of the method in a weak sense. A convergence theorem for the scheme is established and proved. Our analysis and numerical examples show that the proposed scheme of high order is effective and efficient for Monte Carlo simulations for jump-diffusion stochastic delay differential equations.

1. Introduction

Stochastic delay differential equation models play a very important role in the study of many fields such as economics and finance, chemistry, biology, microelectronics, and control theories. Models of this type can more accurately describe many phenomena in the real world by taking into account the effect of time delay and random noise. By time delay, it means a certain period of time is required for the effect of an action to be observed after the moment when the action takes place. This phenomenon exists in most systems in almost any area of science; for example, a patient shows symptoms of an illness days or even weeks after he/she was infected. Similarly, random noises appear in almost all real world phenomena and systems, for example, the motion of molecules and the price of assets in financial markets. Hence, study of SDDEs is an important undertaking in order to understand real world phenomena and systems precisely.

Over the last couple of decades, a lot of work has been carried out to study differential equations with delay and/or random noises, for example, [1–5]. The best-known and well-studied theory and systems include the delay differential equations (DDEs) presented by Kolmanvskii and Myshkis [6] and their stochastic generalizations and the stochastic delay differential equations (SDDEs) established by Mohammed [7, 8], Mao [9, 10], and Mohammed and Scheutzow [11]. Other SDDEs theories of interest include, for instance, the so-called SDDEs with Markovian switching and Poisson jumps. These models have been investigated in the literature [12–14].

Analytical solutions of SDDEs can hardly be obtained. It is thus important to develop and study discrete-time approximation methods for solving SDDEs. Discrete-time approximations may be divided into two categories: weak approximations and strong approximations [15]. Some implicit and explicit numerical approximation methods for SDDEs in strong approximation sense were derived by Küchler and Platen [16]. Weak numerical methods for SDDEs have been studied by Küchler and Platen. The Monte Carlo simulation method has also been developed as a powerful simulation method for SDEs, while weak numerical approximations are required for Monte Carlo simulation [9, 17–20].

In this paper, we extend a fully weak approximation method for SDDEs to the type of jump-diffusion SDDEs:

$$\begin{aligned} d\mathbf{X}(t) &= \mathbf{a}(t, \mathbf{X}(t), \mathbf{X}(t-\gamma))\,dt \\ &\quad + \mathbf{b}(t, \mathbf{X}(t), \mathbf{X}(t-\gamma))\,d\mathbf{W}_t \\ &\quad + \int_{\boldsymbol{\varepsilon}} \mathbf{c}(t, \mathbf{X}(t^-), v)\, p_\varphi(dv, dt) \end{aligned} \tag{1}$$

subject to the initial condition

$$X(\theta) = \chi(\theta) \quad \text{for } \theta \in [-\gamma, 0], \tag{2}$$

where $t \in [0,T]$, γ is the time delay which is assumed to be constant at all time, $\mathbf{W}_t = \{(W_t^1, \ldots, W_t^m),\ t \in [0,T]\}$ is an $\mathscr{A}$-adapted m-dimensional Wiener process, and p_φ denotes the Poisson random measure. Also here we denote by $\mathbf{X}(t^-)$ the almost sure left-hand limit of $\mathbf{X}(t)$. The coefficient $\mathbf{a}(t,x,x^r) : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $\mathbf{c}(t,x,v) : [0,T] \times \mathbb{R}^d \times \boldsymbol{\varepsilon} \to \mathbb{R}^d$ are d-dimensional vectors of Borel measurable functions. Further, $\mathbf{b}(t,x,x^r)$ defined on $[0,T] \times \mathbb{R}^d \times \mathbb{R}^d$ is a $d \times m$-matrix of Borel measurable function.

The main contributions of this work include construction of a numerical approximation method for weak solutions of SDDEs with jump-diffusion, and establishment of a theoretical result for the convergence of the scheme.

The remaining part of this paper is organised as follows. In the following section, we give the conditions for the existence and uniqueness of the solution of the jump-diffusion SDDEs (1) which is applicable to any cases where solution exists, and present various lemmas to be used later for the proof of the convergence theorem. We then introduce, in Section 3, a general weak approximation scheme, where the simplified stochastic Taylor approximation scheme with order β is constructed; followed by a convergence theorem and its proof. In Section 4, we give a numerical example to demonstrate the application and the convergence of the numerical scheme.

2. Preliminaries

In this section, we present some basic concepts and definitions to be used in later sections, then establish the conditions for the existence and uniqueness of solution to the Jump-diffusion SDDEs (1), and then give some lemmas to be used later for the proof of the convergence theorem. In this work, we assume that the d-dimensional vector valued function for the initial condition, $\chi = \{\chi(s),\ s \in [-\gamma, 0]\}$, is right continuous and has left-hand limits.

From (1), we have the following integral form of the jump-diffusion equation SDDE:

$$\begin{aligned} \mathbf{X}(t) &= \mathbf{X}(0) + \int_0^t \mathbf{a}(\tau, \mathbf{X}(\tau), \mathbf{X}(\tau-\gamma))\,d\tau \\ &\quad + \int_0^t \mathbf{b}(\tau, \mathbf{X}(\tau), \mathbf{X}(\tau-\gamma))\,d\mathbf{W}_\tau \\ &\quad + \sum_{i=1}^{p_\varphi(t)} \mathbf{c}(\tau_i, \mathbf{X}(\tau_i^-), \xi_i), \end{aligned} \tag{3}$$

where (τ_i, ξ_i), for $i \in \{1,2,3,\ldots,p_\varphi(t)\}$, are a sequence of pairs of jump times and corresponding values generated by the Poisson random measure p_φ.

Definition 1. Given a filtered probability space $(\Omega, \mathscr{A}, \underline{\mathscr{A}}, P)$, a stochastic process given by $\mathbf{X} = \{\mathbf{X}(t),\ t \in [-\gamma, T]\}$ is known as a solution of (1) subject to the initial condition (2) if $\mathbf{X}$ is $\underline{\mathscr{A}}$-adapted, the integrals in the equation are well defined, and equalities (3) and (2) hold almost surely. Moreover, if any two solution processes $\mathbf{X}^{(i)} = \{\mathbf{X}^{(i)}(t),\ i \in 1,2\}$ are indistinguishable on $[-\gamma, T]$ with the same initial segment χ and the same path on $[0,T]$, and

$$P\left(\sup_{t\in[0,T]} \left\| X^{(1)}(t) - X^{(2)}(t) \right\| > 0\right) = 0, \tag{4}$$

where $\|\cdot\|$ is the Euclidean norm, then if (1) has a solution, it is a unique solution for this initial value problem.

To guarantee the existence of a unique solution of the jump-diffusion SDDE (1), we assume that the coefficients of (1) satisfy the following Lipschitz conditions:

$$\begin{aligned} &\left|\mathbf{a}(t, \mathbf{y}_1, \mathbf{z}_1) - \mathbf{a}(t, \mathbf{y}_2, \mathbf{z}_2)\right| \le C_1 \left(\left|\mathbf{y}_1 - \mathbf{y}_2\right| + \left|\mathbf{z}_1 - \mathbf{z}_2\right|\right), \\ &\left|\mathbf{b}(t, \mathbf{y}_1, \mathbf{z}_1) - \mathbf{b}(t, \mathbf{y}_2, \mathbf{z}_2)\right| \le C_2 \left(\left|\mathbf{y}_1 - \mathbf{y}_2\right| + \left|\mathbf{z}_1 - \mathbf{z}_2\right|\right), \\ &\int_{\boldsymbol{\varepsilon}} \left|\mathbf{c}(t, \mathbf{y}_1, \mathbf{z}_1, v) - \mathbf{c}(t, \mathbf{y}_2, \mathbf{z}_2, v)\right|^2 \varphi(dv) \\ &\quad \le C_3 \left((y_1 - y_2)^2 + (z_1 - z_2)^2\right) \end{aligned} \tag{5}$$

for $t \in [0,T]$ and $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^d$, as well as the growth conditions

$$\begin{aligned} &\left|\mathbf{a}(t, \mathbf{y}, \mathbf{z})\right| \le D_1 \left(1 + |\mathbf{y}| + |\mathbf{z}|\right), \\ &\left|\mathbf{b}(t, \mathbf{y}, \mathbf{z})\right| \le D_2 \left(1 + |\mathbf{y}| + |\mathbf{z}|\right), \\ &\int_{\boldsymbol{\varepsilon}} \left|\mathbf{c}(t, \mathbf{y}, \mathbf{z}, v)\right|^2 \varphi(dv) \le D_3 \left(1 + y^2 + z^2\right) \end{aligned} \tag{6}$$

for $\mathbf{y}, \mathbf{z} \in \mathbb{R}^d$, $t \in [0,T]$.

We denote by $\mathbb{C} = \mathbb{C}([-\gamma, 0], \mathbb{R}^d)$ the Banach space of all d-dimensional continuous functions η on $[-\gamma, 0]$ equipped with the supremum norm $\|\eta\|_{\mathbb{C}} = \sup_{s\in[-\gamma,0]} |\eta(s)|$. Furthermore, we suppose that the set $L_2(\Omega, \mathbb{C}, \mathscr{A}_0)$ of $\mathbb{R}^d$-valued continuous process $\eta = \{\eta(s),\ s \in [-\gamma, 0]\}$ is $\mathscr{A}_0$-measurable with

$$E\left(\|\eta\|_{\mathbb{C}}^2\right) = E\left(\sup_{s\in[-\gamma,0]} |\eta(s)|^2\right) < \infty. \tag{7}$$

Following the work in [7, 10], the following theorem can be established for the existence and uniqueness of a solution to the problem defined by (1) and (2).

Theorem 2. *Suppose that the Lipschitz conditions and the growth conditions are satisfied, and the initial condition χ is in $L_2(\Omega, \mathbb{C}, \mathscr{A}_0)$. Then* (1) *subject to the initial condition* (2) *admits a unique solution.*

Now we present some lemmas to be used later for the proof of the convergence theorem. Consider a right continuous process $Y^{\Delta_l} = \{Y^{\Delta_l}(t),\ t \in [-\gamma, T]\}$. Y^{Δ_l} is called a discrete-time numerical approximation with maximum step size Δ_l, if it is obtained by using a time discretization t_{Δ_l}, and the random variable $Y^{\Delta_l}_{t_n}$ is $\mathscr{F}_{t_n}$-measurable for $n \in \{1, \ldots, N\}$. Further, $Y^{\Delta_l}_{t_{n+1}}$ can be expressed as a function of $Y^{\Delta_l}_{t_{-l}}$, $Y^{\Delta_l}_{t_{-l+1}}, \ldots, Y^{\Delta_l}_{t_n}$ and the discrete-time t_n.

Because of dealing with the approximation of solutions of jump-diffusion SDDEs, we introduce a concept of weak order convergence due to Kloeden and Platen [15].

Definition 3. A discrete-time approximation Y^{Δ_l} converges weakly towards X at time T with order $\beta > 0$ if for each $g \in \mathscr{C}_p$ there is a constant C, independent of Δ_l, such that

$$\left|E\left(g\left(X\left(T\right)\right)\right) - E\left(g\left(Y^{\Delta_l}\left(T\right)\right)\right)\right| \le C(\Delta_l)^{\beta}, \qquad (8)$$

where $\mathscr{C}_p$ denotes the set of all polynomials $g : \mathbb{R}^d \to \mathbb{R}$.

We now give some auxiliary results to prepare for the proof of the weak convergence theorem to be presented.

Lemma 4. *For $n \in \{-l+1, \ldots, 0, 1, \ldots, N\}$ and $\mathbf{z} \in \mathbb{R}^d$, we have*

$$\begin{aligned} E\bigg(u\left(n, \mathbf{X}^{n-1,z}_{n^-}\right) - u\left(n-1, \mathbf{z}\right) \\ + \int_{n-1}^{n} \int_{\boldsymbol{\varepsilon}} \mathbf{L}^{-1}_{v} u\left(\tau, \mathbf{X}^{n-1,z}_{\tau}\right) \varphi\left(dv\right) d\tau \mid \mathscr{A}_{n-1}\bigg) = 0 \end{aligned} \qquad (9)$$

for $(\tau, \mathbf{z}) \in [-\gamma, t] \times \mathbb{R}^d$ and $u(\tau, \mathbf{z}) = E(g(\mathbf{X}^{\tau,z}_{T})\mathscr{A}_{\tau})$.

The proof of the lemma for the case with no delay was established by Platen and Bruti-Liberati [21], and a similar procedure can be used for the proof of this lemma.

Lemma 5. *Given $p \in \{1, 2, 3, \ldots\}$, there is a bounded constant M satisfying*

$$E\left(\left|\mathbf{X}^{n-1,z}_{n^-} - \mathbf{z}\right|^{2q} \mid \widetilde{\mathscr{A}}_{n-1}\right) \le M\left(1 + |\mathbf{z}|^{2q}\right)\left(\Delta_l\right)^{q} \qquad (10)$$

for $q \in \{1, \ldots, p\}$ and $n \in \{-l+1, \ldots, 0, 1, \ldots, N\}$.

The proof of (10) can be obtained by following that of a lemma for SDEs with jumps but with no delay in [15]. The following results are similar to what was given in Mikulevičius and Platen [22].

Lemma 6. *Given $p \in \{1, 2, 3, \ldots\}$, there is a finite constant M satisfying*

$$E\left(\sup_{-\gamma \le t \le T} |\boldsymbol{\zeta}(t)|^{2q}\right) \le M\left(1 + \left|\mathbf{Y}_0\right|^{2q}\right) \qquad (11)$$

for every $q \in \{1, \ldots, p\}$.

Lemma 7. *Given $p \in \{1, 2, \ldots\}$, there is $r \in \{1, 2, 3, \ldots\}$ and a bounded constant M satisfying*

$$\begin{aligned} &\left|E\left(\left|\mathbf{F_p}\left(\boldsymbol{\zeta}\left(z\right) - \mathbf{Y}^{\Delta_l}_{z}\right)\right|^{2q} + \left|\mathbf{F_p}\left(\mathbf{X}^{z,Y^{\Delta_l}_{z}}_{z} - \mathbf{Y}^{\Delta_l}_{z}\right)\right|^{2q} \mid \widetilde{\mathscr{A}}_{z}\right)\right| \\ &\quad \le M\left(1 + \left|\mathbf{Y}^{\Delta_l}_{z}\right|^{2r}\right)\left(\Delta_l\right)^{qk} \end{aligned} \qquad (12)$$

for each $q \in \{1, \ldots, p\}$, $k \in \{1, \ldots, 2(\beta+1)\}$, $\mathbf{p} \in P_k = \{1, 2, \ldots, d\}^k$, and $z \in [-r, T]$, where $F_{\mathbf{p}}(\mathbf{y}) = \prod_{h=1}^{k} y^{p_h}$ for all $\mathbf{y} = (y^1, \ldots, y^d)^T \in \mathfrak{R}^d$ and $\mathbf{p} = (p_1, \ldots, p_k) \in P_k$.

The proof of estimate (12) can be established by following Itô's formula for SDEs with jumps but with no delay as in [15].

Lemma 8. *For $\mathbf{p} \in P_k$, there exist $r \in \{1, 2, \ldots\}$ and a finite constant M satisfying*

$$\begin{aligned} &\left|E\left(\mathbf{F_p}\left(\boldsymbol{\zeta}\left(t\right) - \mathbf{Y}^{\Delta_l}_{n-1}\right) - \mathbf{F_p}\left(\mathbf{X}^{n-1,Y^{\Delta_l}_{n-1}}_{t} - \mathbf{Y}^{\Delta}_{t_{n-1}}\right) \mid \widetilde{\mathscr{A}}_{t}\right)\right| \\ &\quad \le M\left(1 + \left|\mathbf{Y}^{\Delta_l}_{n-1}\right|^{r}\right)\left(\Delta_l\right)^{\beta} \end{aligned} \qquad (13)$$

for each $k \in \{1, \ldots, 2\beta+1\}$, $n \in \{-l+1, \ldots, 0, 1, \ldots, N\}$, and $t \in [t_{n-1}, t_n)$.

3. The Jump-Adapted Weak Taylor Approximation Scheme

In Monte Carlo simulations for functionals of jump-diffusion SDDEs, one uses numerical approximations evaluated only at discretization time. Here, we first give a jump adapted weak approximation Taylor scheme of order β, and then study the basic properties of the discrete Taylor approximation in a weak order sense.

First we define the jump-adapted time discretization. Let $T > r > 0$. The jump adapted time discretization used throughout this paper is

$$(t)_{\Delta} = \{t_i : i = -l, -l+1, \ldots, 0, 1, 2, \ldots, N\} \qquad (14)$$

with the maximum step size $\Delta_l \in (0, 1)$. We choose the time discretization in such a way that all jump times are at the nodes of the time discretization. If the discretization node t_i is not a jump time, then t_i is $\mathscr{A}_{t_{i-1}}$-measurable. Otherwise, t_i is $\mathscr{A}_{t_i^-}$-measurable. Also, throughout the paper, we denote the set of all multiindices α by

$$\begin{aligned} \mathscr{M}_m = &\left\{\left(j_1, \ldots, j_l\right) : j_i \in \{0, 1, 2, \ldots, m\},\ i \in \{1, 2, \ldots, l\}\right. \\ &\left.\text{for } l \in \mathscr{N}\right\} \cup \{v\}, \end{aligned} \qquad (15)$$

where the element $\alpha = (j_1, j_2, \ldots, j_l)$ is called a multiindex of length $l = l(\alpha) \in \mathscr{N}$ and v has zero length. In the following, by a component $j \in \{0, 1, 2, \ldots, m\}$ of a multi-index we refer to the integration with respect to the jth Wiener process in a multiple stochastic integral. A component with $j = 0$ corresponds to integration with respect to time t.

Now define the following operators for the coefficient functions:

$$L^0 = \frac{\partial}{\partial t} + \sum_{i=1}^{d} a^i(t, x, x_l)\frac{\partial}{\partial x^i} + \sum_{i=1}^{d} a^i(t-l, x_l, x_{2l})\frac{\partial}{\partial x_l^i} + \frac{1}{2}\sum_{i,\gamma=1}^{d}\sum_{j=1}^{m} b^{ij}(t, x, x_l) b^{\gamma j}(t, x, x_l)\frac{\partial^2}{\partial x^i \partial x^\gamma} + \frac{1}{2}\sum_{i,\gamma=1}^{d}\sum_{j=1}^{m} b^{ij}(t-l, x_l, x_{2l}) b^{\gamma j}(t-l, x_l, x_{2l})\frac{\partial^2}{\partial x_l^i \partial x_l^\gamma},$$

$$L^k = \sum_{i=1}^{d} b^{ik}(t, x, x_l)\frac{\partial}{\partial x^i} + \sum_{i=1}^{d} b^{ik}(t-l, x_l, x_{2l})\frac{\partial}{\partial x_l^i}. \tag{16}$$

A subset $\mathscr{A} \in \mathscr{M}$ is the hierarchical set, and its corresponding remainder set $\overline{A}(\mathscr{M})$ is defined by $\overline{A}(\mathscr{M}) = \{\alpha \in \mathscr{M}_m \backslash \mathscr{A} : -\alpha \in \mathscr{A}\}$. For each $\beta \in 1, 2, 3, \ldots$, we can then define the hierarchical set $\Gamma_\beta = \{\alpha \in \mathscr{M}_m : l(\alpha) \leq \beta\}$. The weak Taylor method of order β is then constructed as follows:

$$\mathbf{Y}_{(n+1)^-} = \mathbf{Y}_n + \sum_{\alpha \in \Gamma_\beta} f_\alpha(n, \mathbf{Y}_n, \mathbf{Y}_{n-l}) I_\alpha, \qquad \mathbf{Y}_{n+1} = \mathbf{Y}_{(n+1)^-} + \int_\varepsilon c\left(n, \mathbf{Y}_{(n+1)^-}, v\right) p_\varphi(dv, (n+1)), \tag{17}$$

where

$$f_\alpha(t, x, u) = \begin{cases} f(t, x) & \text{if } l(\alpha) = 0 \\ L^{j_1} f_{-\alpha}(t, x, u) & \text{if } l(\alpha) \geq 1, \\ & j_1 \in 0, 1, \ldots, m. \end{cases} \tag{18}$$

The multiple stochastic integral is then defined recursively as follows:

$$I_{\alpha,t} = \begin{cases} t & \text{if } l = 0 \\ \int_0^t I_{\alpha-,z} dz & \text{if } l \geq 1,\ j_l = 0 \\ \int_0^t I_{\alpha-,z} dW_z^{j_l} & \text{if } l \geq 1,\ j_l \in 1, \ldots, m, \end{cases} \tag{19}$$

where $\alpha-$ is obtained from α by deleting its last component, while $-\alpha$ is obtained from α by deleting its first component.

Now, we give the weak convergence theorem of the Taylor approximation with order β.

Theorem 9. *Given $\beta \in \{1, 2, \ldots\}$, let $Y^{\Delta_l} = \{Y_n^{\Delta_l},\ n \in [-l, \ldots, 0, 1, \ldots, N]\}$ be the results obtained from the Taylor scheme (17) corresponding to $(t)_\Delta$ with maximum step size $\Delta_l \in (0, 1)$. Suppose that $E(|\mathbf{X}_\xi|^i) < \infty$ for $\xi \in (-\gamma, 0)$, $i \in \{1, 2, \ldots\}$, and $\mathbf{Y}_\xi^\Delta$ converges to $\mathbf{X}_\xi$ weakly with order $\beta \in \{1, 2, \ldots\}$. Assume that the coefficients, a^k, b^{kj}, c^k, are in the space $\mathscr{C}_P^{2(\beta+1)}(\mathbb{R}^d, \mathbb{R})$, for $j \in \{1, 2, \ldots, m\}$ and $k \in \{1, 2, \ldots, d\}$, and the coefficient functions f_α, with $f(t, \mathbf{y}) = \mathbf{y}$, satisfy the growth condition $|f_\alpha(t, \mathbf{y})| \leq M(1 + |\mathbf{y}|)$, with $M < \infty$, for all $t < T$, $\mathbf{y} \in \mathbb{R}^d$, and $\alpha \in \Gamma_\beta$. Then for any $g \in \mathscr{C}_P^{2(\beta+1)}(\mathbb{R}^d, \mathbb{R})$ there is a positive constant C, which does not depend on Δ, such that*

$$\left|E(g(X(T))) - E\left(g\left(Y^{\Delta_l}(T)\right)\right)\right| \leq C(\Delta_l)^\beta. \tag{20}$$

Proof. For $\beta \in \{1, 2, 3, \ldots\}$ and $g \in \mathscr{C}_P^{2(\beta+1)}(\mathbb{R}^d, \mathbb{R})$, consider the Itô process below:

$$\mathbf{X}_t^{z,y} = \mathbf{y} + \int_z^t \mathbf{a}\left(\mathbf{X}_u^{z,y}\right) du + \int_z^t \mathbf{b}\left(\mathbf{X}_u^{z,y}\right) d\mathbf{W}_u + \int_z^t \int_{\boldsymbol{\varepsilon}} \mathbf{c}\left(\mathbf{X}_{u-}^{z,y}, v\right) p_\varphi(dv, du). \tag{21}$$

Then we can get $u(0, X_0) = E(g(X_T^{0,X_0})) = E(g(X_T))$. Define also the process $\zeta = \zeta(t)$, $t \in (-\gamma, T)$, by

$$\zeta(t) = \zeta(t_n) + \sum_{\alpha \in \Gamma_\beta} I_\alpha\left(f_\alpha\left(n, \boldsymbol{\zeta}_n, \boldsymbol{\zeta}_{n-l}\right)\right) + \int_n^t \int_\varepsilon c\left(n, \boldsymbol{\zeta}_{(n+1)^-}, v\right) p_\varphi(dv, (n+1)) \tag{22}$$

for $n \in \{-l, -l+1, \ldots, 0, 1, \ldots, N-1\}$, $t \in (t_n, t_{n+1}]$, $\zeta(0) = Y_0$, and $\zeta(t_n) = Y_{t_n}$ for $n \in \{-l, \ldots, 0, 1, 2, \ldots, N\}$.

By the definition of the functional u and the terminal condition of the stochastic process X, we have

$$H = \left|E\left(g\left(\mathbf{Y}_T^{\Delta_l}\right)\right) - E\left(g\left(\mathbf{X}_T\right)\right)\right| = \left|E\left(u\left(T, \mathbf{Y}_T^{\Delta_l}\right) - u\left(0, \mathbf{X}_0\right)\right)\right|. \tag{23}$$

Since $\mathbf{Y}_0$ converges towards $\mathbf{X}_0$ weakly with order β, one has

$$H \leq \left|E\left(\sum_{n=-l+1}^{N}\left(u\left(n, \mathbf{Y}_n\right) - u\left(n, \mathbf{Y}_{n^-}\right) + u\left(n, \mathbf{Y}_{n^-}\right) - u\left(n-1, \mathbf{Y}_{n-1}\right)\right)\right)\right| + K(\Delta_l)^\beta. \tag{24}$$

By Lemma 4, we can write

$$H \leq \left|E\left(\sum_{n=-l+1}^{N}\left\{\left[u\left(n, \mathbf{Y}_n\right) - u\left(n, \mathbf{Y}_{n^-}\right) + u\left(n, \mathbf{Y}_{n^-}\right) - u\left(n-1, \mathbf{Y}_{n-1}\right)\right] - \left[u\left(n, \mathbf{X}_{n^-}^{n-1, Y_{n-1}}\right) - u\left(n-1, \mathbf{Y}_{n-1}\right) + \int_{n-1}^{n}\int_{\boldsymbol{\varepsilon}} \mathbf{L}_v^{-1} u\left(z, \mathbf{X}_z^{n-1, Y_{n-1}}\right) \times \varphi(dv)\, dz\right]\right\}\right)\right| + K(\Delta_l)^\beta. \tag{25}$$

From the properties of stochastic integrals, we obtain

$$E\left(\sum_{n=-l}^{N}\left[u\left(n, \mathbf{Y}_n\right) - u\left(n, \mathbf{Y}_{n^-}\right)\right]\right) = E\left(\int_{-r}^{T}\int_{\boldsymbol{\varepsilon}} \mathbf{L}_v^{-1} u\left(z, \boldsymbol{\zeta}(z)\right) \varphi(dv)\, dz\right). \tag{26}$$

Therefore, we have

$$H \leq H_1 + H_2 + K(\Delta_l)^{\beta}, \tag{27}$$

where

$$H_1 = \left| E\left(\sum_{n=-l+1}^{N} \left\{ \left[u\left(n, \mathbf{Y}_{n^-}\right) - u\left(n, \mathbf{Y}_{n-1}\right) \right] - \left[u\left(n, \mathbf{X}_{n^-}^{n-1, Y_{n-1}}\right) - u\left(n, \mathbf{Y}_{n-1}\right) \right] \right\} \right) \right|, \tag{28}$$

$$H_2 = \left| E\left(\int_{-r}^{T} \int_{\boldsymbol{\varepsilon}} \left\{ \left[L_v^{-1} u\left(z, \boldsymbol{\zeta}(z)\right) - L_v^{-1} u\left(z, \mathbf{Y}_z\right) \right] - \left[L_v^{-1} u\left(z, \mathbf{X}_z^{z, Y_z}\right) - L_v^{-1} u\left(z, \mathbf{Y}_z\right) \right] \right\} \times \varphi(dv)\, dz \right) \right|. \tag{29}$$

In the following, we proceed to estimate H_1 and H_2 in Steps 1 and 2, respectively, and then complete the proof in Step 3.

Step 1. Let us assume that u is so smooth that the deterministic Taylor expansion may be applied. Hence, by expanding du in H_1, we get

$$\begin{aligned} H_1 = \Bigg| E\Bigg(\sum_{n=-l+1}^{N} \Bigg\{ \Bigg[& \sum_{k=1}^{2\beta+1} \frac{1}{k!} \sum_{\mathbf{p} \in P_k} \left(\partial_y^{\mathbf{p}} u\left(n, \mathbf{Y}_{n-1}\right) \right) \times F_{\mathbf{p}}\left(\mathbf{Y}_{n^-} - \mathbf{Y}_{n-1}\right) + R_n\left(\mathbf{Y}_{n^-}\right) \Bigg] \\ & - \Bigg[\sum_{k=1}^{2\beta+1} \frac{1}{k!} \sum_{\mathbf{p} \in P_k} \left(\partial_y^{\mathbf{p}} u\left(n, \mathbf{Y}_{n-1}\right) \right) \times F_{\mathbf{p}}\left(\mathbf{X}_{n^-}^{n-1, Y_{n-1}} - \mathbf{Y}_{n-1}\right) + R_n\left(\mathbf{X}_{n^-}^{n-1, Y_{n-1}}\right) \Bigg] \Bigg\} \Bigg) \Bigg|, \end{aligned} \tag{30}$$

where the remainder term is

$$R_n(\mathbf{Z}) = \frac{1}{2(\beta+1)!} \times \sum_{\mathbf{p} \in P_{2(\beta+1)}} \partial_y^{\mathbf{p}} u\left(n, \mathbf{Y}_{n-1} + \boldsymbol{\theta}_{\mathbf{p},n}\left(\mathbf{Z} - \mathbf{Y}_{n-1}\right)\right) \times F_{\mathbf{p}}\left(\mathbf{Z} - \mathbf{Y}_{n-1}\right), \tag{31}$$

where $\boldsymbol{\theta}_{\mathbf{p},n}(\mathbf{Z})$ is a $d \times d$ diagonal matrix with $\theta_{\mathbf{p},n}^{k,k}(\mathbf{Z}) \in (0,1)$ for $k \in \{1, 2, 3, \ldots, d\}$ and $\mathbf{Z} = \mathbf{Y}_{n^-}$ and $\mathbf{X}_{n^-}^{n-1, Y_{n-1}}$, respectively.

Therefore, according to the properties of expectation and absolute value, we get

$$\begin{aligned} H_1 &\leq E\Bigg(\sum_{n=-l+1}^{N} \Bigg\{ \sum_{k=1}^{2\beta+1} \frac{1}{k!} \sum_{\mathbf{p} \in P_k} \left| \left(\partial_y^{\mathbf{p}} u\left(n, \mathbf{Y}_{n-1}\right) \right) \right| \times \left(F_{\mathbf{p}}\left(\mathbf{Y}_{n^-} - \mathbf{Y}_{n-1}\right) - F_{\mathbf{p}}\left(\mathbf{X}_{n^-}^{n-1, Y_{n-1}} - \mathbf{Y}_{n-1}\right) \right) \times \left| R_n\left(\mathbf{Y}_{n^-}\right) \right| + \left| R_n\left(\mathbf{X}_{n^-}^{n-1, Y_{n-1}}\right) \right| \Bigg\} \Bigg) \\ &\leq E\Bigg(\sum_{n=-l+1}^{N} \Bigg\{ \sum_{k=1}^{2\beta+1} \frac{1}{k!} \sum_{\mathbf{p} \in P_k} \left| \left(\partial_y^{\mathbf{p}} u\left(n, \mathbf{Y}_{n-1}\right) \right) \right| \times \left| E\left(F_{\mathbf{p}}\left(\mathbf{Y}_{n^-} - \mathbf{Y}_{n-1}\right) - F_{\mathbf{p}}\left(\mathbf{X}_{n^-}^{n-1, Y_{n-1}} - \mathbf{Y}_{n-1}\right) \mid \widetilde{\mathscr{A}}_{n-1} \right) \right| \\ &\qquad + E\left(\left| R_n\left(\mathbf{Y}_{n^-}\right) \right| \mid \widetilde{\mathscr{A}}_{n-1} \right) + E\left(\left| R_n\left(\mathbf{X}_{n^-}^{n-1, Y_{n-1}}\right) \right| \mid \widetilde{\mathscr{A}}_{n-1} \right) \Bigg\} \Bigg). \end{aligned} \tag{32}$$

By (31), the Hölder inequality, and Lemma 7, we get

$$\begin{aligned} E&\left(\left| R_n\left(\mathbf{Y}_{n^-}\right) \right| \mid \widetilde{\mathscr{A}}_{n-1} \right) \\ &\leq M \sum_{\mathbf{p} \in P_{2(\beta+1)}} \left[E\left(\left| \partial_y^{\mathbf{p}} u\left(n, \mathbf{Y}_{n-1} + \boldsymbol{\theta}_{\mathbf{p},n}\left(\mathbf{Y}_{n^-}\right) \times \left(\mathbf{Y}_{n^-} - \mathbf{Y}_{n-1}\right)\right) \right|^2 \mid \widetilde{\mathscr{A}}_{n-1} \right) \right]^{1/2} \times \left[E\left(\left| F_{\mathbf{p}}\left(\mathbf{Y}_{n^-} - \mathbf{Y}_{n-1}\right) \right|^2 \mid \widetilde{\mathscr{A}}_{n-1} \right) \right]^{1/2} \\ &\leq M \left[E\left(1 + \left| \mathbf{Y}_{n-1} \right|^{2r} + \left| \mathbf{Y}_{n^-} - \mathbf{Y}_{n-1} \right|^{2r} \mid \widetilde{\mathscr{A}}_{n-1} \right) \right]^{1/2} \times \left[E\left(\left| \mathbf{Y}_{n^-} - \mathbf{Y}_{n-1} \right|^{4(\beta+1)} \mid \widetilde{\mathscr{A}}_{n-1} \right) \right]^{1/2} \\ &\leq M \left(1 + \left| \mathbf{Y}_{n-1} \right|^{2r} \right) \left(\Delta_l\right)^{\beta+1}. \end{aligned} \tag{33}$$

Similarly, by Lemma 5 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&E\left(\left|R_n\left(\mathbf{X}_{n^-}^{n-1,Y_{n-1}}\right)\right| \mid \widetilde{\mathscr{A}}_{n-1}\right)\\
&\quad\le M\sum_{\mathbf{p}\in P_{2(\beta+1)}}\left[E\left(\left|\partial_y^{\mathbf{p}}u\left(n,\mathbf{Y}_{n^-}+\boldsymbol{\theta}_{\mathbf{p},n}\left(\mathbf{X}_{n^-}^{n-1,Y_{n-1}}\right.\right.\right.\right.\\
&\qquad\left.\left.\left.\left.\times\left(\mathbf{X}_{n^-}^{n-1,Y_{n-1}}-\mathbf{Y}_{n-1}\right)\right)\right|^2 \mid \widetilde{\mathscr{A}}_{n-1}\right)\right]^{1/2}\\
&\qquad\times\left[E\left(\left|F_{\mathbf{p}}\left(\mathbf{X}_{n^-}^{n-1,Y_{n-1}}-\mathbf{Y}_{n-1}\right)\right|^2 \mid \widetilde{\mathscr{A}}_{n-1}\right)\right]^{1/2}\\
&\quad\le M\left[E\left(1+\left|\mathbf{Y}_{n-1}\right|^{2r}+\left|\mathbf{X}_{n^-}^{n-1,Y_{n-1}}-\mathbf{Y}_{n-1}\right|^{2\gamma} \mid \widetilde{\mathscr{A}}_{n-1}\right)\right]^{1/2}\\
&\qquad\times\left[E\left(\left|\mathbf{X}_{n^-}^{n-1,Y_{n-1}}-\mathbf{Y}_{n-1}\right|^{4(\beta+1)} \mid \widetilde{\mathscr{A}}_{n-1}\right)\right]^{1/2}\\
&\quad\le M\left(1+\left|\mathbf{Y}_{n-1}\right|^{2r}\right)\left(\Delta_l\right)^{\beta+1}.
\end{aligned}\tag{34}$$

Now, from the Cauchy-Schwarz inequality, Lemmas 8 and 6, and inequalities (33) and (34), we obtain

$$\begin{aligned}
H_1 &\le E\left(K\sum_{n=-l+1}^{N}\left(1+\left|\mathbf{Y}_{n-1}\right|^{2\gamma}\right)\left(\Delta_l\right)^{\beta}\right)\\
&\le M(\Delta_l)^{\beta}\left(1+E\left(\max_{-l\le n\le N}\left|\mathbf{Y}_n\right|^{2\gamma}\right)\right)\\
&\le M(\Delta_l)^{\beta}\left(1+\left|\mathbf{Y}_0\right|^{2\gamma}\right)\le K(\Delta_l)^{\beta}.
\end{aligned}\tag{35}$$

Step 2. Now we estimate the term H_2 in inequality (27). By the jump coefficient c and the smooth function u, applying the Taylor expansion yields

$$\begin{aligned}
H_2 = &\left|E\left(\int_{-r}^{T}\int_{\varepsilon}\left\{\left[\left[\sum_{k=1}^{2\beta+1}\frac{1}{k!}\sum_{\mathbf{p}\in P_k}\left(\partial_y^{\mathbf{p}}L_v^{-1}u\left(z,\mathbf{Y}_z\right)\right)\right.\right.\right.\right.\right.\\
&\qquad\times F_{\mathbf{p}}\left(\boldsymbol{\zeta}(z)-\mathbf{Y}_{t_z}\right)\\
&\qquad\left.+R_n\left(\boldsymbol{\zeta}(z)\right)\right]\\
&\quad-\left[\sum_{k=1}^{2\beta+1}\frac{1}{k!}\sum_{\mathbf{p}\in P_k}\left(\partial_y^{\mathbf{p}}L_v^{-1}u\left(z,\mathbf{Y}_z\right)\right)\right.\\
&\qquad\times F_{\mathbf{p}}\left(\mathbf{X}_z^{z,Y_z}-\mathbf{Y}_{t_z}\right)\\
&\qquad\left.\left.\left.\left.+R_n\left(\mathbf{X}_z^{z,Y_z}\right)\right]\right]\right\}\varphi(dv)\,dz\right)\right|\\
\le &\int_{-r}^{T}\int_{\varepsilon}E\left(\sum_{k=1}^{2\beta+1}\frac{1}{k!}\sum_{\mathbf{p}\in P_k}\left|\left(\partial_y^{\mathbf{p}}L_v^{-1}u\left(z,\mathbf{Y}_z\right)\right)\right|\right.\\
&\qquad\times\left|E\left(F_{\mathbf{p}}\left(\boldsymbol{\zeta}(z)-\mathbf{Y}_z\right)\right.\right.\\
&\qquad\qquad\left.\left.-F_{\mathbf{p}}\left(\mathbf{X}_z^{z,Y_z}-\mathbf{Y}_z\right)\mid \widetilde{\mathscr{A}}_z\right)\right|\\
&\qquad+E\left(\left|R_n\left(\boldsymbol{\zeta}(z)\right)\right| \mid \widetilde{\mathscr{A}}_z\right)\\
&\qquad\left.+E\left(\left|R_n\left(\mathbf{X}_z^{z,Y_z}\right)\right| \mid \widetilde{\mathscr{A}}_z\right)\right)\varphi(dv)\,dz.
\end{aligned}\tag{36}$$

Similarly, we can estimate the reminders as follows:

$$\begin{aligned}
E\left(\left|R_n\left(\boldsymbol{\zeta}(z)\right)\right| \mid \widetilde{\mathscr{A}}_z\right)&\le M\left(1+\left|\mathbf{Y}_{t_z}\right|^{2r}\right)\left(z-t_z\right)^{\beta+1},\\
E\left(\left|R_n\left(\mathbf{X}_z^{z,Y_z}\right)\right| \mid \widetilde{\mathscr{A}}_z\right)&\le M\left(1+\left|\mathbf{Y}_{t_z}\right|^{2r}\right)\left(z-t_z\right)^{\beta+1}.
\end{aligned}\tag{37}$$

Then, by applying the Hölder inequality, Lemmas 8 and 6, inequalities (37) to estimate the inequality above, we get

$$\begin{aligned}
H_2 &\le M\int_{-r}^{T}\int_{\varepsilon}E\left(1+\left|\mathbf{Y}_{t_z}\right|^{2r}\right)\left(\Delta_l\right)^{\beta}\varphi(dv)\,dz\\
&\le M(\Delta_l)^{\beta}\int_0^T E\left(1+\max_{0\le n\le n_T}\left|\mathbf{Y}_{t_n}\right|^{2r}\right)\left(z-t_z\right)dz\\
&\le M(\Delta_l)^{\beta}.
\end{aligned}\tag{38}$$

Step 3. Finally, by inequalities (27) and (35) as well as (38), we have

$$\left|E\left(g\left(X(T)\right)\right)-E\left(g\left(Y^{\Delta_l}(T)\right)\right)\right|\le M(\Delta_l)^{\beta}.\tag{39}$$

□

4. A Numerical Example

Here we give an illustrative example to demonstrate the application and the convergence of the proposed numerical scheme. We consider the following linear SDDE with Poisson jumps:

$$\begin{aligned}
dX_t &= \left[\left(\mu-\nu\lambda\right)X_t+\alpha X_{t-1}\right]dt\\
&\qquad+\left[\sigma X_t+\beta X_{t-1}\right]dW_t+\nu X_t dN,\\
X(t) &= t+1,\quad t\in[-1,0],
\end{aligned}\tag{40}$$

where μ, σ, and ν are, respectively, the drift coefficient, the diffusion coefficient and the jump coefficient; λ is the jump intensity; and α and β are the delay coefficients.

Table 1: Convergence results for the linear SDDE with jumps.

Stepsize $1/h$	2^8	2^{10}	2^{12}	2^{14}	2^{16}	2^{18}	2^{20}
Weak error1	0.00352	0.00201	0.00156	0.00114	0.00082	0.00060	0.00031
Weak error2	0.00229	0.00142	0.00061	0.00020	0.00009	0.00004	0.00002

By the method for solving linear stochastic differential equations in [23], the analytical solution for $t \in [0,1]$ is derived as follows:

$$X(t) = \Phi(t)\left(1 + \int_0^t \Phi(s)^{-1}(\alpha - \sigma\beta)\, s\,ds + \int_0^t \Phi(s)^{-1}\beta s\,dW_s\right), \tag{41}$$

where

$$\Phi(t) = (\nu+1)^{N(t)} \exp\left\{\left(\mu - \nu\lambda - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\}. \tag{42}$$

According to the weak Taylor approximation scheme (17)–(19) proposed in Section 3, we now expand it with weak order 1 (well known as Euler scheme):

$$\begin{aligned} Y_{(n+1)^-} &= Y_n + ((\mu - \nu\lambda)Y_n + \alpha hn)h + (\sigma Y_n + \beta hn)\Delta W_n, \\ Y_{n+1} &= Y_{(n+1)^-} + \nu Y_n \Delta N_n. \end{aligned} \tag{43}$$

Here we have used the jump adapted time discretization, and h is the maximum step size.

For higher accuracy and efficiency, one needs to construct higher order numerical schemes. We now give a Taylor scheme of weak order two below:

$$\begin{aligned} Y_{(n+1)^-} = {} & Y_n + ((\mu - \nu\lambda)Y_n + \alpha hn)h \\ & + (\sigma Y_n + \beta hn)\Delta W_n \\ & + ((\mu - \nu\lambda)(\sigma Y_n + \beta hn) \\ & \quad + \sigma((\mu - \nu\lambda)Y_n + \alpha hn))\frac{h}{2}\Delta W_n \\ & + (\mu - \nu\lambda)((\mu - \nu\lambda)Y_n + \alpha hn)\frac{h^2}{2} \\ & + \sigma(\sigma Y_n + \beta hn)\frac{(\Delta W_n)^2 - h}{2}, \\ Y_{n+1} = {} & Y_{(n+1)^-} + \nu Y_n \Delta N_n. \end{aligned} \tag{44}$$

Next, we study the convergence of the two numerical schemes presented above by using the weak errors measured by

$$\varepsilon(h) = |E(X(T)) - E(Y(T))| \tag{45}$$

and compare the results obtained from these two schemes to the explicit exact solution. We estimate the weak errors $\varepsilon(h)$ by running a very large number of simulations. The exact number depends on the implemented scheme. We use the following parameters: $\alpha = 0.01$, $\beta = 0.01$, $\mu = 0.001$, $\sigma = 0.6$, $\nu = 0.002$, and $\lambda = 0.001$.

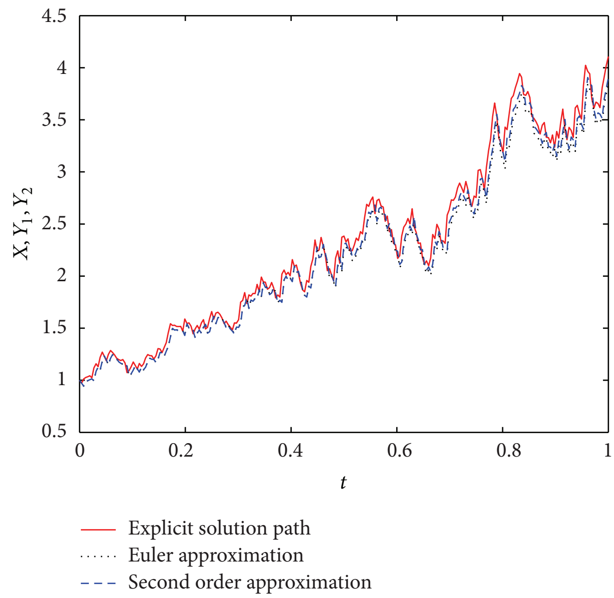

Figure 1: Sample paths of linear SDDE with jumps.

In Figure 1, we give the sample paths under the two approximation schemes and the numerical explicit solution of (40). We can see from the figure that the weak Taylor scheme path is closer to the analytical solution line than the Euler scheme.

Now we present the numerical errors generated by the two numerical schemes presented above. From Table 1, we notice that, for all the step sizes used in the numerical experiments, the weak Taylor method is more accurate. Moreover, the errors of the weak order two Taylor method decrease faster than the Euler scheme.

5. Conclusions

In this work, we have extended previous research on weak convergence to a more general class of stochastic differential equations involving both jumps and time delay. We proved that under the Poisson random measure and a fixed time delay, a simplified Taylor method gives weak convergence rate arbitrarily close to order β. There is much scope for further work in the context of weak solution of jump-diffusion SDDEs. For example, it is clearly of great importance to extend the weak convergence theory to the case where coefficients in the equations are not globally Lipschitz, and to develop and analyze new methods that maintain good properties of convergence and stability.

Acknowledgments

The authors thank the anonymous referees and the editors for their constructive comments and suggestions. Yanli Zhou acknowledges financial support from Curtin International Postgraduate Research Scholarship (CIPRS) and Chinese Scholarship Council (CSC). Acknowledgement is also made to the China National Social Science Foundation (SSF) for the research support (10BJY104).

References

[1] P. Hu and C. Huang, "Stability of stochastic θ-methods for stochastic delay integro-differential equations," *International Journal of Computer Mathematics*, vol. 88, no. 7, pp. 1417–1429, 2011.

[2] J. Tan and H. Wang, "Mean-square stability of the Euler-Maruyama method for stochastic differential delay equations with jumps," *International Journal of Computer Mathematics*, vol. 88, no. 2, pp. 421–429, 2011.

[3] N. Jacob, Y. Wang, and C. Yuan, "Numerical solutions of stochastic differential delay equations with jumps," *Stochastic Analysis and Applications*, vol. 27, no. 4, pp. 825–853, 2009.

[4] H. Zhang, S. Gan, and L. Hu, "The split-step backward Euler method for linear stochastic delay differential equations," *Journal of Computational and Applied Mathematics*, vol. 225, no. 2, pp. 558–568, 2009.

[5] W. Wang and Y. Chen, "Mean-square stability of semi-implicit Euler method for nonlinear neutral stochastic delay differential equations," *Applied Numerical Mathematics*, vol. 61, no. 5, pp. 696–701, 2011.

[6] V. Kolmanovskii and A. Myshkis, *Applied Theory of Fundamental Differential Equations*, Kluwer, New York, NY, USA, 1992.

[7] S. E. A. Mohammed, *Stochastic Functional Differential Equations*, vol. 99 of *Research Notes in Mathematics*, Pitman, Boston, Mass, USA, 1984.

[8] S. E. A. Mohammed, "Lyapunov exponents and stochastic flows of linear and affine hereditary systems," in *Diffusion Processes and Related Problems in Analysis, Volume II*, vol. 27 of *Progress in Probability*, pp. 141–169, Birkhäuser, Boston, Mass, USA, 1992.

[9] X. Mao, *Exponential Stability of Stochastic Differential Equations*, vol. 182, Marcel Dekker, New York, NY, USA, 1994.

[10] X. Mao, *Stochastic Differential Equations and Applications*, Horwood, New York, NY, USA, 1997.

[11] S. E. A. Mohammed and M. K. R. Scheutzow, "Lyapunov exponents and stationary solutions for affine stochastic delay equations," *Stochastics and Stochastics Reports*, vol. 29, no. 2, pp. 259–283, 1990.

[12] A. V. Svishchuk and Y. I. Kazmerchuk, "Stability of stochastic Ito equations delays, with jumps and Markov switchings, and their application in finance," *Theory of Probability and Mathematical Statistics*, no. 64, pp. 141–151, 2002.

[13] J. Luo, "Comparison principle and stability of Ito stochastic differential delay equations with Poisson jump and Markovian switching," *Nonlinear Analysis: Theory, Methods and Applications A*, vol. 64, no. 2, pp. 253–262, 2006.

[14] R. Li and Z. Chang, "Convergence of numerical solution to stochastic delay differential equation with Poisson jump and Markovian switching," *Applied Mathematics and Computation*, vol. 184, no. 2, pp. 451–463, 2007.

[15] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, vol. 23 of *Applied Mathematics, 3rd corrected printing*, Springer, Berlin, Germany, 1999.

[16] U. Küchler and E. Platen, "Strong discrete time approximation of stochastic differential equations with time delay," *Mathematics and Computers in Simulation*, vol. 54, no. 1–3, pp. 189–205, 2000.

[17] C. T. H. Baker and E. Buckwar, "Introduction to the numerical analysis of stochastic delay differential equations," Numberical Analysis Report 345, University of Manchester, Department of Mathematics, Manchester, UK, 1999.

[18] H. Gilsing, "On the stability of the Euler scheme for an affine stochastic differential equations with time delay," Diskussion Paper 20, SFB 373, Humboldt University, Berlin, Germany, 2001.

[19] C. Tudor and M. Tudor, "On approximation of solutions for stochastic delay equations," *Studii şi Cercetări Matematice*, vol. 39, no. 3, pp. 265–274, 1987.

[20] C. Tudor, "Approximation of delay stochastic equations with constant retardation by usual Itô equations," *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 34, no. 1, pp. 55–64, 1989.

[21] E. Platen and N. Bruti-Liberati, *Numerical Solution of Stochastic Differential Equations with Jumps in Finance*, vol. 64 of *Stochastic Modelling and Applied Probability*, Springer, Berlin, Germany, 2010.

[22] R. Mikulevičius and E. Platen, "Time discrete Taylor approximations for Itô processes with jump component," *Mathematische Nachrichten*, vol. 138, pp. 93–104, 1988.

[23] C. L. Evans, *An Introduction to Stochastic Differential Equations*, Lecture Notes, University of California, Department of Mathematics, Berkeley, Calif, USA, 2006.