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Research Article

Multiplicity of Positive Solutions for Semilinear Elliptic Systems

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We study the effect of the coefficient $h(x)$ of the critical nonlinearity on the number of positive solutions for semilinear elliptic systems. Under suitable assumptions for $f(x)$, $g(x)$, and $h(x)$, we should prove that for sufficiently small $\lambda, \mu > 0$, there are at least $k+1$ positive solutions of the semilinear elliptic systems $-\Delta u = \lambda f(x)|u|^{q-2}u + (\alpha/(\alpha+\beta))h(x)|u|^{\alpha-2}u|v|^{\beta}$, $-\Delta v = \mu g(x)|v|^{q-2}v + (\beta/(\alpha+\beta))h(x)|u|^{\alpha}|v|^{\beta-2}v$, where $0 \in \Omega \subset \mathbb{R}^N$ is a bounded domain, $\alpha > 1$, $\beta > 1$, and $N/(N-2) < q < 2 < \alpha+\beta = 2^*$ for $N > 4$.

1. Introduction and Main Results

For $N \geq 3$, $\alpha > 1$, $\beta > 1$, and $1 \leq q < 2 < \alpha+\beta = 2^* = 2N/(N-2)$, consider the semilinear elliptic systems

$$\begin{cases} -\Delta u = \lambda f(x)|u|^{q-2}u + \dfrac{\alpha}{\alpha+\beta} h(x)|u|^{\alpha-2}u|v|^{\beta} & \text{in } \Omega, \\ -\Delta v = \mu g(x)|v|^{q-2}v + \dfrac{\beta}{\alpha+\beta} h(x)|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \qquad (P_{\lambda,\mu})$$

where $\lambda, \mu > 0$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$.

Let f, g, and h satisfy the following conditions.

($H1$) f, g, and h are positive continuous functions in $\overline{\Omega}$ and $\max_{x\in\overline{\Omega}} h(x) = 1$.

($H2$) There exist k points $a_1, a_2, \ldots, a_k \in \Omega$ and some $\sigma \geq N-2$ such that $h(a_i)$ are strict maxima and satisfy

$$h(a_i) = \max_{x\in\overline{\Omega}} h(x) = 1 \quad \forall 1 \leq i \leq k \qquad (1)$$

and $h(x) = h(a_i) + O(|x-a_i|^{\sigma})$ as $x \to a_i$ uniformly in i.

Recent studies [1–10] have investigated the elliptic systems with subcritical or critical exponents and have proved the existence of a ground state solution or the existence of at least two positive solutions for these problems. For the case of $N > 4$, $\alpha > 1$, $\beta > 1$, and $2 < q < \alpha+\beta = 2^* = 2N/(N-2)$, Lin [11] constructs the k compact Palais-Smale sequences that are suitably localized in correspondence of k maximum points of h. Under assumptions ($H1$)-($H2$), she has showed that there are at least k positive solutions of the problem $(P_{\lambda,\mu})$ for sufficiently small $\lambda, \mu > 0$. In this paper, we study the problem $(P_{\lambda,\mu})$ and complement the results of [11] to the case $1 \leq q < 2$. Under assumptions ($H1$)-($H2$), we should prove that there exist at least $k+1$ positive solutions of the problem $(P_{\lambda,\mu})$ for sufficiently small $\lambda, \mu > 0$.

Let $E = H_0^1(\Omega) \times H_0^1(\Omega)$ be the space with the standard norm

$$\|(u,v)\|_E = \left(\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right) dx\right)^{1/2}. \qquad (2)$$

Associated with the problem $(P_{\lambda,\mu})$, we consider the C^1-functional $I_{\lambda,\mu}$, for $(u,v) \in E$,

$$I_{\lambda,\mu}(u,v) = \frac{1}{2}\|(u,v)\|_E^2 - \frac{1}{q}\int_\Omega \left(\lambda f(x)|u|^q + \mu g(x)|v|^q\right) dx - \frac{1}{2^*}\int_\Omega h(x)|u|^{\alpha}|v|^{\beta} dx. \qquad (3)$$

The weak solution $(u, v) \in E$ of the problem $(P_{\lambda,\mu})$ is the critical point of the functional $I_{\lambda,\mu}$; that is, $(u, v) \in E$ satisfies

$$\int_\Omega (\nabla u\nabla\varphi_1 + \nabla v\nabla\varphi_2)\,dx - \lambda\int_\Omega f(x)|u|^{q-2}u\varphi_1 dx - \mu\int_\Omega g(x)|v|^{q-2}v\varphi_2 dx - \frac{\alpha}{2^*}\int_\Omega h(x)|u|^{\alpha-2}u|v|^\beta\varphi_1 dx - \frac{\beta}{2^*}\int_\Omega h(x)|u|^\alpha|v|^{\beta-2}v\varphi_2 dx = 0 \tag{4}$$

for any $(\varphi_1, \varphi_2) \in E$.

Let $D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) \mid \nabla u \in (L^2(\mathbb{R}^N))^N\}$ with the norm $\|u\|^2 = \int_{\mathbb{R}^N}|\nabla u|^2dx$, and let S be the best Sobolev constant defined by

$$S = \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}}\frac{\int_{\mathbb{R}^N}|\nabla u|^2dx}{\left(\int_{\mathbb{R}^N}|u|^{2^*}dx\right)^{2/2^*}} \left(= \inf_{u\in H_0^1(\Omega)\setminus\{0\}}\frac{\int_\Omega|\nabla u|^2dx}{\left(\int_\Omega|u|^{2^*}dx\right)^{2/2^*}}\right) > 0, \tag{5}$$

and let

$$S_{\alpha,\beta} = \inf_{u,v\in H_0^1(\Omega)\setminus\{0\}}\frac{\int_\Omega\left(|\nabla u|^2 + |\nabla v|^2\right)dx}{\left(\int_\Omega|u|^\alpha|v|^\beta dx\right)^{2/(\alpha+\beta)}}; \tag{6}$$

then, by [1, Theorem 5], we have

$$S_{\alpha,\beta} = \left(\left(\frac{\alpha}{\beta}\right)^{\beta/(\alpha+\beta)} + \left(\frac{\beta}{\alpha}\right)^{\alpha/(\alpha+\beta)}\right)S, \tag{7}$$

where $\alpha + \beta = 2^*$.

Set

$$\Lambda_1 = \left(\frac{2-q}{2^*-q}\right)^{2/(2^*-2)}\left(\frac{(2^*-q)\gamma_\infty}{2^*-2}|\Omega|^{(2^*-q)/2^*}\right)^{-2/(2-q)} \times S^{N/2+q/(2-q)} > 0, \tag{8}$$

where $\gamma_\infty = \max\{|f|_{L^\infty(\Omega)}, |g|_{L^\infty(\Omega)}\}$.

The main results of this paper are given as follows.

Theorem 1. *Assume that $(H1)$ holds. If $\lambda, \mu > 0$ satisfy $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_1$, then there exists at least one positive ground state solution of the problem $(P_{\lambda,\mu})$.*

Theorem 2. *Under the assumptions $(H1)$-$(H2)$, and $N/(N-2) < q < 2$ and $N > 4$, there exists a positive number $\Lambda^* \in (0, \Lambda_1)$ such that for $\lambda, \mu > 0$ and $\lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda^*$, the problem $(P_{\lambda,\mu})$ has $k + 1$ positive solutions.*

This paper is organized as follows. In Section 2, we consider the Nehari manifold

$$\mathcal{N}_{\lambda,\mu} = \left\{(u, v) \in E \setminus \{0\} \mid \left\langle I'_{\lambda,\mu}(u, v), (u, v)\right\rangle = 0\right\}, \tag{9}$$

where

$$\left\langle I'_{\lambda,\mu}(u, v), (u, v)\right\rangle = \|(u, v)\|_E^2 - \int_\Omega\left(\lambda f(x)|u|^q + \mu g(x)|v|^q\right)dx - \int_\Omega h(x)|u|^\alpha|v|^\beta dx. \tag{10}$$

Note that $\mathcal{N}_{\lambda,\mu}$ contains all nontrivial weak solution of the problem $(P_{\lambda,\mu})$. Using the argument of Tarantello [12, 13], we split $\mathcal{N}_{\lambda,\mu}$ into two parts $\mathcal{N}^+_{\lambda,\mu}$ and $\mathcal{N}^-_{\lambda,\mu}$ for $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_1$. In Section 3, we prove Theorem 1. In Section 4, since $I_{\lambda,\mu}$ satisfies the $(PS)_\gamma$-condition for $\gamma \in (-\infty, (1/N)(S_{\alpha,\beta})^{N/2} - C_0(\lambda^{2/(2-q)} + \mu^{2/(2-q)}))$, for sufficiently small λ, μ, and some restriction on q and N, we construct the k compact Palais-Smale sequences which are suitably localized in correspondence with the k maximum points of h and which converge to distinct solutions of the problem $(P_{\lambda,\mu})$ belonging to $\mathcal{N}^-_{\lambda,\mu}$. Hence, we prove Theorem 2 (one is the ground state solution belonging to $\mathcal{N}^+_{\lambda,\mu}$ and the others are in $\mathcal{N}^-_{\lambda,\mu}$).

2. Nehari Manifold

Throughout this paper, $(H1)$ will be assumed. First, we give some notations.

Notations. We make use of the following notations.

- $L^p(\Omega)$, $1 \le p \le \infty$, denote Lebesgue spaces; the norm L^p is denoted by $|\cdot|_{L^p(\Omega)}$ for $1 \le p \le \infty$.
- $E = [H_0^1(\Omega)]^2$, endowed with norm $\|z\|_E^2 = \|(u, v)\|_E^2 = |\nabla u|_2^2 + |\nabla v|_2^2$.
- The dual space of a Banach space E will be denoted by E^{-1}.
- $|z| = |(u, v)| = (|u|, |v|)$ and $tz = t(u, v) = (tu, tv)$ for all $z \in E$ and $t \in \mathbb{R}$.
- $z = (u, v)$ is said to be nonnegative in Ω if $u \ge 0$ and $v \ge 0$ in Ω.
- $z = (u, v)$ is said to be positive in Ω if $u > 0$ and $v > 0$ in Ω.
- $|\Omega|$ is the Lebesgue measure of Ω.
- $B_r(a) = \{x \in \mathbb{R}^N \mid |x - a| < r\}$ is a ball in $\mathbb{R}^N$.
- $O(\varepsilon^t)$ denotes $|O(\varepsilon^t)|/\varepsilon^t \le C$ as $\varepsilon \to 0$ for $t \ge 0$.
- $O_1(\varepsilon^t)$ means that there exist the constants $C_1, C_2 > 0$ such that $C_1\varepsilon^t \le O_1(\varepsilon^t) \le C_2\varepsilon^t$ as ε is small.
- $o_n(1)$ denotes $o_n(1) \to 0$ as $n \to \infty$.
- $\gamma_\infty = \max\{|f|_{L^\infty(\Omega)}, |g|_{L^\infty(\Omega)}\}$.
- C, C_i will denote various positive constants, the exact values of which are not important.

Let $K_{\lambda,\mu} : E \to \mathbb{R}$ be the functional defined by

$$K_{\lambda,\mu}(z) = \int_\Omega (\lambda f(x)|u|^q + \mu g(x)|v|^q)\,dx \quad \forall z = (u,v) \in E. \tag{11}$$

We know that $I_{\lambda,\mu}$ is not bounded below on E. From the following lemma, we have that $I_{\lambda,\mu}$ is bounded from below on the Nehari manifold $\mathcal{N}_{\lambda,\mu}$ defined in (9).

Lemma 3. *The energy functional $I_{\lambda,\mu}$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}$.*

Proof. If $z = (u,v) \in \mathcal{N}_{\lambda,\mu}$, then by (10), the Hölder inequality, and the Sobolev embedding theorem, we get

$$I_{\lambda,\mu}(z) = \frac{2^*-2}{2^*2}\|z\|_E^2 - \frac{2^*-q}{2^*q}K_{\lambda,\mu}(z) \tag{12}$$

$$\geq \frac{1}{N}\|z\|_E^2 - \frac{2^*-q}{2^*q}\gamma_\infty S^{-q/2}|\Omega|^{(2^*-q)/2^*} \times \left(\lambda^{2/(2-q)} + \mu^{2/(2-q)}\right)^{(2-q)/2}\|z\|_E^q. \tag{13}$$

Hence, we have that $I_{\lambda,\mu}$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}$. □

Define

$$\Phi_{\lambda,\mu}(z) = \left\langle I'_{\lambda,\mu}(z), z\right\rangle. \tag{14}$$

Then, for $z \in \mathcal{N}_{\lambda,\mu}$,

$$\left\langle \Phi'_{\lambda,\mu}(z), z\right\rangle = 2\|z\|_E^2 - qK_{\lambda,\mu}(z) - 2^*\int_\Omega h(x)|u|^\alpha v^\beta dx \tag{15}$$

$$= (2-q)\|z\|_E^2 - (2^*-q)\int_\Omega h(x)|u|^\alpha v^\beta dx \tag{16}$$

$$= (2^*-q)K_{\lambda,\mu}(z) - (2^*-2)\|z\|_E^2. \tag{17}$$

We apply the method in [12]; let

$$\mathcal{N}^+_{\lambda,\mu} = \left\{z \in \mathcal{N}_{\lambda,\mu} \mid \left\langle \Phi'_{\lambda,\mu}(z), z\right\rangle > 0\right\},$$
$$\mathcal{N}^0_{\lambda,\mu} = \left\{z \in \mathcal{N}_{\lambda,\mu} \mid \left\langle \Phi'_{\lambda,\mu}(z), z\right\rangle = 0\right\}, \tag{18}$$
$$\mathcal{N}^-_{\lambda,\mu} = \left\{z \in \mathcal{N}_{\lambda,\mu} \mid \left\langle \Phi'_{\lambda,\mu}(z), z\right\rangle < 0\right\}.$$

By using equality (17), we get that $K_{\lambda,\mu}(z) > 0$ for $z \in \mathcal{N}^+_{\lambda,\mu}$. Moreover, we have the following results.

Lemma 4. *Let Λ_1 be a constant defined as in (8). If $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_1$, then $\mathcal{N}^0_{\lambda,\mu} = \emptyset$.*

Proof. Assuming the contrary, there exist $\lambda, \mu > 0$ with $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_1$ such that $\mathcal{N}^0_{\lambda,\mu} \neq \emptyset$. Then, by (16) and (17), for $u \in \mathcal{N}^0_{\lambda,\mu}$, we have

$$\|z\|_E^2 = \frac{2^*-q}{2-q}\int_\Omega h(x)|u|^\alpha|v|^\beta dx = \frac{2^*-q}{2^*-2}K_{\lambda,\mu}(z). \tag{19}$$

Using $(H1)$ and both the Hölder and the Sobolev inequalities, we get

$$\|z\|_E \geq \left(\frac{2-q}{2^*-q}S^{2^*/2}\right)^{1/(2^*-2)}, \tag{20}$$

$$\|z\|_E \leq \left(\frac{2^*-q}{2^*-2}S^{-q/2}|\Omega|^{(2^*-q)/2^*}\gamma_\infty\right)^{1/(2-q)} \times \left(\lambda^{2/(2-q)} + \mu^{2/(2-q)}\right)^{1/2}. \tag{21}$$

This implies

$$\lambda^{2/(2-q)} + \mu^{2/(2-q)} \geq \left(\frac{2-q}{2^*-q}\right)^{2/(2^*-2)}\left(\frac{(2^*-q)\gamma_\infty}{2^*-2}|\Omega|^{(2^*-q)/2^*}\right)^{-2/(2-q)} \times S^{N/2+q/(2-q)} = \Lambda_1, \tag{22}$$

which is a contradiction. □

For each $z \in E$ with $\int_\Omega h(x)|u|^\alpha|v|^\beta dx > 0$, we write

$$t_{\max} = \left(\frac{(2-q)\|z\|_E^2}{(2^*-q)\int_\Omega h(x)|u|^\alpha|v|^\beta dx}\right)^{1/(2^*-2)} > 0. \tag{23}$$

Then, the following lemma holds.

Lemma 5. *Suppose that $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_1$, and $z \in E$ with $\int_\Omega h(x)|u|^\alpha|v|^\beta dx > 0$. Then, there exist unique $0 < t^+ < t_{\max} < t^-$ such that $t^+z \in \mathcal{N}^+_{\lambda,\mu}$, $t^-z \in \mathcal{N}^-_{\lambda,\mu}$ and*

$$I_{\lambda,\mu}(t^+z) = \inf_{0\leq t\leq t_{\max}} I_{\lambda,\mu}(tz);$$
$$I_{\lambda,\mu}(t^-z) = \sup_{t\geq 0} I_{\lambda,\mu}(tz). \tag{24}$$

Proof. This is similar to the proof of Hsu [14, Lemma 2.7]. □

Applying Lemma 4 ($\mathcal{N}^0_{\lambda,\mu} = \emptyset$ for $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_1$), we write $\mathcal{N}_{\lambda,\mu} = \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^-_{\lambda,\mu}$ and define

$$\theta_{\lambda,\mu} = \inf_{z\in\mathcal{N}_{\lambda,\mu}} I_{\lambda,\mu}(z);$$
$$\theta^+_{\lambda,\mu} = \inf_{z\in\mathcal{N}^+_{\lambda,\mu}} I_{\lambda,\mu}(z); \tag{25}$$
$$\theta^-_{\lambda,\mu} = \inf_{z\in\mathcal{N}^-_{\lambda,\mu}} I_{\lambda,\mu}(z).$$

The following lemma shows that the minimizers on $\mathcal{N}_{\lambda,\mu}$ are usual critical points for $I_{\lambda,\mu}$.

Lemma 6. *For the case when $\lambda \in (0, \Lambda_1)$, if z_0 is a local minimizer for $I_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$, then $I'_{\lambda,\mu}(z_0) = 0$ in E^{-1}.*

Proof. See Brown and Zhang [15, theorem 2.3]. □

Lemma 7. *(i) If* $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_1$ *and* $z = (u, v) \in \mathcal{N}^+_{\lambda,\mu}$*, then one has*

$$K_{\lambda,\mu}(z) > 0, \qquad I_{\lambda,\mu}(z) < 0. \tag{26}$$

In particular, $\theta_{\lambda,\mu} \le \theta^+_{\lambda,\mu} < 0$.

(ii) If $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < (q/2)^{2/(2-q)}\Lambda_1$ *and* $z = (u, v) \in \mathcal{N}^-_{\lambda,\mu}$*, then one has* $u \not\equiv 0$, $v \not\equiv 0$ *in* Ω,

$$\|z\|_E > \left(\frac{2-q}{2^*-q}\right)^{1/(2^*-2)} S^{N/4}, \tag{27}$$

and $\theta^-_{\lambda,\mu} > d_0$ *for some positive constant* $d_0 = d_0(\lambda, \mu, q, N, S, \gamma_\infty, |\Omega|)$.

Proof. (i) Let $z = (u, v) \in \mathcal{N}^+_{\lambda,\mu}$. By (16) and (17), we have

$$\begin{gathered} K_{\lambda,\mu}(z) > \frac{2^*-2}{2^*-q}\|z\|_E^2 > 0, \\ \frac{2-q}{2^*-q}\|z\|_E^2 > \int_\Omega h(x)|u|^\alpha |v|^\beta dx. \end{gathered} \tag{28}$$

Then,

$$\begin{aligned} I_{\lambda,\mu}(z) &= \left(\frac{1}{2}-\frac{1}{q}\right)\|z\|_E^2 + \left(\frac{1}{q}-\frac{1}{2^*}\right)\int_\Omega h(x)|u|^\alpha|v|^\beta dx \\ &< \left[\left(\frac{1}{2}-\frac{1}{q}\right) + \left(\frac{1}{q}-\frac{1}{2^*}\right)\frac{2-q}{2^*-q}\right]\|z\|_E^2 \\ &= -\frac{2-q}{qN}\|z\|_E^2 < 0. \end{aligned} \tag{29}$$

By the definition of $\theta_{\lambda,\mu}$, $\theta^+_{\lambda,\mu}$, we deduce that $\theta_{\lambda,\mu} \le \theta^+_{\lambda,\mu} < 0$.

(ii) Let $z \in \mathcal{N}^-_{\lambda,\mu}$; by (16) and the Hölder and the Sobolev inequalities, we get

$$\frac{2-q}{2^*-q}\|z\|_E^2 < \int_\Omega h(x)|u|^\alpha|v|^\beta dx \le S^{-2^*/2}\|z\|_E^{2^*}. \tag{30}$$

This implies

$$\begin{aligned} \int_\Omega h(x)|u|^\alpha|v|^\beta dx &> \frac{2^*-q}{2-q}\|z\|_E \\ &> \left(\frac{2-q}{2^*-q}\right)^{\frac{1}{2^*-2}} S^{\frac{N}{4}} \quad \forall z \in \mathcal{N}^-_{\lambda,\mu}. \end{aligned} \tag{31}$$

By (13) and (31), we obtain that $u \not\equiv 0$, $v \not\equiv 0$ in Ω, and

$$\begin{aligned} &I_{\lambda,\mu}(z) \\ &\ge \|z\|_E^q\left[\frac{1}{N}\|z\|_E^{2-q} - \left(\frac{2^*-q}{2^*q}\right)\gamma_\infty S^{-q/2}|\Omega|^{(2^*-q)/2^*} \times \left(\lambda^{2/(2-q)}+\mu^{2/(2-q)}\right)^{(2-q)/2}\right] \\ &> \left(\frac{2-q}{2^*-q}\right)^{q/(2^*-2)} S^{qN/4} \\ &\quad \times \left[\frac{1}{N}\left(\frac{2-q}{2^*-q}\right)^{(2-q)/(2^*-2)} S^{(2-q)N/4} - \left(\frac{2^*-q}{2^*q}\right)\gamma_\infty S^{-q/2}|\Omega|^{(2^*-q)/2^*} \times\left(\lambda^{2/(2-q)}+\mu^{2/(2-q)}\right)^{(2-q)/2}\right]. \end{aligned} \tag{32}$$

Thus, if $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < (q/2)^{2/(2-q)}\Lambda_1$, for all $z \in \mathcal{N}^-_{\lambda,\mu}$, then

$$I_{\lambda,\mu}(z) \ge d_0(\lambda, \mu, q, N, S, \gamma_\infty, |\Omega|) > 0. \tag{33}$$

□

3. Existence of a Ground State Solution

First of all, we define the Palais-Smale (denote by (PS)) sequences and (PS)-condition in E for $I_{\lambda,\mu}$ as follows.

Definition 8. (i) For $\gamma \in \mathbb{R}$, a sequence $\{z_n\}$ is a $(\mathrm{PS})_\gamma$-sequence in E for $I_{\lambda,\mu}$ if $I_{\lambda,\mu}(z_n) = \gamma + o_n(1)$ and $I'_{\lambda,\mu}(z_n) = o_n(1)$ strongly in E^{-1} as $n \to \infty$.

(ii) $I_{\lambda,\mu}$ satisfies the $(\mathrm{PS})_\gamma$-condition in E if any $(\mathrm{PS})_\gamma$-sequence $\{z_n\}$ in E for $I_{\lambda,\mu}$ contains a convergent subsequence.

Proof of Theorem 1. Using the same argument as in Wu [16, Proposition 9] or Hsu [14, Proposition 3.3], there exists a minimizing sequence $\{z_n\}$ for $I_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$ such that

$$I_{\lambda,\mu}(z_n) = \theta_{\lambda,\mu} + o_n(1), \qquad I'_{\lambda,\mu}(z_n) = o_n(1) \text{ in } E^{-1}. \tag{34}$$

Since $I_{\lambda,\mu}$ is coercive on $\mathcal{N}_{\lambda,\mu}$ (see Lemma 3), we get that $\{z_n\}$ is bounded in E. Then, there exist a subsequence $\{z_n = (u_n, v_n)\}$ and $z^1_{\lambda,\mu} = (u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in E$ such that

$$\begin{gathered} u_n \rightharpoonup u^1_{\lambda,\mu}, \quad v_n \rightharpoonup v^1_{\lambda,\mu} \quad \text{weakly in } H^1_0(\Omega), \\ u_n \longrightarrow u^1_{\lambda,\mu}, \quad v_n \longrightarrow v^1_{\lambda,\mu} \quad \text{almost everywhere in } \Omega, \\ u_n \longrightarrow u^1_{\lambda,\mu}, \quad v_n \longrightarrow v^1_{\lambda,\mu} \quad \text{strongly in } L^s(\Omega)\ \forall 1 \le s < 2^*. \end{gathered} \tag{35}$$

This implies

$$K_{\lambda,\mu}(z_n) = K_{\lambda,\mu}(z^1_{\lambda,\mu}) + o_n(1) \quad \text{as } n \longrightarrow \infty. \tag{36}$$

First, we claim that $z^1_{\lambda,\mu}$ is a nontrivial solution of $(P_{\lambda,\mu})$. By (34) and (35), it is easy to verify that $z^1_{\lambda,\mu}$ is a weak solution of $(P_{\lambda,\mu})$. From $z_n \in \mathcal{N}_{\lambda,\mu}$ and (12), we deduce that

$$K_{\lambda,\mu}(z_n) = \frac{q(2^*-2)}{2(2^*-q)}\|z_n\|^2_E - \frac{2^*q}{2^*-q}I_{\lambda,\mu}(z_n). \tag{37}$$

Let $n \to \infty$ in (37); by (34), (36), and $\theta_{\lambda,\mu} < 0$, we get

$$K_{\lambda,\mu}(z^1_{\lambda,\mu}) \geq -\frac{2^*q}{2^*-q}\theta_{\lambda,\mu} > 0. \tag{38}$$

Thus, $z^1_{\lambda,\mu} \in \mathcal{N}_{\lambda,\mu}$ is a nontrivial solution of $(P_{\lambda,\mu})$. Now, we prove that $z_n \to z^1_{\lambda,\mu}$ strongly in E and $I_{\lambda,\mu}(z^1_{\lambda,\mu}) = \theta_{\lambda,\mu}$. By (37), if $z \in \mathcal{N}_{\lambda,\mu}$, then

$$I_{\lambda,\mu}(z) = \frac{1}{N}\|z\|^2_E - \frac{2^*-q}{2^*q}K_{\lambda,\mu}(z). \tag{39}$$

In order to prove that $I_{\lambda,\mu}(z^1_{\lambda,\mu}) = \theta_{\lambda,\mu}$, it suffices to recall that $z^1_{\lambda,\mu} \in \mathcal{N}_{\lambda,\mu}$, by (39) and applying Fatou's lemma to get

$$\begin{aligned}\theta_{\lambda,\mu} \leq I_{\lambda,\mu}(z^1_{\lambda,\mu}) &= \frac{1}{N}\|z^1_{\lambda,\mu}\|^2_E - \frac{2^*-q}{2^*q}K_{\lambda,\mu}(z^1_{\lambda,\mu}) \\ &\leq \liminf_{n\to\infty}\left(\frac{1}{N}\|z_n\|^2_E - \frac{2^*-q}{2^*q}K_{\lambda,\mu}(z_n)\right) \\ &\leq \liminf_{n\to\infty} I_{\lambda,\mu}(z_n) = \theta_{\lambda,\mu}.\end{aligned} \tag{40}$$

This implies that $I_{\lambda,\mu}(z^1_{\lambda,\mu}) = \theta_{\lambda,\mu}$ and $\lim_{n\to\infty}\|z_n\|^2_E = \|z^1_{\lambda,\mu}\|^2_E$. Let $\widetilde{z}_n = z_n - z^1_{\lambda,\mu}$; then Brézis-Lieb lemma [17] implies

$$\|\widetilde{z}_n\|^2_E = \|z_n\|^2_E - \|z^1_{\lambda,\mu}\|^2_E. \tag{41}$$

Therefore, $z_n \to z^1_{\lambda,\mu}$ strongly in E. Since $I_{\lambda,\mu}(z^1_{\lambda,\mu}) = I_{\lambda,\mu}(|z^1_{\lambda,\mu}|) = \theta_{\lambda,\mu}$ and $|z^1_{\lambda,\mu}| \in \mathcal{N}^+_{\lambda,\mu}$, by Lemma 6 we may assume that $z^1_{\lambda,\mu}$ is a nontrivial nonnegative solution of $(P_{\lambda,\mu})$. By an argument of Hsu [18, Lemma 4.2], we can deduce that $u^1_{\lambda,\mu} \not\equiv 0$ and $v^1_{\lambda,\mu} \not\equiv 0$ in Ω. Finally, from the maximum principle [19], we deduce that $z^1_{\lambda,\mu}$ is positive in Ω. □

Remark 9. $z^1_{\lambda,\mu} \in \mathcal{N}^+_{\lambda,\mu}$ and $I_{\lambda,\mu}(z^1_{\lambda,\mu}) = \theta_{\lambda,\mu} = \theta^+_{\lambda,\mu}$.

Proof. We claim that $z^1_{\lambda,\mu} \in \mathcal{N}^+_{\lambda,\mu}$. On the contrary, assume that $z^1_{\lambda,\mu} \in \mathcal{N}^-_{\lambda,\mu}$ ($\mathcal{N}^0_{\lambda,\mu} = \emptyset$ for $\lambda^{2/(2-q)} + \mu^{2/(2-q)} \in (0, \Lambda_1)$); then by Lemma 5, there exist unique t^+_1 and t^-_1 such that $t^+_1 z^1_{\lambda,\mu} \in \mathcal{N}^+_{\lambda,\mu}$ and $t^-_1 z^1_{\lambda,\mu} \in \mathcal{N}^-_{\lambda,\mu}$. In particular, we have $t^+_1 < t^-_1 = 1$. Since

$$\frac{d}{dt}I_{\lambda,\mu}(t^+_1 z^1_{\lambda,\mu}) = 0, \qquad \frac{d^2}{dt^2}I_{\lambda,\mu}(t^+_1 z^1_{\lambda,\mu}) > 0, \tag{42}$$

there exists $t^+_1 < \bar{t} \leq t^-_1$ such that $I_{\lambda,\mu}(t^+_1 z^1_{\lambda,\mu}) < I_{\lambda,\mu}(\bar{t} z^1_{\lambda,\mu})$. By Lemma 5,

$$I_{\lambda,\mu}(t^+_1 z^1_{\lambda,\mu}) < I_{\lambda,\mu}(\bar{t} z^1_{\lambda,\mu}) \leq I_{\lambda,\mu}(t^-_1 z^1_{\lambda,\mu}) = I_{\lambda,\mu}(z^1_{\lambda,\mu}), \tag{43}$$

which is a contradiction. Hence, $z^1_{\lambda,\mu} \in \mathcal{N}^+_{\lambda,\mu}$ and $I_{\lambda,\mu}(z^1_{\lambda,\mu}) = \theta_{\lambda,\mu} = \theta^+_{\lambda,\mu}$. □

4. Existence of $k+1$ Solutions

Throughout this section, $(H1)$-$(H2)$ will be assumed. First of all, we want to show that $I_{\lambda,\mu}$ satisfies the $(PS)_\gamma$-condition in E for $\gamma \in (-\infty, (1/N)(S_{\alpha,\beta})^{N/2} - C_0(\lambda^{2/(2-q)} + \mu^{2/(2-q)}))$, where C_0 is defined in the following lemma.

Lemma 10. *If $\{z_n\} \subset E$ is a $(PS)_\gamma$-sequence for $I_{\lambda,\mu}$ with $z_n \rightharpoonup z$ weakly in E, then $I'_{\lambda,\mu}(z) = 0$ and there exists a constant $C_0 = C_0(q, N, S, \gamma_\infty, |\Omega|) > 0$ such that $I_{\lambda,\mu}(z) \geq -C_0(\lambda^{2/(2-q)} + \mu^{2/(2-q)})$.*

Proof. Let $z_n = (u_n, v_n)$ and $z = (u, v)$. If $\{z_n\}$ is a $(PS)_\gamma$-sequence for $I_{\lambda,\mu}$ with $z_n \rightharpoonup z$ weakly in E, it is easy to check that $I'_{\lambda,\mu}(z) = 0$ in E^{-1}. Then, we get $\langle I'_{\lambda,\mu}(z), z\rangle = 0$; that is, $\int_\Omega h(x)|u|^\alpha|v|^\beta dx = \|z\|^2_E - K_{\lambda,\mu}(z)$. Thus, by (13), the Hölder, the Young, and the Sobolev inequalities, we have

$$\begin{aligned}I_{\lambda,\mu}(z) &\geq \frac{1}{N}\|z\|^2_E - \frac{2^*-q}{2^*q}\gamma_\infty S^{-q/2}|\Omega|^{(2^*-q)/2^*} \\ &\quad \times \left(\lambda^{2/(2-q)} + \mu^{2/(2-q)}\right)^{(2-q)/2}\|z\|^q_E \\ &\geq \frac{1}{N}\|z\|^2_E - \frac{1}{N}\|z\|^2_E - C_0\left(\lambda^{2/(2-q)} + \mu^{2/(2-q)}\right) \\ &= -C_0\left(\lambda^{2/(2-q)} + \mu^{2/(2-q)}\right),\end{aligned} \tag{44}$$

where $C_0 = C_0(q, N, S, \gamma_\infty, |\Omega|) > 0$. □

Lemma 11. *If $\{z_n\} \subset E$ is a $(PS)_\gamma$-sequence for $I_{\lambda,\mu}$, then $\{z_n\}$ is bounded in E.*

Proof. See Hsu and Lin [8, Lemma 2.3]. □

Recall that

$$S_{\alpha,\beta} = \inf_{u,v\in H^1_0(\Omega)\setminus\{0\}} \frac{\|(u,v)\|^2_E}{\left(\int_\Omega |u|^\alpha|v|^\beta dx\right)^{2/(\alpha+\beta)}}, \tag{45}$$

and let

$$c^* = \frac{1}{N}\left(S_{\alpha,\beta}\right)^{N/2} - C_0\left(\lambda^{2/(2-q)} + \mu^{2/(2-q)}\right), \tag{46}$$

where $C_0 > 0$ is given in Lemma 10.

Lemma 12. *$I_{\lambda,\mu}$ satisfies the $(PS)_\gamma$-condition in E for $\gamma \in (-\infty, c^*)$.*

Proof. Let $\{z_n\} \subset E$ be a $(\mathrm{PS})_\gamma$-sequence for $I_{\lambda,\mu}$ with $\gamma \in (-\infty, c^*)$. Write $z_n = (u_n, v_n)$. We know from Lemma 11 that $\{z_n\}$ is bounded in E, and then $z_n \rightharpoonup z = (u, v)$ weakly up to a subsequence; z is a critical point of $I_{\lambda,\mu}$. Furthermore, we may assume that $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ weakly in $H_0^1(\Omega)$ and $u_n \to u$, $v_n \to v$ strongly in $L^s(\Omega)$ for all $1 \le s < 2^*$, and $u_n \to u$, $v_n \to v$ a.e. on Ω. Hence, we have that $I'_{\lambda,\mu}(z) = 0$ and

$$K_{\lambda,\mu}(z_n) = K_{\lambda,\mu}(z) + o_n(1). \tag{47}$$

Let $\widetilde{u}_n = u_n - u$, $\widetilde{v}_n = v_n - v$ and $\widetilde{z}_n = (\widetilde{u}_n, \widetilde{v}_n)$. Then, we obtain

$$\|\widetilde{z}_n\|_E^2 = \|z_n\|_E^2 - \|z\|_E^2 + o_n(1), \tag{48}$$

and by an argument of Han [20, Lemma 2.1],

$$\begin{aligned}\int_\Omega & h(x)|\widetilde{u}_n|^\alpha|\widetilde{v}_n|^\beta dx \\ &= \int_\Omega h(x)|u_n|^\alpha|v_n|^\beta dx \\ &\quad - \int_\Omega h(x)|u|^\alpha|v|^\beta dx + o_n(1).\end{aligned} \tag{49}$$

Since $I_{\lambda,\mu}(z_n) = \gamma + o_n(1)$, $I'_{\lambda,\mu}(z_n) = o_n(1)$ in E^{-1} and (47)–(49), we deduce that

$$\frac{1}{2}\|\widetilde{z}_n\|_E^2 - \frac{1}{2^*}\int_\Omega h(x)|\widetilde{u}_n|^\alpha|\widetilde{v}_n|^\beta dx = \gamma - I_{\lambda,\mu}(z) + o_n(1), \tag{50}$$

$$\|\widetilde{z}_n\|_E^2 - \int_\Omega h(x)|\widetilde{u}_n|^\alpha|\widetilde{v}_n|^\beta dx = o_n(1). \tag{51}$$

Hence, we may assume that

$$\|\widetilde{z}_n\|_E^2 \longrightarrow l, \qquad \int_\Omega h(x)|\widetilde{u}_n|^\alpha|\widetilde{v}_n|^\beta dx \longrightarrow l. \tag{52}$$

Assume that $l \neq 0$; by the definition of $S_{\alpha,\beta}$, $|h|_{L^\infty(\Omega)} = 1$ and (52), we obtain

$$\begin{aligned}S_{\alpha,\beta}l^{2/2^*} &= S_{\alpha,\beta}\lim_{n\to\infty}\left(\int_\Omega h(x)|\widetilde{u}_n|^\alpha|\widetilde{v}_n|^\beta dx\right)^{2/2^*} \\ &\le |h|_{L^\infty(\Omega)}^{2/2^*}\lim_{n\to\infty}\|\widetilde{z}_n\|^2 = l,\end{aligned} \tag{53}$$

which implies that $l \ge (S_{\alpha,\beta})^{N/2}$. In addition, from Lemma 10, (50), and (52), we get

$$\begin{aligned}\gamma &= \left(\frac{1}{2} - \frac{1}{2^*}\right)l + I_{\lambda,\mu}(z) \\ &\ge \frac{1}{N}\left(S_{\alpha,\beta}\right)^{N/2} - C_0\left(\lambda^{2/(2-q)} + \mu^{2/(2-q)}\right),\end{aligned} \tag{54}$$

which is a contradiction. Hence, $l = 0$; that is, $z_n \to z$ strongly in E. □

From assumption $(H2)$, we can choose $r_0 \in (0, 1)$ such that

$$\overline{B_{r_0}(a_i)} \cap \overline{B_{r_0}(a_j)} = \emptyset \quad \text{for } i \neq j,\ 1 \le i,\ j \le k, \tag{55}$$

and $\cup_{i=1}^k \overline{B_{r_0}(a_i)} \subset \Omega$, where $\overline{B_{r_0}(a_i)} = \{x \in \mathbb{R}^N \mid |x - a_i| \le r_0\}$ and $h(a_i) = |h|_\infty = 1$ for $1 \le i \le k$.

Define

$$Q_i(z) = \frac{\int_\Omega \psi_i(x)\left(|\nabla u|^2 + |\nabla v|^2\right)dx}{\int_\Omega\left(|\nabla u|^2 + |\nabla v|^2\right)dx}, \quad z = (u, v) \in E \setminus \{0\}, \tag{56}$$

where $\psi_i(x) = \min\{1, |x - a_i|\}$, $i = 1, 2 \dots, k$.

Then, we have the following separation result.

Lemma 13. *If $Q_i(z) \le r_0/3$ and $Q_j(z) \le r_0/3$ for $z \in E \setminus \{0\}$, then $i = j$.*

Proof. For any $z \in E \setminus \{0\}$ satisfying $Q_i(z) \le r_0/3$ $(1 \le i \le k)$, we get

$$\begin{aligned}\frac{r_0}{3}\|z\|_E^2 &\ge \int_\Omega \psi_i(x)\left(|\nabla u|^2 + |\nabla v|^2\right)dx \\ &\ge \int_{\Omega\setminus B_{r_0}(a_i)} \psi_i(x)\left(|\nabla u|^2 + |\nabla v|^2\right)dx \\ &\ge r_0\int_{\Omega\setminus B_{r_0}(a_i)}\left(|\nabla u|^2 + |\nabla v|^2\right)dx,\end{aligned} \tag{57}$$

which implies that

$$\|z\|_E^2 \ge 3\int_{\Omega\setminus B_{r_0}(a_i)}\left(|\nabla u|^2 + |\nabla v|^2\right)dx, \quad 1 \le i \le k. \tag{58}$$

Hence, from (58), we obtain

$$\begin{aligned}2\|z\|_E^2 &\ge 3\left(\int_{\Omega\setminus B_{r_0}(a_i)}\left(|\nabla u|^2 + |\nabla v|^2\right)dx\right. \\ &\qquad \left.+ \int_{\Omega\setminus B_{r_0}(a_j)}\left(|\nabla u|^2 + |\nabla v|^2\right)dx\right) \\ &\ge 3\|z\|_E^2 \quad \text{if } i \neq j,\end{aligned} \tag{59}$$

which is a contradiction. □

For $i = 1, 2, \dots, k$, we set

$$\begin{aligned}\mathcal{N}_{\lambda,\mu}^i &= \left\{u \in \mathcal{N}_{\lambda,\mu}^- \mid Q_i(z) < \frac{r_0}{3}\right\}, \\ \partial\mathcal{N}_{\lambda,\mu}^i &= \left\{u \in \mathcal{N}_{\lambda,\mu}^- \mid Q_i(z) = \frac{r_0}{3}\right\},\end{aligned} \tag{60}$$

and define

$$\theta_{\lambda,\mu}^i = \inf_{\mathcal{N}_{\lambda,\mu}^i} I_{\lambda,\mu}(z), \qquad \widetilde{\theta}_{\lambda,\mu}^i = \inf_{\partial\mathcal{N}_{\lambda,\mu}^i} I_{\lambda,\mu}(z). \tag{61}$$

Recall that the best Sobolev constant S is defined as

$$S = \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}|\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N}|u|^{2^*}dx\right)^{2/2^*}}. \tag{62}$$

It is well known that

$$U(x) = \frac{[N(N-2)]^{(N-2)/4}}{\left[1+|x|^2\right]^{(N-2)/2}} \tag{63}$$

is a minimizer of S, and $|\nabla U|^2_{L^2(\mathbb{R}^N)} = |U|^{2^*}_{L^{2^*}(\mathbb{R}^N)} = S^{N/2}$. Fix a maximum point a_i of h ($1 \le i \le k$). Let $\eta_i \in C_0^\infty(\Omega)$ be a cut-off function such that $0 \le \eta_i \le 1$, $|\nabla \eta_i| \le C$, and $\eta_i(x) = 1$ for $|x-a_i| < r_0/2$, $\eta_i(x) = 0$ for $|x - a_i| > r_0$. We define

$$u^i_\varepsilon(x) = \varepsilon^{(2-N)/2}\eta_i(x)\,U\left(\frac{x-a_i}{\varepsilon}\right) = \frac{c_1\varepsilon^{(N-2)/2}\eta_i(x)}{\left[\varepsilon^2 + \left|x-a_i\right|^2\right]^{(N-2)/2}}, \tag{64}$$

where $c_1 = [N(N-2)]^{(N-2)/4}$ and $\varepsilon > 0$.

From now on, we assume that $N/(N-2) < q < 2$ and $N > 4$.

Lemma 14. *There exist $\varepsilon_0 > 0$, $\Lambda_2 \in (0, (q/2)^{2/(2-q)}\Lambda_1)$, such that for $\varepsilon \in (0, \varepsilon_0)$ and $\lambda^{2/(2-q)} + \mu^{2/(2-q)} \in (0, \Lambda_2)$, one has*

$$\sup_{t\ge 0} I_{\lambda,\mu}\left(t\sqrt{\alpha}u^i_\varepsilon, t\sqrt{\beta}u^i_\varepsilon\right) < c^* \quad \text{uniformly in } i, \tag{65}$$

where c^* is the positive constant given in Lemma 12.

In particular, $0 < \theta^-_{\lambda,\mu} \le \theta^i_{\lambda,\mu} < c^*$ for all $1 \le i \le k$.

Proof. It is well known that (or see Brézis and Nirenberg [21], Cheng and Ma [22, Lemma 3.2], Struwe [23], and Willem [24, Lemma 1.46]) as $\varepsilon \to 0^+$,

$$\left|u^i_\varepsilon\right|^2_{L^{2^*}(\Omega)} = |U|^2_{L^{2^*}(\mathbb{R}^N)} + O\left(\varepsilon^{N-2}\right), \tag{66}$$

$$\left|\nabla u^i_\varepsilon\right|^2_{L^2(\Omega)} |\nabla U|^2_{L^2(\mathbb{R}^N)} + O\left(\varepsilon^{N-2}\right). \tag{67}$$

For $N/(N-2) < q < 2$, $N > 4$ and $\varepsilon < r_0/2$,

$$\begin{aligned} \left|u^i_\varepsilon\right|^q_{L^q(\Omega)} &= \int_{B_{r_0/2}(a_i)} \left[\varepsilon^{(2-N)/2} U\left(\frac{x-a_i}{\varepsilon}\right)\right]^q dx + O\left(\varepsilon^{N-2}\right) \\ &\ge C\varepsilon^\theta + O\left(\varepsilon^{N-2}\right), \quad \text{where } \theta = N - \frac{(N-2)q}{2}. \end{aligned} \tag{68}$$

Set $\overline{z}^i_\varepsilon = (\sqrt{\alpha}u^i_\varepsilon, \sqrt{\beta}u^i_\varepsilon)$. By Lemma 5, there exists $t^i_\varepsilon > 0$ such that $z^i_\varepsilon = t^i_\varepsilon \overline{z}^i_\varepsilon \in \mathcal{N}^-_{\lambda,\mu}$ for $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_1$. Furthermore,

$$\begin{aligned} Q_i\left(z^i_\varepsilon\right) &= \frac{\int_\Omega \psi_i(x)\left|\nabla u^i_\varepsilon\right|^2 dx}{\int_\Omega \left|\nabla u^i_\varepsilon\right|^2 dx} \\ &= \frac{\int_{(\Omega-a_i)/\varepsilon} \psi_i(a_i+\varepsilon y)\left|\nabla\left(\eta_i(a_i+\varepsilon y)U(y)\right)\right|^2 dy}{\int_{(\Omega-a_i)/\varepsilon} \left|\nabla\left(\eta_i(a_i+\varepsilon y)U(y)\right)\right|^2 dy} \\ &\longrightarrow \psi_i(a_i) = 0 \quad \text{as } \varepsilon \longrightarrow 0. \end{aligned} \tag{69}$$

Hence, there exists $\overline{\varepsilon}_0 > 0$ for any

$$\varepsilon \in (0, \overline{\varepsilon}_0), \qquad Q_i\left(z^i_\varepsilon\right) < \frac{r_0}{3}, \tag{70}$$

which implies

$$z^i_\varepsilon = t^i_\varepsilon \overline{z}^i_\varepsilon \in \mathcal{N}^i_{\lambda,\mu} \quad \text{for } \varepsilon \in (0, \overline{\varepsilon}_0), \tag{71}$$

and then

$$\theta^-_{\lambda,\mu} \le \theta^i_{\lambda,\mu} \le I_{\lambda,\mu}\left(z^i_\varepsilon\right) \le \sup_{t\ge 0} I_{\lambda,\mu}\left(t t^i_\varepsilon \overline{z}^i_\varepsilon\right) = \sup_{t\ge 0} I_{\lambda,\mu}\left(t\overline{z}^i_\varepsilon\right). \tag{72}$$

First, we consider the functional $I_{0,0} : E \to \mathbb{R}$ defined by

$$I_{0,0}(u,v) = \frac{1}{2}\|(u,v)\|^2_E - \frac{1}{2^*}\int_\Omega h(x)\,|u|^\alpha |v|^\beta dx. \tag{73}$$

Step I. Show that $\sup_{t\ge 0} I_{0,0}(\overline{z}^i_\varepsilon) \le (1/N)(S_{\alpha,\beta})^{N/2} + O(\varepsilon^{N-2})$.

According to condition $(H2)$, we conclude that

$$\begin{aligned} &\left|\int_\Omega h(x)\left|u^i_\varepsilon(x)\right|^{2^*} dx - \int_\Omega h(a_i)\left|u^i_\varepsilon(x)\right|^{2^*} dx\right| \\ &\quad \le \int_\Omega \left|h(x) - h(a_i)\right|\left|u^i_\varepsilon(x)\right|^{2^*} dx \\ &\quad = O\left(\int_{B_{r_0}(a_i)} \left|x-a_i\right|^\sigma \left|u^i_\varepsilon(x)\right|^{2^*} dx\right) \\ &\quad = O\left(\varepsilon^\sigma\right). \end{aligned} \tag{74}$$

From (66), (74), $h(a_i) = 1$, and $\sigma \ge N-2$, we can deduce that

$$\begin{aligned} &\left(\int_\Omega h(x)\left|u^i_\varepsilon(x)\right|^{2^*} dx\right)^{2/2^*} \\ &\quad = \left(\left|u^i_\varepsilon\right|^{2^*}_{L^{2^*}(\Omega)} + O\left(\varepsilon^\sigma\right)\right)^{2/2^*} \\ &\quad = \left|u^i_\varepsilon\right|^2_{L^{2^*}(\Omega)} + O\left(\varepsilon^\sigma\right) \\ &\quad = |U|^2_{L^{2^*}(\mathbb{R}^N)} + O\left(\varepsilon^{N-2}\right). \end{aligned} \tag{75}$$

Using (67) and (75), then

$$\begin{aligned} &\frac{\left|\nabla u^i_\varepsilon\right|^2_{L^2(\Omega)}}{\left(\int_\Omega h(x)\left|u^i_\varepsilon(x)\right|^{2^*} dx\right)^{2/2^*}} \\ &\quad = \frac{|\nabla U|^2_{L^2(\mathbb{R}^N)} + O\left(\varepsilon^{N-2}\right)}{|U|^2_{L^{2^*}(\mathbb{R}^N)} + O\left(\varepsilon^{N-2}\right)} \\ &\quad = S + O\left(\varepsilon^{N-2}\right). \end{aligned} \tag{76}$$

Since

$$\begin{aligned} &\sup_{t\ge 0}\left(\frac{A}{2}t^2 - \frac{B}{2^*}t^{2^*}\right) \\ &\quad = \frac{1}{N}\left(\frac{A}{B^{2/2^*}}\right)^{N/2}, \quad \text{for any } A > 0,\ B > 0, \end{aligned} \tag{77}$$

by (7) and (76), we conclude that

$$\begin{aligned}&\sup_{t\geq 0} I_{0,0}\left(t\overline{z}_\varepsilon^i\right)\\&\quad=\frac{1}{N}\left(\frac{(\alpha+\beta)\left|\nabla u_\varepsilon^i\right|_{L^2(\Omega)}^2}{\left(\alpha^{\alpha/2}\beta^{\beta/2}\int_\Omega h(x)\left|u_\varepsilon^i(x)\right|^{2^*}dx\right)^{2/2^*}}\right)^{N/2}\\&\quad\leq\frac{1}{N}\left(S_{\alpha,\beta}\right)^{N/2}+O\left(\varepsilon^{N-2}\right).\end{aligned}\tag{78}$$

Step II. Let C_0 be the positive constant given in Lemma 10. We can choose $\delta_1>0$ such that for all $0<\lambda^{2/(2-q)}+\mu^{2/(2-q)}<\delta_1$, we have

$$c^*=\frac{1}{N}\left(S_{\alpha,\beta}\right)^{N/2}-C_0\left(\lambda^{2/(2-q)}+\mu^{2/(2-q)}\right)>0.\tag{79}$$

Since $I_{\lambda,\mu}$ is continuous in E, $I_{\lambda,\mu}(0)=0$, and $\{\overline{z}_\varepsilon^i\}$ is uniformly bounded in E for any $0<\varepsilon<\min\{\overline{\varepsilon}_0,r_0/2\}$ (see (67)), then there exists $t_0>0$ (independent of ε) such that for any $0<\varepsilon<\min\{\overline{\varepsilon}_0,r_0/2\}$,

$$\begin{gathered}\sup_{0\leq t\leq t_0} I_{\lambda,\mu}\left(t\overline{z}_\varepsilon^i\right)<c^*,\quad \text{uniformly in } i,\\ \forall 0<\lambda^{2/(2-q)}+\mu^{2/(2-q)}<\delta_1.\end{gathered}\tag{80}$$

According to condition $(H1)$, $f_{\min}=\min_{x\in\overline{\Omega}}f(x)>0$ and $g_{\min}=\min_{x\in\overline{\Omega}}g(x)>0$. Applying the results of Step I and (68), we have that for $N/(N-2)<q<2$ and $N>4$,

$$\begin{aligned}&\sup_{t\geq t_0} I_{\lambda,\mu}\left(t\overline{z}_\varepsilon^i\right)\\&\quad=\sup_{t\geq t_0}\left(I_{0,0}\left(t\overline{z}_\varepsilon^i\right)-\frac{t^q}{q}K_{\lambda,\mu}\left(t\overline{z}_\varepsilon^i\right)\right)\\&\quad\leq\frac{1}{N}\left(S_{\alpha,\beta}\right)^{N/2}+O\left(\varepsilon^{N-2}\right)-\frac{t_0^q}{q}m(\lambda+\mu)\\&\qquad\times\int_{B_{r_0/2}(a_i)}\left|u_\varepsilon^i\right|^q dx\\&\quad\leq\frac{1}{N}\left(S_{\alpha,\beta}\right)^{N/2}+O\left(\varepsilon^{N-2}\right)-(\lambda+\mu)O_1\left(\varepsilon^\theta\right),\end{aligned}\tag{81}$$

where $m=\min\{\alpha^{q/2}f_{\min},\beta^{q/2}g_{\min}\}$ and $\theta=N-((N-2)q)/2$.

Therefore, we can choose $\lambda=O_1(\varepsilon^{\tau_1})$ and $\mu=O_1(\varepsilon^{\tau_2})$ such that

$$\frac{2-q}{q}\theta<\tau_1,\qquad \tau_2<(N-2)-\theta.\tag{82}$$

This implies that

$$\begin{gathered}\min\{\tau_1,\tau_2\}+\theta<\frac{2}{2-q}\min\left(\tau_1,\tau_2\right),\\ \min\{\tau_1,\tau_2\}+\theta<N-2,\\ (\lambda+\mu)O_1\left(\varepsilon^\theta\right)=O_1\left(\varepsilon^{\min\{\tau_1,\tau_2\}+\theta}\right),\\ \lambda^{2/(2-q)}+\mu^{2/(2-q)}=O_1\left(\varepsilon^{2/(2-q)\min\{\tau_1,\tau_2\}}\right).\end{gathered}\tag{83}$$

There exist $\delta_2>0$, $\varepsilon_0\in(0,\min\{\overline{\varepsilon}_0,r_0/2\})$ such that for all $0<\lambda^{2/(2-q)}+\mu^{2/(2-q)}<\delta_2$ and $0<\varepsilon<\varepsilon_0$, we have

$$O\left(\varepsilon^{N-2}\right)-(\lambda+\mu)O_1\left(\varepsilon^\theta\right)<-C_0\left(\lambda^{2/(2-q)}+\mu^{2/(2-q)}\right).\tag{84}$$

Thus, we can choose $\Lambda_2=\min\{(q/2)^{2/(2-q)}\Lambda_1,\delta_1,\delta_2\}>0$. Then, for all $\lambda^{2/(2-q)}+\mu^{2/(2-q)}\in(0,\Lambda_2)$, there holds

$$\sup_{t\geq 0} I_{\lambda,\mu}\left(t\overline{z}_\varepsilon^i\right)<c^*\quad \text{uniformly in } i.\tag{85}$$

Step III. For $0<\lambda^{2/(2-q)}+\mu^{2/(2-q)}<\Lambda_2$ and $0<\varepsilon<\varepsilon_0$, by Lemma 7, (72), and (85), we get

$$0<\theta_{\lambda,\mu}^-\leq\theta_{\lambda,\mu}^i\leq I_{\lambda,\mu}\left(t\overline{z}_\varepsilon^i\right)<c^*\quad\forall 1\leq i\leq k.\tag{86}$$

□

To proceed, we need to quote the concentration-compactness principle (see [24, 25]) about the case of systems.

Lemma 15. *Let $\{u_n,v_n\}\subset H_0^1(\Omega)\times H_0^1(\Omega)$ be a sequence such that*

$$\begin{gathered}u_n\rightharpoonup u,\quad v_n\rightharpoonup v\quad \text{weakly in } H_0^1(\Omega);\\ u_n\longrightarrow u,\quad v_n\longrightarrow v\quad \text{a.e. on } \Omega,\\ \left|\nabla(u_n-u)\right|^2+\left|\nabla(v_n-v)\right|^2\rightharpoonup\widetilde{\mu}\\ \text{weakly in the sense of measures},\\ \left|u_n-u\right|^\alpha\left|v_n-v\right|^\beta\rightharpoonup\widetilde{\nu}\\ \text{weakly in the sense of measures}.\end{gathered}\tag{87}$$

Then, it follows that

$$\begin{gathered}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left(\left|\nabla u_n\right|^2+\left|\nabla v\right|_n^{\,2}\right)dx\\ =\int_{\mathbb{R}^N}\left(|\nabla u|^2+|\nabla v|^2\right)dx+\|\widetilde{\mu}\|,\\ \limsup_{n\to\infty}\int_{\mathbb{R}^N}\left|u_n\right|^\alpha\left|v_n\right|^\beta dx=\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta dx+\|\widetilde{\nu}\|,\\ \|\widetilde{\nu}\|^{2/(\alpha+\beta)}\leq S_{\alpha,\beta}^{-1}\|\widetilde{\mu}\|.\end{gathered}\tag{88}$$

Moreover, if $u\equiv v\equiv 0$ and $\|\widetilde{\nu}\|^{2/(\alpha+\beta)}=S_{\alpha,\beta}^{-1}\|\widetilde{\mu}\|$, then $\widetilde{\mu}$ and $\widetilde{\nu}$ concentrate at a single point.

Proof. See Han [20, Lemma 2.2]. □

Lemma 16. *For any* $i \in \{1,2,\ldots,k\}$, *there exist* $\widetilde{\Lambda}_i > 0$ *such that*

$$\widetilde{\theta}^i_{\lambda,\mu} > \frac{1}{N}\left(S_{\alpha,\beta}\right)^{N/2} \quad \forall 0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \widetilde{\Lambda}_i. \tag{89}$$

Proof. Fix $i \in \{1,2,\ldots,k\}$. Assume the contrary. There then exists a sequence $\{(\lambda_n,\mu_n)\}$ with $(\lambda_n,\mu_n) \to (0,0)$ as $n \to \infty$ such that $\widetilde{\theta}^i_{\lambda_n,\mu_n} \to c \le (1/N)(S_{\alpha,\beta})^{N/2}$ as $n \to \infty$. Consequently, there exists a sequence $\{z_n = (u_n,v_n)\} \subset \partial\mathcal{N}^i_{\lambda_n,\mu_n}$ such that as $n \to \infty$,

$$\begin{aligned}\int_\Omega \left(|\nabla u_n|^2 + |\nabla v_n|^2\right)dx &= \int_\Omega \left(\lambda_n f(x)|u_n|^q + \mu_n g(x)|v_n|^q\right)dx \\ &\quad + \int_\Omega h(x)|u_n|^\alpha |v_n|^\beta dx,\end{aligned} \tag{90}$$

$$I_{\lambda_n,\mu_n}(z_n) \longrightarrow c \le \frac{1}{N}\left(S_{\alpha,\beta}\right)^{N/2} \quad \text{as } n \longrightarrow \infty. \tag{91}$$

It then follows easily that $\{z_n\}$ is uniformly bounded in E, and since f and g are continuous on $\overline{\Omega}$, we obtain

$$\begin{aligned}K_{\lambda_n,\mu_n}(z_n) &= \int_\Omega \left(\lambda_n f(x)|u_n|^q + \mu_n g(x)|v_n|^q\right)dx \\ &= o_n(1) \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{92}$$

From (90), and by the Hölder and the Sobolev inequalities, we can fix $m_0 > 0$ such that

$$\begin{gathered}\int_\Omega \left(|\nabla u_n|^2 + |\nabla v_n|^2\right)dx \ge m_0, \\ \int_\Omega h(x)|u_n|^\alpha |v_n|^\beta dx \ge m_0.\end{gathered} \tag{93}$$

Thus, up to a subsequence, we infer that

$$\begin{aligned}&\lim_{n\to\infty}\int_\Omega \left(|\nabla u_n|^2 + |\nabla v_n|^2\right)dx \\ &\quad = \lim_{n\to\infty}\int_\Omega h(x)|u_n|^\alpha |v_n|^\beta dx = l > 0.\end{aligned} \tag{94}$$

Furthermore, by $|h|_{L^\infty(\Omega)} = 1$, we deduce

$$\begin{aligned}l &= \lim_{n\to\infty}\int_\Omega h(x)|u_n|^\alpha |v_n|^\beta dx \\ &\le |h|_{L^\infty(\Omega)} \lim_{n\to\infty}\int_\Omega |u_n|^\alpha |v_n|^\beta dx \\ &\le S_{\alpha,\beta}^{-2^*/2} \lim_{n\to\infty}\left(\int_\Omega \left(|\nabla u_n|^2 + |\nabla v_n|^2\right)dx\right)^{2^*/2} \\ &\le S_{\alpha,\beta}^{-2^*/2} l^{2^*/2},\end{aligned} \tag{95}$$

which implies

$$l \ge \left(S_{\alpha,\beta}\right)^{N/2}. \tag{96}$$

On the other hand, we have, as $n \to \infty$,

$$\begin{aligned}\frac{1}{N}l &= \frac{1}{2}\|z_n\|^2 dx - \frac{1}{2^*}\int_\Omega h(x)|u_n|^\alpha |v_n|^\beta dx \\ &\quad - \frac{1}{q}K_{\lambda_n,\mu_n}(z_n) + o_n(1) \\ &= I_{\lambda_n,\mu_n}(z_n) + o_n(1) \\ &\le \frac{1}{N}\left(S_{\alpha,\beta}\right)^{N/2}.\end{aligned} \tag{97}$$

Hence, together with (96), we get

$$l = \left(S_{\alpha,\beta}\right)^{N/2}, \tag{98}$$

and then from (95), we also have

$$\lim_{n\to\infty}\int_\Omega h(x)|u_n|^\alpha |v_n|^\beta dx = \lim_{n\to\infty}\int_\Omega |u_n|^\alpha |v_n|^\beta dx = l. \tag{99}$$

Therefore,

$$\lim_{n\to\infty}\int_\Omega |u_n|^\alpha |v_n|^\beta dx = l. \tag{100}$$

Set $\widetilde{z}_n = (\widetilde{u}_n,\widetilde{v}_n) = z_n/\|z_n\|$; then, we have $\|\widetilde{z}_n\| = 1$. Moreover, by (94),(98), and (100), we get

$$\lim_{n\to\infty}\int_\Omega |\widetilde{u}_n|^\alpha |\widetilde{v}_n|^\beta dx = \lim_{n\to\infty}\frac{\int_\Omega |u_n|^\alpha |v_n|^\beta dx}{\|z_n\|^{2^*}} = S_{\alpha,\beta}^{-N/(N-2)}. \tag{101}$$

Thus, up to a subsequence, we may assume that

$$\begin{gathered}\widetilde{u}_n \rightharpoonup u, \quad \widetilde{v}_n \rightharpoonup v \quad \text{weakly in } H_0^1(\Omega); \\ \widetilde{u}_n \longrightarrow u, \quad \widetilde{v}_n \longrightarrow v \quad \text{a.e. on } \Omega, \\ |\nabla(\widetilde{u}_n - u)|^2 + |\nabla(\widetilde{v}_n - v)|^2 \rightharpoonup \widetilde{\mu} \\ \text{weakly in the sense of measures,} \\ |\widetilde{u}_n - u|^\alpha |\widetilde{v}_n - v|^\beta \rightharpoonup \widetilde{\nu} \\ \text{weakly in the sense of measures.}\end{gathered} \tag{102}$$

Since Ω is bounded, from (101) and Lemma 15, we deduce that

$$1 = \int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right)dx + \|\widetilde{\mu}\|, \tag{103}$$

$$S_{\alpha,\beta}^{-N/(N-2)} = \int_\Omega |u|^\alpha |v|^\beta dx + \|\widetilde{\nu}\|, \tag{104}$$

$$\|\widetilde{\nu}\|^{2/(\alpha+\beta)} \le S_{\alpha,\beta}^{-1}\|\widetilde{\mu}\|. \tag{105}$$

If $\int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx \neq 0$ and $\|\widetilde{\mu}\| \neq 0$, we deduce that

$$\begin{aligned} 1 &= \left(\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2 \right) dx + \|\widetilde{\mu}\| \right)^{(\alpha+\beta)/2} \\ &> \left(\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2 \right) dx \right)^{(\alpha+\beta)/2} + \|\widetilde{\mu}\|^{(\alpha+\beta)/2} \\ &\geq S_{\alpha,\beta}^{(\alpha+\beta)/2} \int_\Omega |u|^\alpha |v|^\beta dx + S_{\alpha,\beta}^{(\alpha+\beta)/2} \|\widetilde{\nu}\| \\ &= S_{\alpha,\beta}^{(\alpha+\beta)/2} \cdot S_{\alpha,\beta}^{-N/(N-2)} \\ &= 1, \end{aligned} \tag{106}$$

which is a contradiction.

Thus, $\int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx = 0$ or $\|\widetilde{\mu}\| = 0$. If $\|\widetilde{\mu}\| = 0$, from (103)–(105), we get $\int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx = 1$ and $\int_\Omega |u|^\alpha |v|^\beta dx = S_{\alpha,\beta}^{-N/(N-2)}$. Then,

$$\frac{\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2 \right) dx}{\left(\int_\Omega |u|^\alpha |v|^\beta dx \right)^{2/(\alpha+\beta)}} = S_{\alpha,\beta}, \tag{107}$$

which means that $S_{\alpha,\beta}$ is achieved by (u, v). It is impossible since $S_{\alpha,\beta}$ cannot be achieved on any bounded domain Ω. Hence,

$$\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2 \right) dx = 0, \qquad \|\widetilde{\mu}\| = 1. \tag{108}$$

Then, $u \equiv v \equiv 0$ on Ω, and from (103), (104), we easily have $\|\widetilde{\nu}\|^{2/(\alpha+\beta)} = S_{\alpha,\beta}^{-1} = S_{\alpha,\beta}^{-1}\|\widetilde{\mu}\|$. By Lemma 15, we conclude that $x_0 \in \overline{\Omega}$ such that

$$\begin{gathered} \left|\nabla \widetilde{u}_n\right|^2 + \left|\nabla \widetilde{v}\right|^2 \rightharpoonup \delta_{x_0} \\ \text{weakly in the sense of measures,} \\ \left|\widetilde{u}_n\right|^\alpha \left|\widetilde{v}_n\right|^\beta \rightharpoonup S_{\alpha,\beta}^{-N/(N-2)} \delta_{x_0} \\ \text{weakly in the sense of measures.} \end{gathered} \tag{109}$$

Observe that $Q_i(\widetilde{z}_n) = Q_i(z_n) = r_0/3$;

$$\begin{aligned} \frac{r_0}{3} &= \lim_{n\to\infty} Q_i(z_n) \\ &= \lim_{n\to\infty} \frac{\int_\Omega \psi_i(x) \left(\left|\nabla \widetilde{u}_n\right|^2 + \left|\nabla \widetilde{v}_n\right|^2 \right) dx}{\int_\Omega \left(\left|\nabla \widetilde{u}_n\right|^2 + \left|\nabla \widetilde{v}_n\right|^2 \right) dx} = \psi_i(x_0), \end{aligned} \tag{110}$$

which implies that $x_0 \neq a_i$ by the definition of $\psi_i(x)$. On the other hand, from (95) and (101), we get

$$\begin{aligned} S_{\alpha,\beta}^{-N/(N-2)} h(x_0) &= \lim_{n\to\infty} \int_\Omega h(x) \left|\widetilde{u}_n\right|^\alpha \left|\widetilde{v}_n\right|^\beta dx \\ &= \lim_{n\to\infty} \frac{\int_\Omega h(x) \left|u_n\right|^\alpha \left|v_n\right|^\beta dx}{\left\|z_n\right\|^{2^*}} \\ &= \lim_{n\to\infty} \frac{\int_\Omega \left|u_n\right|^\alpha \left|v_n\right|^\beta dx}{\left\|z_n\right\|^{2^*}} \\ &= \lim_{n\to\infty} \int_\Omega \left|\widetilde{u}_n\right|^\alpha \left|\widetilde{v}_n\right|^\beta dx \\ &= S_{\alpha,\beta}^{-N/(N-2)}, \end{aligned} \tag{111}$$

which is impossible, because $h(x)$ is not a constant function by condition $(H2)$. □

Throughout this section, take $\Lambda^* = \min\{\Lambda_2, \min_{1\leq i\leq k} \widetilde{\Lambda}_i\}$; Λ_2 and $\widetilde{\Lambda}_i$ are as in Lemmas 14 and 16. Using the idea of Tarantello [12], we have the following results. For $z = (u, v)$, $\varphi = (\varphi_1, \varphi_2) \in E$, we define

$$\begin{aligned} z - \varphi &= (u - \varphi_1, v - \varphi_2), \\ \langle z, \varphi \rangle &= \int_\Omega (\nabla u \nabla \varphi_1 + \nabla v \nabla \varphi_2) dx, \\ G_{\lambda,\mu}(z, \varphi) &= \int_\Omega \left(\lambda f(x) |u|^{q-2} u \varphi_1 + \mu g(x) |v|^{q-2} v \varphi_2 \right) dx, \\ H(z, \varphi) &= \frac{\alpha}{\alpha+\beta} \int_\Omega h(x) |u|^{\alpha-2} u |v|^\beta \varphi_1 dx \\ &\quad + \frac{\beta}{\alpha+\beta} \int_\Omega h(x) |u|^\alpha |v|^{\beta-2} v \varphi_2 dx. \end{aligned} \tag{112}$$

Lemma 17. *For each $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda^*$ and $z = (u, v) \in \mathcal{N}_{\lambda,\mu}^i$ $(1 \leq i \leq k)$, there exist $\epsilon > 0$ and a differentiable function $\xi : B_\epsilon(0) \subset E \to \mathbb{R}^+$ such that $\xi(0) = 1$, $\xi(\phi)(z-\phi) \in \mathcal{N}_{\lambda,\mu}^i$ for all $\phi \in B_\epsilon(0)$ and*

$$\left\langle \xi'(0), \varphi \right\rangle = \frac{2\langle z, \varphi \rangle - q G_{\lambda,\mu}(z, \varphi) - 2^* H(z, \varphi)}{(2-q)\|z\|_E^2 - (2^* - q) H(z, z)}, \tag{113}$$

for all $\varphi = (\varphi_1, \varphi_2) \in E$.

Proof. For $z \in \mathcal{N}_{\lambda,\mu}^i$, define a function $F_z : \mathbb{R} \times E \to \mathbb{R}$ by

$$\begin{aligned} F_z(\xi, \phi) &= \left\langle I'_{\lambda,\mu}(\xi(z-\phi)), \xi(z-\phi) \right\rangle \\ &= \xi^2 \|z - \phi\|^2 - \xi^q G_{\lambda,\mu}(z-\phi, z-\phi) \\ &\quad - \xi^{\alpha+\beta} H(z-\phi, z-\phi). \end{aligned} \tag{114}$$

Then, $F_u(1, 0) = \langle I'_{\lambda,\mu}(z), z \rangle = 0$ and

$$\begin{aligned} \frac{d}{d\xi} F_z(1, 0) &= 2\|z\|_E^2 - q G_{\lambda,\mu}(z, z) - (\alpha+\beta) H(z, z) \\ &= (2-q)\|z\|_E^2 - (2^* - q) H(z, z) < 0. \end{aligned} \tag{115}$$

According to the implicit function theorem, there exist $\epsilon > 0$ and a differentiable function $\xi : B_\epsilon(0) \subset E \to \mathbb{R}$ such that $\xi(0) = 1$;

$$\left\langle \xi'(0), \varphi \right\rangle = \frac{2\langle z, \varphi\rangle - qG_{\lambda,\mu}(z,\varphi) - 2^* H(z,\varphi)}{(2-q)\|z\|_E^2 - (2^*-q)H(z,z)}, \qquad (116)$$
$$F_z(\xi(\varphi), \varphi) = 0 \quad \forall \varphi \in B_\epsilon(0),$$

which is equivalent to

$$\left\langle I'_{\lambda,\mu}(\xi(\varphi)(z-\varphi)), \xi(\varphi)(z-\varphi) \right\rangle = 0 \quad \forall \varphi \in B(0;\epsilon); \qquad (117)$$

that is, $\xi(\varphi)(z-\varphi) \in \mathcal{N}_{\lambda,\mu}$ for all $\varphi \in B_\epsilon(0)$. Furthermore, by the continuity of the functions ξ and Q_i, we have that

$$(2-q)\|\xi(\varphi)(z-\varphi)\|^2 - (2^*-q)H(\xi(\varphi)(z-\varphi), \xi(\varphi)(z-\varphi)) < 0, \qquad (118)$$
$$Q_i(\xi(\varphi)(z-\varphi)) < \frac{r_0}{3}$$

still holds if ϵ is sufficiently small. This implies that $\xi(\varphi)(z-\varphi) \in \mathcal{N}^i_{\lambda,\mu}$. □

Proposition 18. *If $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda^*$, then there exists a $(PS)_{\theta^i_{\lambda,\mu}}$-sequence $\{z^i_n\} \subset \mathcal{N}^i_{\lambda,\mu}$ in E for $I_{\lambda,\mu}$.*

Proof. If $\overline{\mathcal{N}^i_{\lambda,\mu}}$ denotes the closure of $\mathcal{N}^i_{\lambda,\mu}$, at first we note that $\overline{\mathcal{N}^i_{\lambda,\mu}} = \mathcal{N}^i_{\lambda,\mu} \cup \partial\mathcal{N}^i_{\lambda,\mu}$ for all $i = 1, 2, \ldots, k$. It then follows from Lemmas 14 and 16, that

$$\theta^i_{\lambda,\mu} < \widetilde{\theta}^i_{\lambda,\mu} \quad \text{for } i = 1,2,\ldots,k,\ 0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda^*. \qquad (119)$$

Hence,

$$\theta^i_{\lambda,\mu} = \inf\left\{ I_{\lambda,\mu}(z) \mid z \in \overline{\mathcal{N}^i_{\lambda,\mu}} \right\} \quad \text{for } i = 1,2,\ldots,k. \qquad (120)$$

Now, we fix $i \in \{1, 2, \ldots, k\}$. Applying the Ekeland variational principle [26], there exists a minimizing sequence $\{z^i_n\} \subset \overline{\mathcal{N}^i_{\lambda,\mu}}$ such that

$$I_{\lambda,\mu}(z^i_n) < \theta^i_{\lambda,\mu} + \frac{1}{n},$$
$$I_{\lambda,\mu}(z^i_n) \le I_{\lambda,\mu}(\varphi) + \frac{1}{n}\left\|\varphi - z^i_n\right\|_E \quad \text{for each } \varphi \in \overline{\mathcal{N}^i_{\lambda,\mu}}. \qquad (121)$$

Using (119), we may assume that $z^i_n \in \mathcal{N}^i_{\lambda,\mu}$ for n sufficiently large. Applying Lemma 17 with $z = z^i_n$, we obtain the function $\xi_n : B_{\epsilon_n}(0) \to \mathbb{R}$ for some $\epsilon_n > 0$ such that $\xi_n(\varphi)(z^i_n - \varphi) \in \mathcal{N}^i_{\lambda,\mu}$ for all $\varphi \in B_{\epsilon_n}(0)$. Let $0 < \delta < \epsilon_n$ and $z \in E \setminus \{0\}$; we set

$$\varphi_\delta = \frac{\delta z}{\|z\|_E} \qquad (122)$$

and $z_\delta = \xi_n(\varphi_\delta)(z^i_n - \varphi_\delta)$. Since $z_\delta \in \mathcal{N}^i_{\lambda,\mu}$, we deduce from (121) that

$$I_{\lambda,\mu}(z_\delta) - I_{\lambda,\mu}(z^i_n) \ge -\frac{1}{n}\left\|z_\delta - z^i_n\right\|_E. \qquad (123)$$

By the mean-value theorem, we obtain

$$\left\langle I'_{\lambda,\mu}(z^i_n), (z_\delta - z^i_n) \right\rangle + o\left(\left\|z_\delta - z^i_n\right\|_E\right) \ge -\frac{1}{n}\left\|z_\delta - z^i_n\right\|_E. \qquad (124)$$

Therefore,

$$\left\langle I'_{\lambda,\mu}(z^i_n), -\varphi_\delta \right\rangle + (\xi_n(\varphi_\delta) - 1)\left\langle I'_{\lambda,\mu}(z^i_n), (z^i_n - \varphi_\delta) \right\rangle \ge -\frac{1}{n}\left\|z_\delta - z^i_n\right\|_E + o\left(\left\|z_\delta - z^i_n\right\|_E\right). \qquad (125)$$

Now, we observe that $\xi_n(\varphi_\delta)(z^i_n - \varphi_\delta) \in \mathcal{N}^i_{\lambda,\mu}$, and consequently we get from (125) that

$$\begin{aligned} &-\delta\left\langle I'_{\lambda,\mu}(z^i_n), \frac{z}{\|z\|_E} \right\rangle \\ &\quad + \frac{(\xi_n(\varphi_\delta) - 1)}{\xi_n(\varphi_\delta)}\left\langle I'_{\lambda,\mu}(z_\delta), \xi_n(\varphi_\delta)(z^i_n - \varphi_\delta) \right\rangle \\ &\quad + (\xi_n(\varphi_\delta) - 1)\left\langle I'_{\lambda,\mu}(z^i_n) - I'_{\lambda,\mu}(z_\delta), (z^i_n - \varphi_\delta) \right\rangle \\ &\ge -\frac{1}{n}\left\|z_\delta - z^i_n\right\|_E + o\left(\left\|z_\delta - z^i_n\right\|_E\right). \end{aligned} \qquad (126)$$

Then, we write the pervious inequality in the following form:

$$\begin{aligned} &\left\langle I'_{\lambda,\mu}(z^i_n), \frac{z}{\|z\|_E} \right\rangle \\ &\le \frac{\left\|z_\delta - z^i_n\right\|_E}{\delta n} + \frac{o\left(\left\|z_\delta - z^i_n\right\|_E\right)}{\delta} \\ &\quad + \frac{(\xi_n(\varphi_\delta) - 1)}{\delta}\left\langle I'_{\lambda,\mu}(z^i_n) - I'_{\lambda,\mu}(z_\delta), (z^i_n - \varphi_\delta) \right\rangle. \end{aligned} \qquad (127)$$

We can find a constant $C > 0$ independent of δ such that

$$\left\|z_\delta - z^i_n\right\| \le \delta + C\left(\left|\xi_n(\varphi_\delta) - 1\right|\right),$$
$$\lim_{\delta \to 0}\frac{\left|\xi_n(\varphi_\delta) - 1\right|}{\delta} \le \left\|\xi'_n(0)\right\| \le C. \qquad (128)$$

For a fixed n, let $\delta \to 0$ in (127). Using the fact that

$$\lim_{\delta \to 0}\left\|z_\delta - z^i_n\right\|_E = 0, \qquad (129)$$

we obtain

$$\left\langle I'_{\lambda,\mu}(z^i_n), \frac{z}{\|z\|_E} \right\rangle \le \frac{C}{n}. \qquad (130)$$

This implies

$$I_{\lambda,\mu}\left(z_n^i\right) = \theta_{\lambda,\mu}^i + o_n(1), \quad I'_{\lambda,\mu}\left(z_n^i\right) = o_n(1) \quad \text{in } E^{-1}. \tag{131}$$

□

Now, we complete the proof of Theorem 2. By Lemmas 12, 14 and Proposition 18, for all $\lambda^{2/(2-q)} + \mu^{2/(2-q)} \in (0, \Lambda^*)$, there exists a sequence $\{z_n^i\} \subset \mathcal{N}_{\lambda,\mu}^i$ and $z_0^i = (u_0^i, v_0^i) \in E$, $1 \le i \le k$, such that

$$\begin{aligned} I_{\lambda,\mu}\left(z_n^i\right) &= \theta_{\lambda,\mu}^i + o_n(1), \\ I'_{\lambda,\mu}\left(z_n^i\right) &= o_n(1) \quad \text{in } E^{-1}, \\ z_n^i &\longrightarrow z_0^i \quad \text{strongly in } E. \end{aligned} \tag{132}$$

Moreover, $\{z_n^i\} \subset \mathcal{N}_{\lambda,\mu}^-$, and by Lemma 7 (ii), we get $z_0^i \in \mathcal{N}_{\lambda,\mu}^-$, $u_0^i \not\equiv 0$, $v_0^i \not\equiv 0$ in Ω,

$$\begin{aligned} \left\|z_0^i\right\|_E &> \left(\frac{2-q}{2^*-q}\right)^{1/(2^*-2)} S^{N/4}, \\ \theta_{\lambda,\mu}^i &\ge \theta_{\lambda,\mu}^- > 0 \quad \text{for } i = 1, 2, \ldots, k. \end{aligned} \tag{133}$$

Thus, z_0^i is a nontrivial solution of the problem $\left(P_{\lambda,\mu}\right)$ and $I_{\lambda,\mu}(z_0^i) = \theta_{\lambda,\mu}^i$ for $i = 1, 2 \ldots, k$. Set $u_+ = \max\{u, 0\}$ and $v_+ = \max\{v, 0\}$. Replace the terms $\int_\Omega h(x)|u|^\alpha |v|^\beta dx$ and $\int_\Omega (\lambda f(x)|u|^q + \mu g(x)|v|^q) dx$ of the functional $I_{\lambda,\mu}$ by $\int_\Omega h(x) u_+^\alpha v_+^\beta dx$ and $\int_\Omega (\lambda f(x) u_+^q + \mu g(x) v_+^q) dx$, respectively. It then follows that z_0^i is a nonnegative solution of the problem $\left(P_{\lambda,\mu}\right)$. Applying the maximum principle [19], z_0^i is a positive solution of the problem $\left(P_{\lambda,\mu}\right)$. Since $Q_i(z_0^i) < r_0/3$,

$$z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}^+, \qquad z_0^i \in \mathcal{N}_{\lambda,\mu}^i \subset \mathcal{N}_{\lambda,\mu}^- \quad \text{for } i = 1, 2, \ldots, k, \tag{134}$$

where $z_{\lambda,\mu}^1$ is a positive solution of equation $\left(P_{\lambda,\mu}\right)$ as in Theorem 1. From Lemma 13, we conclude that $\mathcal{N}_{\lambda,\mu}^i$ are disjoint for $i = 1, 2 \ldots, k$. This implies that z_0^i $(1 \le i \le k)$ and $z_{\lambda,\mu}^1$ are distinct positive solutions of the problem $\left(P_{\lambda,\mu}\right)$.

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