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Research Article

A New Hybrid Extragradient Iterative Method for Approximating the Common Solutions of a System of Variational Inequalities, a Mixed Equilibrium Problem, and a Hierarchical Fixed Point Problem

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We suggest and analyze an iterative scheme for finding the approximate element of the common set of solutions of a system of variational inequalities, a mixed equilibrium problem, and a hierarchical fixed point problem in a real Hilbert space. Strong convergence of the proposed method is proved under some conditions. The results presented in this paper extend and improve some well-known results in the literature.

1. Introduction

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. Let C be a nonempty closed convex subset of H. We consider the system of variational inequalities of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \mu_1 B_1 y^* + x^* - y^*, x - x^* \rangle \geq 0; \quad \forall x \in C,\ \mu_1 > 0, \\ &\langle \mu_2 B_2 x^* + y^* - x^*, x - y^* \rangle \geq 0; \quad \forall x \in C,\ \mu_2 > 0, \end{aligned} \tag{1}$$

where $B_i : C \to C$ is a nonlinear mapping for each $i = 1, 2$. The solution set of (1) is denoted by S^*.

If $B_1 = B_2 = B$, then the problem (1) reduces finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \mu_1 B y^* + x^* - y^*, x - x^* \rangle \geq 0; \quad \forall x \in C,\ \mu_1 > 0, \\ &\langle \mu_2 B x^* + y^* - x^*, x - y^* \rangle \geq 0; \quad \forall x \in C,\ \mu_2 > 0, \end{aligned} \tag{2}$$

which has been introduced and studied by Verma [1, 2].

If $x^* = y^*$ and $\mu_1 = \mu_2$, then the problem (2) collapses to the classical variational inequality finding $x^* \in C$, such that

$$\langle Bx^*, x - x^* \rangle \geq 0, \quad \forall x \in C \tag{3}$$

is called the classical variational inequality problem, which was introduced by Stampacchia [3] in 1964. For the recent applications, numerical techniques, and physical formulation, see [1–33]. We now have a variety of techniques to suggest and analyze various iterative algorithms for solving the system of variational inequalities (1); see [1, 2, 7, 8, 12, 14, 24, 28, 30].

We introduce the following definitions which are useful in the following analysis.

Definition 1. The mapping $T : C \to H$ is said to be

(a) monotone, if

$$\langle Tx - Ty, x - y \rangle \geq 0, \quad \forall x, y \in C; \tag{4}$$

(b) strongly monotone, if there exists an $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C; \tag{5}$$

(c) α-inverse strongly monotone, if there exists an $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|Tx - Ty\|^2, \quad \forall x, y \in C; \tag{6}$$

(d) nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C; \tag{7}$$

(e) k-Lipschitz continuous, if there exists a constant $k > 0$ such that

$$\|Tx - Ty\| \leq k\|x - y\|, \quad \forall x, y \in C; \tag{8}$$

(f) contraction on C, if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\| \leq k\|x - y\|, \quad \forall x, y \in C. \tag{9}$$

It is easy to observe that every α-inverse strongly monotone T is monotone and Lipschitz continuous. A mapping $T : C \to H$ is called k-strict pseudocontraction, if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{10}$$

The fixed-point problem for the mapping T is to find $x \in C$ such that

$$Tx = x. \tag{11}$$

We denote by $F(T)$ the set of solutions of (11). It is well-known that the class of strict pseudocontractions strictly includes the class of nonexpansive mappings; then $F(T)$ is closed and convex and $P_{F(T)}$ is well defined (see [33]).

The mixed equilibrium problem, denoted by MEP, is to find $x \in C$ such that

$$F(x, y) + \langle Dx, y - x\rangle \geq 0, \quad \forall y \in C, \tag{12}$$

where $F : C \times C \to \mathbb{R}$ is bifunction and $D : C \to H$ is a nonlinear mapping. This problem was introduced and studied by Moudafi and Théra [21] and Moudafi [22]. The set of solutions of (12) is denoted by

$$\text{MEP}(F) := \{x \in C : F(x, y) + \langle Dx, y - x\rangle \geq 0,\ \forall y \in C\}. \tag{13}$$

If $D = 0$, then it is reduced to the equilibrium problem is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{14}$$

The solution set of (14) is denoted by EP(F). Numerous problems in physics, optimization, and economics reduce to find a solution of (14); see [9, 13, 25, 26]. In 1997, Flåm and Antipin [10] introduced an iterative scheme of finding the best approximation to the initial data when EP(F) is nonempty. Recently, Plubtieng and Punpaeng [25] introduced an iterative method for finding the common element of the set $F(T) \cap \Omega^* \cap \text{EP}(F)$.

Let $S : C \to H$ be a nonexpansive mapping. The following problem is called a hierarchical fixed point problem: Find $x \in F(T)$ such that

$$\langle x - Sx, y - x\rangle \geq 0, \quad \forall y \in F(T). \tag{15}$$

It is known that the hierarchical fixed-point problem (15) links with some monotone variational inequalities and convex programming problems; see [11, 31]. Various methods have been proposed to solve the hierarchical fixed point problem; see Moudafi [23], Maingé and Moudafi in [17], Marino and Xu in [19], and Cianciaruso et al. [6]. Very recently, Yao et al. [31] introduced the following strong convergence iterative algorithm to solve the problem (15):

$$\begin{aligned} y_n &= \beta_n S x_n + (1 - \beta_n) x_n \\ x_{n+1} &= P_C[\alpha_n f(x_n) + (1 - \alpha_n) T y_n], \quad \forall n \geq 0, \end{aligned} \tag{16}$$

where $f : C \to H$ is a contraction mapping and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$. Under some certain restrictions on parameters, Yao et al. proved that the sequence $\{x_n\}$ generated by (16) converges strongly to $z \in F(T)$, which is the unique solution of the following variational inequality:

$$\langle (I - f) z, y - z\rangle \geq 0, \quad \forall y \in F(T). \tag{17}$$

By changing the restrictions on parameters, the authors obtained another result on the iterative scheme (16); the sequence $\{x_n\}$ generated by (16) converges strongly to a point $z \in F(T)$, which is the unique solution of the following variational inequality:

$$\left\langle \frac{1}{\tau}(I - f) z + (I - S) z, y - z \right\rangle \geq 0, \quad \forall y \in F(T). \tag{18}$$

Let $S : C \to H$ be a nonexpansive mapping and $\{T_i\}_{i=1}^{\infty} : C \to C$ a countable family of nonexpansive mappings. Very recently, Gu et al. [11] introduced the following iterative algorithm:

$$\begin{aligned} y_n &= P_C[\beta_n S x_n + (1 - \beta_n) x_n] \\ x_{n+1} &= P_C\left[\alpha_n f(x_n) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) T_i y_n\right], \quad \forall n \geq 1, \end{aligned} \tag{19}$$

where $\alpha_0 = 1$, $\{\alpha_n\}$ is a strictly decreasing sequence in $(0, 1)$, and $\{\beta_n\}$ is a sequence in $(0, 1)$. Under some certain conditions on parameters, Gu et al. proved that the sequence $\{x_n\}$ generated by (19) converges strongly to $z \in \bigcap_{i=1}^{\infty} F(T_i)$, which is unique solution of one of the variational inequalities (17) and (18).

In this paper, motivated by the work of Yao et al. [31] and Gu et al. [11] and by the recent work going in this direction, we give an iterative method for finding the approximate element of the common set of solutions of (1), (12), and (15) for a strictly pseudocontraction mapping in real Hilbert space. We establish a strong convergence theorem based on this method. The presented method improves and generalizes many known results for solving system of variational inequality problems, mixed equilibrium problems, and hierarchical fixed point problems; see, for example [6, 11, 17, 31] and relevant references cited therein.

2. Preliminaries

In this section, we list some fundamental lemmas that are useful in the consequent analysis. The first lemma provides some basic properties of projection onto C.

Lemma 2. *Let P_C denote the projection of H onto C. Then, one has the following inequalities:*

$$\langle z - P_C[z], P_C[z] - v\rangle \geq 0, \quad \forall z \in H,\ v \in C; \tag{20}$$

$$\langle u - v, P_C[u] - P_C[v]\rangle \geq \|P_C[u] - P_C[v]\|^2, \quad \forall u, v \in H; \tag{21}$$

$$\|P_C[u] - P_C[v]\| \leq \|u - v\|, \quad \forall u, v \in H;$$

$$\|u - P_C[z]\|^2 \leq \|z - u\|^2 - \|z - P_C[z]\|^2, \quad \forall z \in H,\ u \in C. \tag{22}$$

Lemma 3 (see [7]). *For any $(x^*, y^*) \in C \times C, (x^*, y^*)$ is a solution of (1) if and only if x^* is a fixed point of the mapping $Q : C \to C$ de fined by*

$$Q(x) = P_C[P_C[x - \mu_2 B_2 x] - \mu_1 B_1 P_C[x - \mu_2 B_2 x]], \quad \forall x \in C, \tag{23}$$

where $y^* = P_C[x^* - \mu_2 B_2 x^*], \mu_i \in (0, 2\theta_i)$, and $B_i : C \to C$ is θ_i-inverse strongly monotone mappings for each $i = 1, 2$.

Lemma 4 (see [5]). *Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions.*

(i) *$F(x, x) = 0, \forall x \in C$.*
(ii) *F is monotone; that is, $F(x, y) + F(y, x) \leq 0, \forall x, y \in C$.*
(iii) *For each $x, y, z \in C, \lim_{t\to 0} F(tz + (1 - t)x, y) \leq F(x, y)$.*
(iv) *For each $x \in C, y \to F(x, y)$ is convex and lower semicontinuous.*

Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{24}$$

Lemma 5 (see [10]). *Assume that $F : C \times C \to \mathbb{R}$ satisfies assumptions (i)–(iv) of Lemma 4. For $r > 0$ and forall $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\right\}. \tag{25}$$

Then, the following hold.

(i) *T_r is single valued.*
(ii) *T_r is firmly nonexpansive; that is,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \quad \forall x, y \in H. \tag{26}$$

(iii) *$F(T_r) = \mathrm{EP}(F)$.*
(iv) *$\mathrm{EP}(F)$ is closed and convex.*

Lemma 6 (see [32]). *Let C be a nonempty closed convex subset of a real Hilbert space H. If $T : C \to C$ is a k-strict pseudocontraction, then*

(i) *the mapping $I - T$ is demiclosed at 0; that is, if $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to 0, then $(I - T)x = 0$;*
(ii) *the set $F(T)$ of T is closed and convex so that the projection $P_{F(T)}$ is well defined.*

Lemma 7 (see [29]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n) a_n + \delta_n, \tag{27}$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and δ_n is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(2) $\lim\sup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 8 (see [4]). *Let C be a closed convex subset of H. Let $\{x_n\}$ be a bounded sequence in H. Assume that*

(i) *the weak w-limit set $w_w(x_n) \subset C$ where $w_w(x_n) = \{x : x_{n_i} \rightharpoonup x\}$,*
(ii) *for each $z \in C, \lim_{n\to\infty} \|x_n - z\|$ exists.*

Then, $\{x_n\}$ is weakly convergent to a point in C.

Lemma 9 (see [33]). *Let H be a Hilbert space, C a closed and convex subset of H, and $T : C \to C$ a k-strict pseudocontraction mapping. Define a mapping $V : C \to H$ by $Vx = \lambda x + (1 - \lambda)Tx$, for all $x \in C$. Then, as $k \leq \lambda < 1, V$ is a nonexpansive mapping such that $F(V) = F(T)$.*

Lemma 10 (see [11]). *Let H be a Hilbert space, C a closed and convex subset of H, and $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \emptyset$. Then,*

$$\|Tx - x\|^2 \leq 2\left\langle x - Tx, x - x'\right\rangle, \quad \forall x' \in F(T),\ \forall x \in C. \tag{28}$$

3. The Proposed Method and Some Properties

In this section, we suggest and analyze our method for finding the common solutions of the system of variational inequality problem (1), the mixed equilibrium problem (12), and the hierarchical fixed point problem (15).

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $D, B_i : C \to H$ be η, θ_i-inverse strongly monotone mappings for each $i = 1, 2$, respectively. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the assumptions (i)–(iv) of Lemma 4, $S : C \to H$ a nonexpansive mapping, and $\{T_i\}_{i=1}^{\infty} : C \to C$ a countable family of k_i-strict pseudocontraction mappings such that $S^* \cap \mathrm{MEP}(F) \cap F(T) \neq \emptyset$, where $F(T) := \bigcap_{i=1}^{\infty} F(T_i) = \bigcap_{i=1}^{\infty} F(V_i)$. Let f be a ρ-contraction mapping.

Algorithm 11. For a given $x_0 \in C$ arbitrarily, let the iterative sequences $\{u_n\}, \{x_n\}, \{y_n\}$, and $\{z_n\}$ be generated by

$$\begin{gathered} F(u_n, y) + \langle Dx_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C; \\ z_n = P_C [P_C [u_n - \mu_2 B_2 u_n] - \mu_1 B_1 P_C [u_n - \mu_2 B_2 u_n]]; \\ y_n = P_C [\beta_n S x_n + (1 - \beta_n) z_n]; \\ x_{n+1} = P_C \left[\alpha_n f(x_n) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) V_i y_n \right], \quad \forall n \geq 0, \end{gathered} \tag{29}$$

where $V_i = k_i I + (1 - k_i) T_i$, $0 \leq k_i < 1$, $\mu_i \in (0, 2\theta_i)$ for each $i = 1, 2$, $\{r_n\} \subset (0, 2\eta)$, $\alpha_0 = 1$, $\{\alpha_n\}$ is a strictly decreasing sequence in $(0, 1)$, and $\{\beta_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions:

(a) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(b) $\lim_{n \to \infty} (\beta_n / \alpha_n) = 0$,

(c) $\sum_{n=1}^{\infty} |\alpha_{n-1} - \alpha_n| < \infty$ and $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$,

(d) $\liminf_{n \to \infty} r_n < \limsup_{n \to \infty} r_n < 2\eta$ and $\sum_{n=1}^{\infty} |r_{n-1} - r_n| < \infty$.

Lemma 12. *Let $x^* \in S^* \cap \mathrm{MEP}(F) \cap F(T)$. Then, $\{x_n\}, \{u_n\}, \{z_n\}$, and $\{y_n\}$ are bounded.*

Proof. First, we show that the mapping $(I - r_n D)$ is nonexpansive. For any $x, y \in C$,

$$\begin{aligned} \|(I - r_n D) x - (I - r_n D) y\|^2 &= \|(x - y) - r_n (Dx - Dy)\|^2 \\ &= \|x - y\|^2 - 2r_n \langle x - y, Dx - Dy \rangle + r_n^2 \|Dx - Dy\|^2 \leq \|x - y\|^2 - r_n (2\eta - r_n) \|Dx - Dy\|^2 \\ &\leq \|x - y\|^2. \end{aligned} \tag{30}$$

Similarly, we can show that the mapping $(I - \mu_i B_i)$ is nonexpansive for each $i = 1, 2$. It follows from Lemma 5 that $u_n = T_{r_n}(x_n - r_n D x_n)$. Let $x^* \in S^* \cap \mathrm{MEP}(F) \cap F(T)$; we have $x^* = T_{r_n}(x^* - r_n D x^*)$, and it follows that

$$\begin{aligned} \|u_n - x^*\|^2 &= \|T_{r_n}(x_n - r_n D x_n) - T_{r_n}(x^* - r_n D x^*)\|^2 \\ &\leq \|(x_n - r_n D x_n) - (x^* - r_n D x^*)\|^2 \\ &\leq \|x_n - x^*\|^2 - r_n (2\eta - r_n) \|D x_n - D x^*\|^2 \\ &\leq \|x_n - x^*\|^2. \end{aligned} \tag{31}$$

Let $x^* \in S^* \cap \mathrm{MEP}(F) \cap F(T)$; we have

$$x^* = P_C [y^* - \mu_1 B_1 y^*], \tag{32}$$

where

$$y^* = P_C [x^* - \mu_2 B_2 x^*]. \tag{33}$$

Setting $v_n = P_C[u_n - \mu_2 B_2 u_n]$. Since B_2 is θ_2-inverse strongly monotone mapping, it follows that

$$\begin{aligned} \|v_n - y^*\|^2 &= \|P_C [u_n - \mu_2 B_2 u_n] - P_C [x^* - \mu_2 B_2 x^*]\|^2 \\ &\leq \|u_n - u^* - \mu_2 (B_2 u_n - B_2 x^*)\|^2 \\ &\leq \|x_n - x^*\|^2 - \mu_2 (2\theta - \mu_2) \|B_2 u_n - B_2 x^*\|^2 \\ &\leq \|x_n - x^*\|^2. \end{aligned} \tag{34}$$

Since B_i is θ_i-inverse strongly monotone mappings for each $i = 1, 2$, we get

$$\begin{aligned} &\|z_n - x^*\|^2 \\ &= \| P_C [P_C [u_n - \mu_2 B_2 u_n] - \mu_1 B_1 P_C [u_n - \mu_2 B_2 u_n]] - P_C [P_C [x^* - \mu_2 B_2 x^*] - \mu_1 B_1 P_C [x^* - \mu_2 B_2 x^*]]\|^2 \\ &\leq \|P_C [u_n - \mu_2 B_2 u_n] - \mu_1 B_1 P_C [u_n - \mu_2 B_2 u_n] - (P_C[x^* - \mu_2 B_2 x^*] - \mu_1 B_1 P_C[x^* - \mu_2 B_2 x^*])\|^2 \\ &= \|P_C [u_n - \mu_2 B_2 u_n] - P_C [x^* - \mu_2 B_2 x^*] - \mu_1 (B_1 P_C[u_n - \mu_2 B_2 u_n] - B_1 P_C[x^* - \mu_2 B_2 x^*])\|^2 \\ &\leq \|P_C [u_n - \mu_2 B_2 u_n] - P_C [x^* - \mu_2 B_2 x^*]\|^2 - \mu_1 (2\theta_1 - \mu_1) \|B_1 P_C [u_n - \mu_2 B_2 u_n] - B_1 P_C[x^* - \mu_2 B_2 x^*]\|^2 \end{aligned}$$

$$\begin{aligned}
&\leq \|(u_n - \mu_2 B_2 u_n) - (x^* - \mu_2 B_2 x^*)\|^2 \\
&\quad - \mu_1 (2\theta_1 - \mu_1) \|B_1 P_C [u_n - \mu_2 B_2 u_n] \\
&\qquad - B_1 P_C [x^* - \mu_2 B_2 x^*]\|^2 \\
&\leq \|u_n - x^*\|^2 - \mu_2 (2\theta_2 - \mu_2) \|B_2 u_n - B_2 x^*\|^2 \\
&\quad - \mu_1 (2\theta_1 - \mu_1) \|B_1 v_n - B_1 y^*\|^2 \\
&\leq \|u_n - x^*\|^2 \leq \|x_n - x^*\|^2.
\end{aligned} \tag{35}$$

Next, we prove that the sequence $\{x_n\}$ is bounded; without loss of generality we can assume that $\beta_n \leq \alpha_n$ for all $n \geq 1$. From (29), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq \left\| \alpha_n f(x_n) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) V_i y_n \right. \\
&\qquad \left. - \alpha_n x^* - \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) V_i x^* \right\| \\
&\leq \alpha_n \|f(x_n) - f(x^*)\| + \alpha_n \|f(x^*) - x^*\| \\
&\quad + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \|V_i y_n - V_i x^*\| \\
&\leq \alpha_n \|f(x_n) - f(x^*)\| + \alpha_n \|f(x^*) - x^*\| \\
&\quad + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \|y_n - x^*\| \\
&\leq \alpha_n \|f(x_n) - f(x^*)\| + \alpha_n \|f(x^*) - x^*\| \\
&\quad + (1 - \alpha_n) \|\beta_n S x_n + (1 - \beta_n) z_n - x^*\| \\
&\leq \alpha_n \|f(x_n) - f(x^*)\| + \alpha_n \|f(x^*) - x^*\| \\
&\quad + (1 - \alpha_n) (\beta_n \|S x_n - S x^*\| + \beta_n \|S x^* - x^*\| \\
&\qquad + (1 - \beta_n) \|z_n - x^*\|) \\
&\leq \alpha_n \rho \|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| \\
&\quad + (1 - \alpha_n) (\beta_n \|x_n - x^*\| + \beta_n \|S x^* - x^*\| \\
&\qquad + (1 - \beta_n) \|x_n - x^*\|) \\
&= (1 - \alpha_n (1 - \rho)) \\
&\quad \times \|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| \\
&\quad + (1 - \alpha_n) \beta_n \|S x^* - x^*\| \leq (1 - \alpha_n (1 - \rho)) \\
&\quad \times \|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| \\
&\quad + \beta_n \|S x^* - x^*\| \\
&\leq (1 - \alpha_n (1 - \rho)) \|x_n - x^*\| \\
&\quad + \alpha_n (\|f(x^*) - x^*\| + \|S x^* - x^*\|) \\
&= (1 - \alpha_n (1 - \rho)) \|x_n - x^*\| + \frac{\alpha_n (1 - \rho)}{1 - \rho} \\
&\quad \times (\|f(x^*) - x^*\| + \|S x^* - x^*\|) \\
&\leq \max \left\{ \|x_n - x^*\|, \frac{1}{1 - \rho} (\|f(x^*) - x^*\| \right. \\
&\qquad \left. + \|S x^* - x^*\|) \right\}.
\end{aligned} \tag{36}$$

By induction on n, we obtain $\|x_n - x^*\| \leq \max\{\|x_0 - x^*\|, (1/(1-\rho))(\|f(x^*) - x^*\| + \|Sx^* - x^*\|)\}$, for $n \geq 0$ and $x_0 \in C$. Hence, $\{x_n\}$ is bounded and, consequently, we deduce that $\{u_n\}, \{z_n\}, \{v_n\}$, and $\{y_n\}$ are bounded. □

Lemma 13. *Let $x^* \in S^* \cap \mathrm{MEP}(F) \cap F(T)$ and $\{x_n\}$ be the sequence generated by Algorithm 11. Then one has*

(a) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

(b) *The weak w-limit set $w_w(x_n) \subset F(T)$, $(w_w(x_n) = \{x : x_{n_i} \rightharpoonup x\})$.*

Proof. Since $u_n = T_{r_n}(x_n - r_n D x_n)$ and $u_{n-1} = T_{r_{n-1}}(x_{n-1} - r_{n-1} D x_{n-1})$, we have

$$F(u_n, y) + \langle D x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \tag{37}$$

$$\begin{aligned}
&F(u_{n-1}, y) + \langle D x_{n-1}, y - u_{n-1} \rangle \\
&\quad + \frac{1}{r_{n-1}} \langle y - u_{n-1}, u_{n-1} - x_{n-1} \rangle \geq 0, \\
&\qquad \forall y \in C.
\end{aligned} \tag{38}$$

Take $y = u_{n-1}$ in (37) and $y = u_n$ in (38), we get

$$\begin{aligned}
&F(u_n, u_{n-1}) + \langle D x_n, u_{n-1} - u_n \rangle \\
&\quad + \frac{1}{r_n} \langle u_{n-1} - u_n, u_n - x_n \rangle \geq 0, \\
&F(u_{n-1}, u_n) + \langle D x_{n-1}, u_n - u_{n-1} \rangle \\
&\quad + \frac{1}{r_{n-1}} \langle u_n - u_{n-1}, u_{n-1} - x_{n-1} \rangle \geq 0.
\end{aligned} \tag{39}$$

Adding (39) and using the monotonicity of F, we have

$$\begin{aligned}
&\langle D x_{n-1} - D x_n, u_n - u_{n-1} \rangle \\
&\quad + \left\langle u_n - u_{n-1}, \frac{u_{n-1} - x_{n-1}}{r_{n-1}} - \frac{u_n - x_n}{r_n} \right\rangle \geq 0,
\end{aligned} \tag{40}$$

which implies that

$$\begin{aligned}
0 \leq & \left\langle u_n - u_{n-1}, r_n\left(Dx_{n-1} - Dx_n\right) + \frac{r_n}{r_{n-1}}\left(u_{n-1} - x_{n-1}\right) \right. \\
& \left. - \left(u_n - x_n\right) \right\rangle \\
= & \left\langle u_{n-1} - u_n, u_n - u_{n-1} \right. \\
& + \left(1 - \frac{r_n}{r_{n-1}}\right) u_{n-1} + \left(x_{n-1} - r_n Dx_{n-1}\right) \\
& \left. - \left(x_n - r_n Dx_n\right) - x_{n-1} + \frac{r_n}{r_{n-1}} x_{n-1} \right\rangle \\
= & \left\langle u_{n-1} - u_n, \left(1 - \frac{r_n}{r_{n-1}}\right) u_{n-1} + \left(x_{n-1} - r_n Dx_{n-1}\right) \right. \\
& \left. - \left(x_n - r_n Dx_n\right) - x_{n-1} + \frac{r_n}{r_{n-1}} x_{n-1} \right\rangle - \left\|u_n - u_{n-1}\right\|^2 \\
= & \left\langle u_{n-1} - u_n, \left(1 - \frac{r_n}{r_{n-1}}\right)\left(u_{n-1} - x_{n-1}\right) \right. \\
& \left. + \left(x_{n-1} - r_n Dx_{n-1}\right) - \left(x_n - r_n Dx_n\right) \right\rangle \\
& - \left\|u_n - u_{n-1}\right\|^2 \\
\leq & \left\|u_{n-1} - u_n\right\| \\
& \times \left\{ \left|1 - \frac{r_n}{r_{n-1}}\right| \left\|u_{n-1} - x_{n-1}\right\| \right. \\
& \left. + \left\|\left(x_{n-1} - r_n Dx_{n-1}\right) - \left(x_n - r_n Dx_n\right)\right\| \right\} \\
& - \left\|u_n - u_{n-1}\right\|^2 \\
\leq & \left\|u_{n-1} - u_n\right\| \\
& \times \left\{ \left|1 - \frac{r_n}{r_{n-1}}\right| \left\|u_{n-1} - x_{n-1}\right\| + \left\|x_{n-1} - x_n\right\| \right\} \\
& - \left\|u_n - u_{n-1}\right\|^2
\end{aligned} \tag{41}$$

and then

$$\begin{aligned}
\left\|u_{n-1} - u_n\right\| \leq & \left|1 - \frac{r_n}{r_{n-1}}\right| \left\|u_{n-1} - x_{n-1}\right\| \\
& + \left\|x_{n-1} - x_n\right\|.
\end{aligned} \tag{42}$$

Without loss of generality, let us assume that there exists a real number μ such that $r_n > \mu > 0$, for all positive integers n. Then, we get

$$\begin{aligned}
\left\|u_{n-1} - u_n\right\| \leq & \left\|x_{n-1} - x_n\right\| \\
& + \frac{1}{\mu}\left|r_{n-1} - r_n\right| \left\|u_{n-1} - x_{n-1}\right\|.
\end{aligned} \tag{43}$$

Next, we estimate

$$\begin{aligned}
& \left\|z_n - z_{n-1}\right\|^2 \\
& = \left\| P_C\left[P_C\left[u_n - \mu_2 B_2 u_n\right] - \mu_1 B_1 P_C\left[u_n - \mu_2 B_2 u_n\right]\right] \right. \\
& \quad - P_C\left[P_C\left[u_{n-1} - \mu_2 B_2 u_{n-1}\right] \right. \\
& \quad \left.\left. - \mu_1 B_1 P_C\left[u_{n-1} - \mu_2 B_2 u_{n-1}\right]\right]\right\|^2 \\
& \leq \left\| P_C\left[u_n - \mu_2 B_2 u_n\right] - \mu_1 B_1 P_C\left[u_n - \mu_2 B_2 u_n\right] \right. \\
& \quad - \left(P_C\left[u_{n-1} - \mu_2 B_2 u_{n-1}\right] \right. \\
& \quad \left.\left. - \mu_1 B_1 P_C\left[u_{n-1} - \mu_2 B_2 u_{n-1}\right]\right)\right\|^2 \\
& = \left\| P_C\left[u_n - \mu_2 B_2 u_n\right] - P_C\left[u_{n-1} - \mu_2 B_2 u_{n-1}\right] \right. \\
& \quad - \mu_1\left(B_1 P_C\left[u_n - \mu_2 B_2 u_n\right] \right. \\
& \quad \left.\left. - B_1 P_C\left[u_{n-1} - \mu_2 B_2 u_{n-1}\right]\right)\right\|^2 \\
& \leq \left\| P_C\left[u_n - \mu_2 B_2 u_n\right] - P_C\left[u_{n-1} - \mu_2 B_2 u_{n-1}\right]\right\|^2 \\
& \quad - \mu_1\left(2\theta_1 - \mu_1\right) \\
& \quad \times \left\| B_1 P_C\left[u_n - \mu_2 B_2 u_n\right] \right. \\
& \quad \left. - B_1 P_C\left[u_{n-1} - \mu_2 B_2 u_{n-1}\right]\right\|^2 \\
& \leq \left\| P_C\left[u_n - \mu_2 B_2 u_n\right] - P_C\left[u_{n-1} - \mu_2 B_2 u_{n-1}\right]\right\|^2 \\
& \leq \left\|\left(u_n - u_{n-1}\right) - \mu_2\left(B_2 u_n - B_2 u_{n-1}\right)\right\|^2 \\
& \leq \left\|u_n - u_{n-1}\right\|^2 - \mu_2\left(2\theta_2 - \mu_2\right)\left\|B_2 u_n - B_2 u_{n-1}\right\|^2 \\
& \leq \left\|u_n - u_{n-1}\right\|^2.
\end{aligned} \tag{44}$$

It follows from (43) and (44) that

$$\begin{aligned}
\left\|z_n - z_{n-1}\right\| \leq & \left\|x_{n-1} - x_n\right\| \\
& + \frac{1}{\mu}\left|r_{n-1} - r_n\right| \left\|u_{n-1} - x_{n-1}\right\|.
\end{aligned} \tag{45}$$

From (29) and the previous inequality, we get

$$\begin{aligned}
& \left\|y_n - y_{n-1}\right\| \\
& \leq \left\|\beta_n S x_n + \left(1 - \beta_n\right) z_n - \left(\beta_{n-1} S x_{n-1} + \left(1 - \beta_{n-1}\right) z_{n-1}\right)\right\| \\
& = \left\|\beta_n\left(S x_n - S x_{n-1}\right) + \left(\beta_n - \beta_{n-1}\right) S x_{n-1} \right. \\
& \quad \left. + \left(1 - \beta_n\right)\left(z_n - z_{n-1}\right) + \left(\beta_{n-1} - \beta_n\right) z_{n-1}\right\| \\
& \leq \beta_n\left\|x_n - x_{n-1}\right\| + \left(1 - \beta_n\right)\left\|z_n - z_{n-1}\right\| \\
& \quad + \left|\beta_n - \beta_{n-1}\right|\left(\left\|S x_{n-1}\right\| + \left\|z_{n-1}\right\|\right) \\
& \leq \beta_n\left\|x_n - x_{n-1}\right\| + \left(1 - \beta_n\right)
\end{aligned}$$

$$\begin{aligned}
&\times\left\{\|x_{n-1}-x_n\|+\frac{1}{\mu}|r_{n-1}-r_n|\|u_{n-1}-x_{n-1}\|\right\}\\
&+|\beta_n-\beta_{n-1}|(\|Sx_{n-1}\|+\|z_{n-1}\|)\\
\leq{}&\|x_n-x_{n-1}\|+\frac{1}{\mu}|r_{n-1}-r_n|\|u_{n-1}-x_{n-1}\|\\
&+|\beta_n-\beta_{n-1}|(\|Sx_{n-1}\|+\|z_{n-1}\|).
\end{aligned}\tag{46}$$

Next, we estimate

$$\begin{aligned}
\|x_{n+1}-x_n\|\leq{}&\left\|\alpha_n f(x_n)+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_i y_n\right.\\
&\left.-\left(\alpha_{n-1}f(x_{n-1})+\sum_{i=1}^{n-1}(\alpha_{i-1}-\alpha_i)V_i y_{n-1}\right)\right\|\\
={}&\|\alpha_n(f(x_n)-f(x_{n-1}))+(\alpha_n-\alpha_{n-1})f(x_{n-1})\\
&+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(V_i y_n-V_i y_{n-1})\\
&+(\alpha_{n-1}-\alpha_n)V_n y_{n-1}\|\\
\leq{}&\alpha_n\|f(x_n)-f(x_{n-1})\|\\
&+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|V_i y_n-V_i y_{n-1}\|\\
&+|\alpha_n-\alpha_{n-1}|(\|f(x_{n-1})\|+\|V_n y_{n-1}\|)\\
\leq{}&\alpha_n\rho\|x_n-x_{n-1}\|+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|y_n-y_{n-1}\|\\
&+|\alpha_n-\alpha_{n-1}|(\|f(x_{n-1})\|+\|V_n y_{n-1}\|)\\
={}&\alpha_n\rho\|x_n-x_{n-1}\|+(1-\alpha_n)\|y_n-y_{n-1}\|\\
&+|\alpha_n-\alpha_{n-1}|(\|f(x_{n-1})\|+\|V_n y_{n-1}\|).
\end{aligned}\tag{47}$$

From (46) and (47), we have

$$\begin{aligned}
\|x_{n+1}-x_n\|\leq{}&\alpha_n\rho\|x_n-x_{n-1}\|+(1-\alpha_n)\\
&\times\left\{\|x_n-x_{n-1}\|+\frac{1}{\mu}|r_{n-1}-r_n|\|u_{n-1}-x_{n-1}\|\right.\\
&\left.+|\beta_n-\beta_{n-1}|(\|Sx_{n-1}\|+\|z_{n-1}\|)\right\}\\
&+|\alpha_n-\alpha_{n-1}|(\|f(x_{n-1})\|+\|V_n y_{n-1}\|)\\
\leq{}&(1-(1-\rho)\alpha_n)\|x_n-x_{n-1}\|+\frac{1}{\mu}|r_{n-1}-r_n|\\
&\times\|u_{n-1}-x_{n-1}\|\\
&+|\beta_n-\beta_{n-1}|(\|Sx_{n-1}\|+\|z_{n-1}\|)\\
&+|\alpha_n-\alpha_{n-1}|(\|f(x_{n-1})\|+\|V_n y_{n-1}\|)\\
\leq{}&(1-(1-\rho)\alpha_n)\|x_n-x_{n-1}\|+M\\
&\times\left(\frac{1}{\mu}|r_{n-1}-r_n|+|\beta_n-\beta_{n-1}|+|\alpha_n-\alpha_{n-1}|\right),
\end{aligned}\tag{48}$$

where

$$\begin{aligned}
M=\max\Big\{&\sup_{n\geq1}\|u_{n-1}-x_{n-1}\|,\sup_{n\geq1}(\|Sx_{n-1}\|+\|z_{n-1}\|),\\
&\sup_{n\geq1}(\|f(x_{n-1})\|+\|V_n y_{n-1}\|)\Big\}.
\end{aligned}\tag{49}$$

It follows by conditions (a)–(d) of Algorithm 11 and Lemma 7 that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.\tag{50}$$

Next, we show that $\lim_{n\to\infty}\|u_n-x_n\|=0$. Since $x^*\in S^*\cap \mathrm{MEP}(F)\cap F(T)$ and $\alpha_n+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)=1$, by using (31) and (35), we obtain

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&\leq\left\|\alpha_n f(x_n)+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_i y_n-\alpha_n x^*\right.\\
&\qquad\left.-\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_i x^*\right\|^2\\
&\leq\alpha_n\|f(x_n)-x^*\|^2+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|V_i y_n-V_i x^*\|^2\\
&\leq\alpha_n\|f(x_n)-x^*\|^2+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|y_n-x^*\|^2\\
&\leq\alpha_n\|f(x_n)-x^*\|^2+(1-\alpha_n)\\
&\quad\times\left(\beta_n\|Sx_n-x^*\|^2+(1-\beta_n)\|z_n-x^*\|^2\right)\\
&\leq\alpha_n\|f(x_n)-x^*\|^2+(1-\alpha_n)\beta_n\|Sx_n-x^*\|^2\\
&\quad+(1-\alpha_n)(1-\beta_n)\\
&\quad\times\left\{\|u_n-x^*\|^2-\mu_2(2\theta_2-\mu_2)\right.\\
&\qquad\left.\times\|B_2u_n-B_2x^*\|^2-\mu_1(2\theta_1-\mu_1)\|B_1v_n-B_1y^*\|^2\right\}\\
&\leq\alpha_n\|f(x_n)-x^*\|^2+\beta_n\|Sx_n-x^*\|^2\\
&\quad+(1-\alpha_n)(1-\beta_n)\\
&\quad\times\left\{\|x_n-x^*\|^2-r_n(2\eta-r_n)\|Dx_n-Dx^*\|^2\right.\\
&\qquad-\mu_2(2\theta_2-\mu_2)\|B_2u_n-B_2x^*\|^2\\
&\qquad\left.-\mu_1(2\theta_1-\mu_1)\|B_1v_n-B_1y^*\|^2\right\}
\end{aligned}$$

$$\begin{aligned}&\le \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|Sx_n-x^*\|^2+\|x_n-x^*\|^2\\&\quad-(1-\alpha_n)(1-\beta_n)\{r_n(2\eta-r_n)\|Dx_n-Dx^*\|^2\\&\qquad+\mu_2(2\theta_2-\mu_2)\|B_2u_n-B_2x^*\|^2\\&\qquad+\mu_1(2\theta_1-\mu_1)\|B_1v_n-B_1y^*\|^2\}.\end{aligned}\tag{51}$$

Then, from the previous inequality, we get

$$\begin{aligned}&(1-\alpha_n)(1-\beta_n)\{r_n(2\eta-r_n)\|Dx_n-Dx^*\|^2\\&\qquad+\mu_2(2\theta_2-\mu_2)\|B_2u_n-B_2x^*\|^2\\&\qquad+\mu_1(2\theta_1-\mu_1)\|B_1v_n-B_1y^*\|^2\}\\&\quad\le \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|Sx_n-x^*\|^2\\&\qquad+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\&\quad\le \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|Sx_n-x^*\|^2\\&\qquad+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|.\end{aligned}\tag{52}$$

Since $\liminf_{n\to\infty}r_n\le\limsup_{n\to\infty}r_n<2\eta, \lim_{n\to\infty}\|x_{n+1}-x_n\|=0, \alpha_n\to 0$, and $\beta_n\to 0$, we obtain $\lim_{n\to\infty}\|B_2u_n-B_2x^*\|=0, \lim_{n\to\infty}\|B_1v_n-B_1y^*\|=0$, and $\lim_{n\to\infty}\|Dx_n-Dx^*\|=0$.

Since T_{r_n} is firmly nonexpansive, we have

$$\begin{aligned}\|u_n-x^*\|^2&=\|T_{r_n}(x_n-r_nDx_n)-T_{r_n}(x^*-r_nDx^*)\|^2\\&\le\langle u_n-x^*,(x_n-r_nDx_n)-(x^*-r_nDx^*)\rangle\\&=\frac{1}{2}\{\|u_n-x^*\|^2\\&\quad+\|(x_n-r_nDx_n)-(x^*-r_nDx^*)\|^2\\&\quad-\|u_n-x^*\\&\quad-[(x_n-r_nDx_n)-(x^*-r_nDx^*)]\|^2\}.\end{aligned}\tag{53}$$

Hence,

$$\begin{aligned}\|u_n-x^*\|^2&\le\|(x_n-r_nDx_n)-(x^*-r_nDx^*)\|^2\\&\quad-\|u_n-x_n+r_n(Dx_n-Dx^*)\|^2\\&\le\|x_n-x^*\|^2-\|u_n-x_n+r_n(Dx_n-Dx^*)\|^2\\&\le\|x_n-x^*\|^2-\|u_n-x_n\|^2\\&\quad+2r_n\|u_n-x_n\|\|Dx_n-Dx^*\|.\end{aligned}\tag{54}$$

From (51), (35), and the previous inequality, we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2&\le\alpha_n\|f(x_n)-x^*\|^2\\&\quad+(1-\alpha_n)\big(\beta_n\|Sx_n-x^*\|^2\\&\qquad+(1-\beta_n)\|z_n-x^*\|^2\big)\\&\le\alpha_n\|f(x_n)-x^*\|^2+(1-\alpha_n)\\&\quad\times\big(\beta_n\|Sx_n-x^*\|^2+(1-\beta_n)\|u_n-x^*\|^2\big)\\&\le\alpha_n\|f(x_n)-x^*\|^2+(1-\alpha_n)\\&\quad\times\{\beta_n\|Sx_n-x^*\|^2+(1-\beta_n)\\&\qquad\times(\|x_n-x^*\|^2-\|u_n-x_n\|^2\\&\qquad+2r_n\|u_n-x_n\|\|Dx_n-Dx^*\|)\}\\&\le\alpha_n\|f(x_n)-x^*\|^2+\beta_n\|Sx_n-x^*\|^2\\&\quad+\|x_n-x^*\|^2-(1-\alpha_n)(1-\beta_n)\|u_n-x_n\|^2\\&\quad+2r_n\|u_n-x_n\|\|Dx_n-Dx^*\|.\end{aligned}\tag{55}$$

Hence,

$$\begin{aligned}(1-\alpha_n)(1-\beta_n)\|u_n-x_n\|^2&\le\alpha_n\|f(x_n)-x^*\|^2\\&\quad+\beta_n\|Sx_n-x^*\|^2+\|x_n-x^*\|^2\\&\quad-\|x_{n+1}-x^*\|^2\\&\quad+2r_n\|u_n-x_n\|\|Dx_n-Dx^*\|\\&\le\alpha_n\|f(x_n)-x^*\|^2+\beta_n\|Sx_n-x^*\|^2\\&\quad+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|\\&\quad+2r_n\|u_n-x_n\|\|Dx_n-Dx^*\|.\end{aligned}\tag{56}$$

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $\alpha_n\to 0$, $\beta_n\to 0$, and $\lim_{n\to\infty}\|Dx_n-Dx^*\|=0$, we obtain

$$\lim_{n\to\infty}\|u_n-x_n\|=0.\tag{57}$$

From (21), we get

$$\begin{aligned}&\|v_n-y^*\|^2\\&\quad=\|P_C[u_n-\mu_2B_2u_n]-P_C[x^*-\mu_2B_2x^*]\|^2\\&\quad\le\langle v_n-y^*,(u_n-\mu_2B_2u_n)-(x^*-\mu_2B_2x^*)\rangle\\&\quad=\frac{1}{2}\{\|v_n-y^*\|^2+\|u_n-x^*-\mu_2(B_2u_n-B_2x^*)\|^2\\&\qquad-\|u_n-x^*-\mu_2(B_2u_n-B_2x^*)-(v_n-y^*)\|^2\}\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2}\left\{\|v_n - y^*\|^2 + \|u_n - x^*\|^2\right.\\
&\quad - \mu_2\left(2\theta_2 - \mu_2\right)\|B_2u_n - B_2x^*\|^2\\
&\quad - \|u_n - x^* - \mu_2\left(B_2u_n - B_2x^*\right)\\
&\quad \left. -(v_n - y^*)\|^2\right\}\\
&\leq \frac{1}{2}\left\{\|v_n - y^*\|^2 + \|u_n - x^*\|^2\right.\\
&\quad \left. -\|u_n - v_n - \mu_2\left(B_2u_n - B_2x^*\right) - (x^* - y^*)\|^2\right\}\\
&\leq \frac{1}{2}\left\{\|v_n - y^*\|^2 + \|u_n - x^*\|^2\right.\\
&\quad - \| u_n - v_n - (x^* - y^*)\|^2\\
&\quad + 2\mu_2\left\langle u_n - v_n - (x^* - y^*), B_2u_n - B_2x^*\right\rangle\\
&\quad \left. -\mu_2^2\|B_2u_n - B_2x^*\|^2\right\}\\
&\leq \frac{1}{2}\left\{\|v_n - y^*\|^2 + \|u_n - x^*\|^2\right.\\
&\quad - \|u_n - v_n - (x^* - y^*)\|^2\\
&\quad \left. +2\mu_2\|u_n - v_n - (x^* - y^*)\|\|B_2u_n - B_2x^*\|\right\}.
\end{aligned} \tag{58}$$

Hence,

$$\begin{aligned}
\|v_n - y^*\|^2 &\leq \|u_n - x^*\|^2 - \|u_n - v_n - (x^* - y^*)\|^2\\
&\quad + 2\mu_2\|u_n - v_n - (x^* - y^*)\|\|B_2u_n - B_2x^*\|\\
&\leq \|x_n - x^*\|^2 - \|u_n - x_n\|^2\\
&\quad + 2r_n\|u_n - x_n\|\|Dx_n - Dx^*\|\\
&\quad - \|u_n - v_n - (x^* - y^*)\|^2\\
&\quad + 2\mu_2\|u_n - v_n - (x^* - y^*)\|\|B_2u_n - B_2x^*\|,
\end{aligned} \tag{59}$$

where the last inequality follows from (54). On the other hand, from (29) and (21), we obtain

$$\begin{aligned}
\|z_n - x^*\|^2 &= \|P_C[v_n - \mu_1B_1v_n] - P_C[y^* - \mu_1B_1y^*]\|^2\\
&\leq \left\langle z_n - x^*, (v_n - \mu_1B_1v_n) - (y^* - \mu_1B_1y^*)\right\rangle\\
&= \frac{1}{2}\left\{\|z_n - x^*\|^2 + \|v_n - y^* - \mu_1(B_1v_n - B_1y^*)\|^2\right.\\
&\quad - \|v_n - y^* - \mu_1\left(B_1v_n - B_1y^*\right)\\
&\quad \left. - (z_n - x^*)\|^2\right\}\\
&= \frac{1}{2}\left\{\|z_n - x^*\|^2 + \|v_n - y^*\|^2\right.\\
&\quad - 2\mu_1\left\langle v_n - y^*, B_1v_n - B_1y^*\right\rangle\\
&\quad + \mu_1^2\|B_1v_n - B_1y^*\|^2\\
&\quad - \|v_n - y^* - \mu_1\left(B_1v_n - B_1y^*\right)\\
&\quad \left. - (z_n - x^*)\|^2\right\}\\
&\leq \frac{1}{2}\left\{\|z_n - x^*\|^2 + \|v_n - y^*\|^2\right.\\
&\quad - \mu_1\left(2\theta_1 - \mu_1\right)\|B_1v_n - B_1y^*\|^2\\
&\quad - \|v_n - y^* - \mu_1\left(B_1v_n - B_1y^*\right)\\
&\quad \left. - (z_n - x^*)\|^2\right\}\\
&\leq \frac{1}{2}\left\{\|z_n - x^*\|^2 + \|v_n - y^*\|^2\right.\\
&\quad - \|v_n - z_n - \mu_1\left(B_1v_n - B_1y^*\right)\\
&\quad \left. + (x^* - y^*)\|^2\right\}\\
&\leq \frac{1}{2}\left\{\|z_n - x^*\|^2 + \|v_n - y^*\|^2\right.\\
&\quad - \|v_n - z_n + (x^* - y^*)\|^2\\
&\quad \left. +2\mu_1\left\langle v_n - z_n + (x^* - y^*), B_1v_n - B_1y^*\right\rangle\right\}\\
&\leq \frac{1}{2}\left\{\|z_n - x^*\|^2 + \|v_n - y^*\|^2\right.\\
&\quad - \|v_n - z_n + (x^* - y^*)\|^2\\
&\quad + 2\mu_1\|v_n - z_n + (x^* - y^*)\|\\
&\quad \left. \times\|B_1v_n - B_1y^*\|\right\}.
\end{aligned} \tag{60}$$

Hence,

$$\begin{aligned}
\|z_n - x^*\|^2 &\leq \|v_n - y^*\|^2 - \|v_n - z_n + (x^* - y^*)\|^2\\
&\quad + 2\mu_1\|v_n - z_n + (x^* - y^*)\|\|B_1v_n - B_1y^*\|\\
&\leq \|x_n - x^*\|^2 - \|u_n - x_n\|^2\\
&\quad + 2r_n\|u_n - x_n\|\|Dx_n - Dx^*\|\\
&\quad - \|u_n - v_n - (x^* - y^*)\|^2\\
&\quad + 2\mu_2\|u_n - v_n - (x^* - y^*)\|\|B_2u_n - B_2x^*\|\\
&\quad - \|v_n - z_n + (x^* - y^*)\|^2\\
&\quad + 2\mu_1\|v_n - z_n + (x^* - y^*)\|\|B_1v_n - B_1y^*\|,
\end{aligned} \tag{61}$$

where the last inequality follows from (59). From (51) and the previous inequality, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \alpha_n\|f(x_n) - x^*\|^2 + (1 - \alpha_n)\\
&\quad \times\left(\beta_n\|Sx_n - x^*\|^2 + (1 - \beta_n)\|z_n - x^*\|^2\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n \left\| f\left(x_n\right) - x^* \right\|^2 + \left(1-\alpha_n\right) \\
&\quad \times \left\{ \beta_n \left\| Sx_n - x^* \right\|^2 \right. \\
&\quad + \left(1-\beta_n\right) \left(\left\| x_n - x^* \right\|^2 - \left\| u_n - x_n \right\|^2 \right. \\
&\qquad \left. + 2r_n \left\| u_n - x_n \right\| \left\| Dx_n - Dx^* \right\| \right) \\
&\quad + \left(1-\beta_n\right) \left(- \left\| u_n - v_n - \left(x^* - y^*\right) \right\|^2 \right. \\
&\qquad + 2\mu_2 \left\| u_n - v_n - \left(x^* - y^*\right) \right\| \\
&\qquad \left. \times \left\| B_2 u_n - B_2 x^* \right\| \right) \\
&\quad + \left(1-\beta_n\right) \left(- \left\| v_n - z_n + \left(x^* - y^*\right) \right\|^2 \right. \\
&\qquad + 2\mu_1 \left\| v_n - z_n + \left(x^* - y^*\right) \right\| \\
&\qquad \left. \left. \times \left\| B_1 v_n - B_1 y^* \right\| \right) \right\} \\
&\leq \alpha_n \left\| f(x_n) - x^* \right\|^2 + \beta_n \left\| Sx_n - x^* \right\|^2 \\
&\quad + \left\| x_n - x^* \right\|^2 + 2r_n \left\| u_n - x_n \right\| \\
&\quad \times \left\| Dx_n - Dx^* \right\| \\
&\quad + 2\mu_2 \left\| u_n - v_n - \left(x^* - y^*\right) \right\| \\
&\quad \times \left\| B_2 u_n - B_2 x^* \right\| + 2\mu_1 \left\| v_n - z_n + \left(x^* - y^*\right) \right\| \\
&\quad \times \left\| B_1 v_n - B_1 y^* \right\| - \left(1-\alpha_n\right)\left(1-\beta_n\right) \\
&\quad \times \left\{ \left\| u_n - x_n \right\|^2 + \left\| u_n - v_n - \left(x^* - y^*\right) \right\|^2 \right. \\
&\qquad \left. + \left\| v_n - z_n + \left(x^* - y^*\right) \right\|^2 \right\}.
\end{aligned} \tag{62}$$

Hence,

$$\begin{aligned}
&\left(1-\alpha_n\right)\left(1-\beta_n\right) \\
&\quad \times \left\{ \left\| u_n - x_n \right\|^2 + \left\| u_n - v_n - \left(x^* - y^*\right) \right\|^2 \right. \\
&\qquad \left. + \left\| v_n - z_n + \left(x^* - y^*\right) \right\|^2 \right\} \\
&\leq \alpha_n \left\| f(x_n) - x^* \right\|^2 + \beta_n \left\| Sx_n - x^* \right\|^2 \\
&\quad + \left\| x_n - x^* \right\|^2 - \left\| x_{n+1} - x^* \right\|^2 \\
&\quad + 2r_n \left\| u_n - x_n \right\| \left\| Dx_n - Dx^* \right\| \\
&\quad + 2\mu_2 \left\| u_n - v_n - \left(x^* - y^*\right) \right\| \left\| B_2 u_n - B_2 x^* \right\| \\
&\quad + 2\mu_1 \left\| v_n - z_n + \left(x^* - y^*\right) \right\| \left\| B_1 v_n - B_1 y^* \right\| \\
&\leq \alpha_n \left\| f(x_n) - x^* \right\|^2 + \beta_n \left\| Sx_n - x^* \right\|^2 \\
&\quad + \left(\left\| x_n - x^* \right\| + \left\| x_{n+1} - x^* \right\| \right) \left\| x_{n+1} - x_n \right\| \\
&\quad + 2r_n \left\| u_n - x_n \right\| \left\| Dx_n - Dx^* \right\| \\
&\quad + 2\mu_2 \left\| u_n - v_n - \left(x^* - y^*\right) \right\| \left\| B_2 u_n - B_2 x^* \right\| \\
&\quad + 2\mu_1 \left\| v_n - z_n + \left(x^* - y^*\right) \right\| \left\| B_1 v_n - B_1 y^* \right\|.
\end{aligned} \tag{63}$$

Since $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, $\alpha_n \to 0, \beta_n \to 0$, and $\lim_{n\to\infty}\|Dx_n - Dx^*\| = 0$, $\lim_{n\to\infty}\|B_2 u_n - B_2 x^*\| = 0$, $\lim_{n\to\infty}\|B_1 v_n - B_1 y^*\| = 0$, we obtain

$$\begin{gathered}
\lim_{n\to\infty} \left\| u_n - v_n - \left(x^* - y^*\right) \right\| = 0, \\
\lim_{n\to\infty} \left\| v_n - z_n + \left(x^* - y^*\right) \right\| = 0.
\end{gathered} \tag{64}$$

Since

$$\begin{aligned}
\left\| u_n - z_n \right\| \leq &\left\| u_n - v_n - \left(x^* - y^*\right) \right\| \\
&+ \left\| v_n - z_n + \left(x^* - y^*\right) \right\|,
\end{aligned} \tag{65}$$

we get

$$\lim_{n\to\infty} \left\| u_n - z_n \right\| = 0. \tag{66}$$

It follows from (57) and (66) that

$$\lim_{n\to\infty} \left\| x_n - z_n \right\| = 0. \tag{67}$$

Now, let $z \in F(T) \cap S^* \cap \mathrm{MEP}(F)$; since for each $i \geq 1, V_i x_n \in C$, and $\alpha_n + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i) = 1$, we have $\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)V_i x_n + \alpha_n z \in C$. And

$$\begin{aligned}
&\sum_{i=1}^{n} \left(\alpha_{i-1} - \alpha_i\right)\left(x_n - V_i x_n\right) \\
&\quad = P_C \left[\alpha_n f\left(x_n\right) + \sum_{i=1}^{n} \left(\alpha_{i-1} - \alpha_i\right) V_i y_n \right] + \left(1-\alpha_n\right) x_n \\
&\qquad - \left(\sum_{i=1}^{n} \left(\alpha_{i-1} - \alpha_i\right) V_i x_n + \alpha_n z \right) + \alpha_n z - x_{n+1} \\
&\quad = P_C \left[\alpha_n f\left(x_n\right) + \sum_{i=1}^{n} \left(\alpha_{i-1} - \alpha_i\right) V_i y_n \right] + \alpha_n \left(z - x_{n+1}\right) \\
&\qquad - P_C \left[\sum_{i=1}^{n} \left(\alpha_{i-1} - \alpha_i\right) V_i x_n + \alpha_n z \right] \\
&\qquad + \left(1-\alpha_n\right)\left(x_n - x_{n+1}\right).
\end{aligned} \tag{68}$$

It follows that

$$\begin{aligned}
&\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle x_n-V_ix_n,x_n-x^*\rangle \\
&=\Big\langle P_C\Big[\alpha_n f(x_n)+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_iy_n\Big] \\
&\quad -P_C\Big[\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_ix_n+\alpha_n z\Big],x_n-x^*\Big\rangle \\
&\quad +\alpha_n\langle z-x_{n+1},x_n-x^*\rangle \\
&\quad +(1-\alpha_n)\langle x_n-x_{n+1},x_n-x^*\rangle \\
&\le\Big\|\alpha_n(f(x_n)-z)+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(V_iy_n-V_ix_n)\Big\| \\
&\quad\times\|x_n-x^*\|+\alpha_n\|z-x_{n+1}\|\|x_n-x^*\| \\
&\quad +(1-\alpha_n)\|x_n-x_{n+1}\|\|x_n-x^*\| \\
&\le\alpha_n\|f(x_n)-z\|\|x_n-x^*\| \\
&\quad +\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|y_n-x_n\|\|x_n-x^*\| \\
&\quad +\alpha_n\|z-x_{n+1}\|\|x_n-x^*\| \\
&\quad +(1-\alpha_n)\|x_n-x_{n+1}\|\|x_n-x^*\| \\
&=\alpha_n\|f(x_n)-z\|\|x_n-x^*\| \\
&\quad +(1-\alpha_n)\|y_n-x_n\|\|x_n-x^*\| \\
&\quad +\alpha_n\|z-x_{n+1}\|\|x_n-x^*\| \\
&\quad +(1-\alpha_n)\|x_n-x_{n+1}\|\|x_n-x^*\| \\
&\le\alpha_n\|f(x_n)-z\|\|x_n-x^*\| \\
&\quad +(1-\alpha_n)\|\beta_nSx_n+(1-\beta_n)z_n-x_n\| \\
&\quad\times\|x_n-x^*\|+\alpha_n\|z-x_{n+1}\|\|x_n-x^*\| \\
&\quad +(1-\alpha_n)\|x_n-x_{n+1}\|\|x_n-x^*\| \\
&\le\alpha_n\|f(x_n)-z\|\|x_n-x^*\| \\
&\quad +(1-\alpha_n)\beta_n\|Sx_n-x_n\|\|x_n-x^*\| \\
&\quad +(1-\alpha_n)(1-\beta_n)\|z_n-x_n\|\|x_n-x^*\| \\
&\quad +\alpha_n\|z-x_{n+1}\|\|x_n-x^*\| \\
&\quad +(1-\alpha_n)\|x_n-x_{n+1}\|\|x_n-x^*\|.
\end{aligned} \tag{69}$$

From Lemma 10 and the previous inequality, we get

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|x_n-V_ix_n\|^2 \\
&\le\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle x_n-V_ix_n,x_n-x^*\rangle \\
&\le\alpha_n\|f(x_n)-z\|\|x_n-x^*\| \\
&\quad +(1-\alpha_n)\beta_n\|Sx_n-x_n\|\|x_n-x^*\| \\
&\quad +(1-\alpha_n)(1-\beta_n)\|z_n-x_n\|\|x_n-x^*\| \\
&\quad +\alpha_n\|z-x_{n+1}\|\|x_n-x^*\| \\
&\quad +(1-\alpha_n)\|x_n-x_{n+1}\|\|x_n-x^*\|.
\end{aligned} \tag{70}$$

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $\alpha_n\to 0$, $\beta_n\to 0$, and $\lim_{n\to\infty}\|x_n-z_n\|=0$, we obtain

$$\lim_{n\to\infty}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|x_n-V_ix_n\|^2=0. \tag{71}$$

Since $(\alpha_{i-1}-\alpha_i)\|x_n-V_ix_n\|^2\le\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|x_n-V_ix_n\|^2$ and $\{\alpha_n\}$ is strictly decreasing, we have

$$\lim_{n\to\infty}\|x_n-V_ix_n\|=0. \tag{72}$$

Hence, we obtain

$$\lim_{n\to\infty}\|x_n-T_ix_n\|=\lim_{n\to\infty}\frac{\|x_n-V_ix_n\|}{(1-k_i)}=0,\quad\forall i\ge 1. \tag{73}$$

Since $\{x_n\}$ is bounded, without loss of generality, we can assume that $x_n\rightharpoonup w\in C$. It follows from Lemma 6 that $w\in F(T)$. Therefore, $w_w(x_n)\subset F(T)$. □

Theorem 14. *The sequence $\{x_n\}$ generated by Algorithm 11 converges strongly to $z=P_{S^*\cap \mathrm{MEP}(F)\cap F(T)}f(z)$, which is the unique solution of the variational inequality*

$$\langle (I-f)z,x-z\rangle\ge 0,\quad\forall x\in S^*\cap\mathrm{MEP}(F)\cap F(T). \tag{74}$$

Proof. Since $\{x_n\}$ is bounded $x_n\rightharpoonup w$ and from Lemma 13, we have $w\in F(T)$. Next, we show that $w\in\mathrm{MEP}(F)$. Since $u_n=T_{r_n}(x_n-r_nDx_n)$, we have

$$F(u_n,y)+\langle Dx_n,y-u_n\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge 0,\quad\forall y\in C. \tag{75}$$

It follows from monotonicity of F that

$$\begin{aligned}
&\langle Dx_n,y-u_n\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge F(y,u_n),\quad\forall y\in C, \\
&\langle Dx_{n_k},y-u_{n_k}\rangle+\Big\langle y-u_{n_k},\frac{u_{n_k}-x_{n_k}}{r_{n_k}}\Big\rangle\ge F(y,u_{n_k}),\quad\forall y\in C.
\end{aligned} \tag{76}$$

Since $\lim_{n\to\infty}\|u_n - x_n\| = 0$ and $x_n \rightharpoonup w$, it easy to observe that $u_{n_k} \to w$. For any $0 < t \le 1$ and $y \in C$, let $y_t = ty + (1-t)w$; we have $y_t \in C$. Then from (76), we obtain

$$\begin{aligned}\langle Dy_t, y_t - u_{n_k}\rangle &\ge \langle Dy_t, y_t - u_{n_k}\rangle - \langle Dx_{n_k}, y_t - u_{n_k}\rangle \\ &\quad - \left\langle y_t - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}}\right\rangle + F(y_t, u_{n_k}) \\ &= \langle Dy_t - Du_{n_k}, y_t - u_{n_k}\rangle \\ &\quad + \langle Du_{n_k} - Dx_{n_k}, y_t - u_{n_k}\rangle \\ &\quad - \left\langle y_t - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}}\right\rangle + F(y_t, u_{n_k}).\end{aligned} \tag{77}$$

Since D is Lipschitz continuous and $\lim_{n\to\infty}\|u_n - x_n\| = 0$, we obtain $\lim_{k\to\infty}\|Du_{n_k} - Dx_{n_k}\| = 0$. From the monotonicity of D and $u_{n_k} \to w$, it follows from (77) that

$$\langle Dy_t, y_t - w\rangle \ge F(y_t, w). \tag{78}$$

Hence, from assumptions (i)–(iv) of Lemma 4 and (78), we have

$$\begin{aligned}0 = F(y_t, y_t) &\le tF(y_t, y) + (1-t)F(y_t, w) \\ &\le tF(y_t, y) + (1-t)\langle Dy_t, y_t - w\rangle \\ &\le tF(y_t, y) + (1-t)t\langle Dy_t, y - w\rangle\end{aligned} \tag{79}$$

which implies that $F(y_t, y) + (1-t)\langle Dy_t, y - w\rangle \ge 0$. Letting $t \to 0_+$, we have

$$F(w, y) + \langle Dw, y - w\rangle \ge 0, \quad \forall y \in C, \tag{80}$$

which implies that $w \in \mathrm{MEP}(F)$. Next, we show that $w \in S^*$. Since $\lim_{n\to\infty}\|x_n - z_n\| = 0$ and there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to w$, it easy to observe that $z_{n_k} \to w$. For any $x, y \in C$, using (23), we have

$$\begin{aligned}&\|Q(x) - Q(y)\|^2 \\ &\quad = \|P_C[P_C[x - \mu_2 B_2 x] - \mu_1 B_1 P_C[x - \mu_2 B_2 x]] \\ &\qquad - P_C[P_C[y - \mu_2 B_2 y] - \mu_1 B_1 P_C[y - \mu_2 B_2 y]]\|^2 \\ &\quad \le \|(P_C[x - \mu_2 B_2 x] - P_C[y - \mu_2 B_2 y]) \\ &\qquad - \mu_1(B_1 P_C[x - \mu_2 B_2 x] - B_1 P_C[y - \mu_2 B_2 y])\|^2 \\ &\quad \le \|P_C[x - \mu_2 B_2 x] - P_C[y - \mu_2 B_2 y]\|^2 \\ &\qquad - \mu_1(2\theta_1 - \mu_1)\|P_C[x - \mu_2 B_2 x] - P_C[y - \mu_2 B_2 y]\|^2 \\ &\quad \le \|P_C[x - \mu_2 B_2 x] - P_C[y - \mu_2 B_2 y]\|^2 \\ &\quad \le \|(x - \mu_2 B_2 x) - (y - \mu_2 B_2 y)\|^2 \\ &\quad \le \|x - y\|^2 \\ &\qquad - \mu_2(2\theta_2 - \mu_2)\|B_2 x - B_2 y\|^2 \\ &\quad \le \|x - y\|^2.\end{aligned} \tag{81}$$

This implies that $Q : C \to C$ is nonexpansive. On the other hand,

$$\begin{aligned}&\|z_n - Q(z_n)\|^2 \\ &\quad = \|P_C[P_C[u_n - \mu_2 B_2 u_n] - \mu_1 B_1 P_C[u_n - \mu_2 B_2 u_n]] \\ &\qquad - Q(z_n)\|^2 \\ &\quad = \|Q(u_n) - Q(z_n)\|^2 \\ &\quad \le \|u_n - z_n\|^2.\end{aligned} \tag{82}$$

Since $\lim_{n\to\infty}\|u_n - z_n\| = 0$ (see (66)), we have $\lim_{n\to\infty}\|z_n - Q(z_n)\| = 0$. It follows from Lemma 6 that $w = Q(w)$, which implies from Lemma 3 that $w \in S^*$.

Thus, we have

$$w \in S^* \cap \mathrm{MEP}(F) \cap F(T). \tag{83}$$

Next, we claim that $\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle \le 0$ where $z = P_{S^*\cap \mathrm{MEP}(F)\cap F(T)} f(z)$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\begin{aligned}\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle &= \limsup_{k\to\infty}\langle f(z) - z, x_{n_k} - z\rangle \\ &= \langle f(z) - z, w - z\rangle \le 0.\end{aligned} \tag{84}$$

Next, we show that $x_n \to z$.

One has

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \Big\langle x_{n+1} - \alpha_n f(x_n) \\ &\qquad - \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\, V_i y_n, x_{n+1} - z\Big\rangle \\ &\quad + \Big\langle \alpha_n f(x_n) \\ &\qquad + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i) V_i y_n - z, x_{n+1} - z\Big\rangle\end{aligned}$$

$$\begin{aligned}
&\leq \left\langle \alpha_n f(x_n) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) V_i y_n - z, x_{n+1} - z \right\rangle \\
&= \alpha_n \langle f(x_n) - f(z), x_{n+1} - z\rangle \\
&\quad + \alpha_n \langle f(z) - z, x_{n+1} - z\rangle \\
&\quad + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \langle V_i y_n - z, x_{n+1} - z\rangle \\
&\leq \alpha_n \|f(x_n) - f(z)\| \|x_{n+1} - z\| \\
&\quad + \alpha_n \langle f(z) - z, x_{n+1} - z\rangle \\
&\quad + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \|V_i y_n - z\| \|x_{n+1} - z\| \\
&\leq \alpha_n \rho \|x_n - z\| \|x_{n+1} - z\| \\
&\quad + \alpha_n \langle f(z) - z, x_{n+1} - z\rangle \\
&\quad + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \|y_n - z\| \|x_{n+1} - z\| \\
&\leq \alpha_n \rho \|x_n - z\| \|x_{n+1} - z\| \\
&\quad + \alpha_n \langle f(z) - z, x_{n+1} - z\rangle + (1 - \alpha_n) \\
&\quad \times \{\beta_n \|Sx_n - Sz\| \\
&\qquad + \beta_n \|Sz - z\| + (1 - \beta_n) \|z_n - z\|\} \\
&\quad \times \|x_{n+1} - z\| \\
&\leq \alpha_n \rho \|x_n - z\| \|x_{n+1} - z\| \\
&\quad + \alpha_n \langle f(z) - z, x_{n+1} - z\rangle \\
&\quad \times (1 - \alpha_n) \{\beta_n \|x_n - z\| + \beta_n \|Sz - z\| \\
&\qquad + (1 - \beta_n) \|x_n - z\|\} \|x_{n+1} - z\| \\
&\leq (1 - \alpha_n (1 - \rho)) \|x_n - z\| \\
&\quad \times \|x_{n+1} - z\| + \alpha_n \langle f(z) - z, x_{n+1} - z\rangle \\
&\quad + (1 - \alpha_n) \beta_n \|Sz - z\| \|x_{n+1} - z\| \\
&\leq \frac{1 - \alpha_n (1 - \rho)}{2} \left(\|x_n - z\|^2 + \|x_{n+1} - z\|^2\right) \\
&\quad + \alpha_n \langle f(z) - z, x_{n+1} - z\rangle \\
&\quad + (1 - \alpha_n) \beta_n \|Sz - z\| \|x_{n+1} - z\|
\end{aligned} \tag{85}$$

which implies that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \left(1 - \frac{2\alpha_n (1 - \rho)}{1 + \alpha_n (1 - \rho)}\right) \|x_n - z\|^2 \\
&\quad + \frac{2\alpha_n}{1 + \alpha_n (1 - \rho)} \langle f(z) - z, x_{n+1} - z\rangle \\
&\quad + \frac{2(1 - \alpha_n) \beta_n}{1 + \alpha_n (1 - \rho)} \|Sz - z\| \|x_{n+1} - z\| \\
&\leq \left(1 - \frac{2\alpha_n (1 - \rho)}{1 + \alpha_n (1 - \rho)}\right) \|x_n - z\|^2 \\
&\quad + \frac{2\alpha_n (1 - \rho)}{1 + \alpha_n (1 - \rho)} \left\{\frac{1}{1 - \rho} \langle f(z) - z, x_{n+1} - z\rangle \right. \\
&\qquad + \frac{(1 - \alpha_n) \beta_n}{\alpha_n (1 - \rho)} \|Sz - z\| \\
&\qquad \left. \times \|x_{n+1} - z\| \right\}.
\end{aligned} \tag{86}$$

Let $\gamma_n = 2\alpha_n(1-\rho)/(1+\alpha_n(1-\rho))$ and $\delta_n = (2\alpha_n(1-\rho)/(1+\alpha_n(1-\rho)))\{(1/(1-\rho))\langle f(z)-z, x_{n+1}-z\rangle + ((1-\alpha_n)\beta_n/(\alpha_n(1-\rho)))\|Sz-z\|\|x_{n+1}-z\|\}$.

Since

$$\begin{gathered}
\sum_{n=1}^{\infty} \alpha_n = \infty, \qquad 1 + \alpha_n (1 - \rho) \leq 2, \\
\limsup_{n \to \infty} \left\{\frac{1}{1 - \rho} \langle f(z) - z, x_{n+1} - z\rangle + \frac{(1 - \alpha_n) \beta_n}{\alpha_n (1 - \rho)} \|Sz - z\| \|x_{n+1} - z\|\right\} \leq 0.
\end{gathered} \tag{87}$$

It follows that

$$\sum_{n=1}^{\infty} \gamma_n = \infty \qquad \limsup_{n \to \infty} \frac{\delta_n}{\gamma_n} \lim \leq 0. \tag{88}$$

Thus, all the conditions of Lemma 7 are satisfied. Hence, we deduce that $x_n \to z$.

Since $P_{S^* \cap \mathrm{MEP}(F) \cap F(T)} f$ is a contraction, there exists a unique $z \in C$ such that $z = P_{S^* \cap \mathrm{MEP}(F) \cap F(T)} f(z)$. From (20), it follows that z is the unique solution of the problem (74). This completes the proof. □

Theorem 15. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $D, B_i : C \to H$ be η, θ_i-inverse strongly monotone mappings for each $i = 1, 2$, respectively. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the assumptions (i)–(iv) of Lemma 4, $S : C \to H$ a nonexpansive mapping, and $\{T_i\}_{i=1}^{\infty} : C \to C$ a countable family of k_i-strict pseudocontraction mappings such that $S^* \cap \mathrm{MEP}(F) \cap F(T) \neq \emptyset$, where $F(T) := \bigcap_{i=1}^{\infty} F(T_i) = \bigcap_{i=1}^{\infty} F(V_i)$. Let f be a ρ-contraction mapping. For a given $x_0 \in C$ arbitrarily, let the iterative sequences $\{u_n\}, \{x_n\}, \{y_n\}$, and $\{z_n\}$ be generated by*

$$\begin{gathered}
F(u_n, y) + \langle Dx_n, y - u_n\rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n\rangle \geq 0, \\
\forall y \in C; \\
z_n = P_C [P_C [u_n - \mu_2 B_2 u_n] - \mu_1 B_1 P_C [u_n - \mu_2 B_2 u_n]];
\end{gathered}$$

$$y_n = \beta_n S x_n + (1-\beta_n) z_n;$$
$$x_{n+1} = P_C\left[\alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i) V_i y_n\right], \quad \forall n \geq 0, \tag{89}$$

where $V_i = k_i I + (1-k_i)T_i$, $0 \leq k_i < 1$, $\mu_i \in (0, 2\theta_i)$ *for each* $i = 1, 2, \{r_n\} \subset (0, 2\eta)$, $\alpha_0 = 1$, $\{\alpha_n\}$ *is a strictly decreasing sequence in* $(0,1)$*, and* $\{\beta_n\}$ *is a sequence in* $(0,1)$ *satisfying the following conditions:*

(a) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(b) $\lim_{n\to\infty}(\beta_n/\alpha_n) = \tau \in (0,\infty)$,

(c) $\sum_{n=1}^{\infty}(\alpha_{n-1}-\alpha_n) < \infty$ *and* $\sum_{n=1}^{\infty}|\beta_{n-1}-\beta_n| < \infty$,

(d) $\lim_{n\to\infty}((1/\mu)|r_{n-1}-r_n| + |\alpha_{n-1}-\alpha_n| + |\beta_{n-1}-\beta_n|/\alpha_n\beta_n) = 0$,

(e) *there exists a constant* $K > 0$ *such that* $(1/\alpha_n)|(1/\beta_n) - (1/\beta_{n-1})| \leq K$,

(f) $\liminf_{n\to\infty} r_n < \limsup_{n\to\infty} r_n < 2\eta$ *and* $\sum_{n=1}^{\infty}|r_{n-1}-r_n| < \infty$.

Then, sequence $\{x_n\}$ *generated by Algorithm* (89) *converges strongly to* $x^* \in S^* \cap \mathrm{MEP}(F) \cap F(T)$*, which is the unique solution of the variational inequality*

$$\left\langle \frac{1}{\tau}(I-f)x^* + (I-S)x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in S^* \cap \mathrm{MEP}(F) \cap F(T). \tag{90}$$

Proof. From $\lim_{n\to\infty}(\beta_n/\alpha_n) = \tau \in (0,\infty)$, without loss of generality, we can assume that $\beta_n \leq (1+\tau)\alpha_n$ for all $n \geq 1$. Hence $\beta_n \to 0$. By similar argument as that lemmas 12 and 13, we can deduce that $\{x_n\}$ is bounded, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$, $\lim_{n\to\infty}\|x_n - z_n\| = 0$ (see (67)), and $\|(I-V_i)x_n\| \to 0$. Then, we have

$$\|y_n - x_n\| \leq \beta_n \|x_n - Sx_n\| + (1-\beta_n)\|x_n - z_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{91}$$

It follows that, for all $i \geq 1$,

$$\|y_n - V_i x_n\| \leq \|y_n - x_n\| + \|x_n - V_i x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{92}$$

From (91) and (92), we have

$$\|y_n - V_i y_n\| \leq \|y_n - V_i x_n\| + \|V_i x_n - V_i y_n\| \leq \|y_n - V_i x_n\| + \|y_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{93}$$

Set $w_n = \alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_i y_n$. From (47) and (48), we obtain

$$\begin{aligned}
\frac{\|x_{n+1}-x_n\|}{\beta_n} &\leq \frac{\|w_n - w_{n-1}\|}{\beta_n} \\
&\leq (1-(1-\rho)\alpha_n)\frac{\|x_n - x_{n-1}\|}{\beta_n} + M\left(\frac{(1/\mu)|r_n - r_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_n}\right) \\
&= (1-(1-\rho)\alpha_n)\frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} + (1-(1-\rho)\alpha_n)\|x_n - x_{n-1}\|\left(\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}\right) \\
&\quad + M\left(\frac{(1/\mu)|r_n - r_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_n}\right) \\
&\leq (1-(1-\rho)\alpha_n)\frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} + \|x_n - x_{n-1}\|\left|\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}\right| \\
&\quad + M\left(\frac{(1/\mu)|r_n - r_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_n}\right) \\
&\leq (1-(1-\rho)\alpha_n)\frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} + \alpha_n K\|x_n - x_{n-1}\| \\
&\quad + M\left(\frac{(1/\mu)|r_n - r_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_n}\right) \\
&\leq (1-(1-\rho)\alpha_n)\frac{\|w_{n-1} - w_{n-2}\|}{\beta_{n-1}} + \alpha_n K\|x_n - x_{n-1}\| \\
&\quad + M\left(\frac{(1/\mu)|r_n - r_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_n}\right).
\end{aligned} \tag{94}$$

Let $\gamma_n = (1-\rho)\alpha_n$ and $\delta_n = \alpha_n K\|x_n - x_{n-1}\| + M((1/\mu)|r_n - r_{n-1}|/\beta_n + |\beta_n - \beta_{n-1}|/\beta_n + |\alpha_n - \alpha_{n-1}|/\beta_n)$. From conditions (a) and (d), we have

$$\sum_{n=1}^{\infty}\gamma_n = \infty, \qquad \lim_{n\to\infty}\frac{\delta_n}{\gamma_n} = 0. \tag{95}$$

By Lemma 7, we obtain

$$\begin{gathered}\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\beta_n} = 0,\\ \lim_{n\to\infty}\frac{\|w_{n+1}-w_n\|}{\beta_n} = \lim_{n\to\infty}\frac{\|w_{n+1}-w_n\|}{\alpha_n} = 0.\end{gathered} \tag{96}$$

From (89), we have

$$\begin{aligned}x_{n+1} &= P_C[w_n] - w_n + \alpha_n f(x_n)\\ &\quad + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(V_i y_n - y_n) + (1-\alpha_n) y_n.\end{aligned} \tag{97}$$

Hence, it follows that

$$\begin{aligned}x_n - x_{n+1} &= (1-\alpha_n)x_n + \alpha_n x_n\\ &\quad - \Bigg(P_C[w_n] - w_n + \alpha_n f(x_n)\\ &\quad + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(V_i y_n - y_n) + (1-\alpha_n) y_n\Bigg)\\ &= (1-\alpha_n)[\beta_n(x_n - Sx_n) + (1-\beta_n)(x_n - z_n)]\\ &\quad + (w_n - P_C[w_n])\\ &\quad + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(y_n - V_i y_n) + \alpha_n(x_n - f(x_n))\end{aligned} \tag{98}$$

and hence

$$\begin{aligned}\frac{x_n - x_{n+1}}{(1-\alpha_n)\beta_n} &= x_n - Sx_n + \frac{(1-\beta_n)}{\beta_n}(x_n - z_n)\\ &\quad + \frac{1}{(1-\alpha_n)\beta_n}(w_n - P_C[w_n])\\ &\quad + \frac{1}{(1-\alpha_n)\beta_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(y_n - V_i y_n)\\ &\quad + \frac{\alpha_n}{(1-\alpha_n)\beta_n}(x_n - f(x_n)).\end{aligned} \tag{99}$$

Let $v_n = (x_n - x_{n+1})/(1-\alpha_n)\beta_n$. For any $z \in S^* \cap \mathrm{MEP}(F) \cap F(T)$, we have

$$\begin{aligned}&\langle v_n, x_n - z\rangle\\ &\quad = \frac{1}{(1-\alpha_n)\beta_n}\langle w_n - P_C[w_n], P_C[w_{n-1}] - z\rangle\\ &\qquad + \frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)x_n, x_n - z\rangle\\ &\qquad + \langle x_n - Sx_n, x_n - z\rangle\\ &\qquad + \frac{(1-\beta_n)}{\beta_n}\langle x_n - z_n, x_n - z\rangle\\ &\qquad + \frac{1}{(1-\alpha_n)\beta_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle y_n - V_i y_n, x_n - z\rangle.\end{aligned} \tag{100}$$

Since S is nonexpansive mapping, f is ρ-contraction mapping and V_i is k_i-strict pseudocontraction mapping. Then, $(I-S)$ and $(I-V_i)$ are monotones, and f is strongly monotone with coefficient $(1-\rho)$. We can deduce that

$$\begin{aligned}&\langle x_n - Sx_n, x_n - z\rangle\\ &\quad = \langle (I-S)x_n - (I-S)z, x_n - z\rangle\\ &\qquad + \langle (I-S)z, x_n - z\rangle\\ &\quad \ge \langle (I-S)z, x_n - z\rangle,\\ &\langle (I-f)x_n, x_n - z\rangle\\ &\quad = \langle (I-f)x_n - (I-f)z, x_n - z\rangle\\ &\qquad + \langle (I-f)z, x_n - z\rangle \ge (1-\rho)\|x_n - z\|^2\\ &\qquad + \langle (I-f)z, x_n - z\rangle,\\ &\langle (I-V_i)y_n, x_n - z\rangle\\ &\quad = \langle (I-V_i)y_n - (I-V_i)z, x_n - y_n\rangle\\ &\qquad + \langle (I-V_i)y_n - (I-V_i)z, y_n - z\rangle\\ &\quad \ge \langle (I-V_i)y_n - (I-V_i)z, x_n - y_n\rangle\\ &\quad = \langle (I-V_i)y_n, x_n - y_n\rangle\\ &\quad = \langle (I-V_i)y_n, \beta_n(x_n - Sx_n) + (1-\beta_n)(x_n - z_n)\rangle.\end{aligned} \tag{101}$$

From (20), we get

$$\begin{aligned}&\langle w_n - P_C[w_n], P_C[w_{n-1}] - z\rangle\\ &\quad = \langle w_n - P_C[w_n], P_C[w_{n-1}] - P_C[w_n]\rangle\\ &\qquad + \langle w_n - P_C[w_n], P_C[w_n] - z\rangle\\ &\quad \ge \langle w_n - P_C[w_n], P_C[w_{n-1}] - P_C[w_n]\rangle.\end{aligned} \tag{102}$$

Then, from (100)–(102), we have

$$\begin{aligned}\langle v_n, x_n - z\rangle \geq{}& \frac{1}{(1-\alpha_n)\beta_n}\\ &\times\langle w_n - P_C[w_n], P_C[w_{n-1}] - P_C[w_n]\rangle\\ &+\frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)z, x_n - z\rangle\\ &+\langle (I-S)z, x_n - z\rangle\\ &+\frac{(1-\beta_n)}{\beta_n}\langle x_n - z_n, x_n - z\rangle\\ &+\frac{(1-\beta_n)}{(1-\alpha_n)\beta_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\\ &\times\langle (I-V_i)y_n, x_n - z_n\rangle\\ &+\frac{1}{(1-\alpha_n)}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\\ &\qquad\times\langle (I-V_i)y_n, x_n - Sx_n\rangle\\ &+\frac{(1-\rho)\alpha_n}{(1-\alpha_n)\beta_n}\|x_n - z\|^2.\end{aligned} \tag{103}$$

Then, we obtain

$$\begin{aligned}\|x_n - z\|^2 \leq{}& \frac{1}{(1-\rho)\alpha_n}\|w_n - P_C[w_n]\|\|w_{n-1} - w_n\|\\ &-\frac{1}{(1-\rho)}\langle (I-f)z, x_n - z\rangle\\ &+\frac{(1-\alpha_n)\beta_n}{(1-\rho)\alpha_n}(\langle v_n, x_n - z\rangle\\ &\qquad-\langle (I-S)z, x_n - z\rangle)\\ &-\frac{(1-\beta_n)(1-\alpha_n)}{(1-\rho)\alpha_n}\langle x_n - z_n, x_n - z\rangle\\ &-\frac{(1-\beta_n)}{(1-\rho)\alpha_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle (I-V_i)y_n, x_n - z_n\rangle\\ &-\frac{\beta_n}{(1-\rho)\alpha_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\\ &\qquad\times\langle (I-V_i)y_n, x_n - Sx_n\rangle\\ \leq{}& \frac{\|w_{n-1} - w_n\|}{(1-\rho)\alpha_n}\|w_n - P_C[w_n]\|\\ &-\frac{1}{(1-\rho)}\langle (I-f)z, x_n - z\rangle\\ &+\frac{(1-\alpha_n)\beta_n}{(1-\rho)\alpha_n}\\ &\times(\langle v_n, x_n - z\rangle - \langle (I-S)z, x_n - z\rangle)\\ &+\frac{1}{(1-\rho)}\frac{(1-\beta_n)}{\beta_n}\frac{\beta_n}{\alpha_n}\|x_n - z_n\|\|x_n - z\|\\ &+\frac{1}{(1-\rho)}\frac{(1-\beta_n)}{\beta_n}\frac{\beta_n}{\alpha_n}\\ &\times\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|(I-V_i)y_n\|\|x_n - z_n\|\\ &-\frac{\beta_n}{(1-\rho)\alpha_n}\\ &\times\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle (I-V_i)y_n, x_n - Sx_n\rangle.\end{aligned} \tag{104}$$

By condition (e) of Theorem 15, there exists a constant $N > 0$ such that $((1-\beta_n)/\beta_n) \leq N$. Since $\lim_{n\to\infty}\|x_n - z_n\| = 0$, $v_n \to 0, (I-V_i)y_n \to 0$, and $\|w_{n-1} - w_n\|/\alpha_n \to 0$ as $n \to \infty$, then every weak cluster point of $\{x_n\}$ is also a strong cluster point. Since $\{x_n\}$ is bounded, by Lemma 13 there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to a point $x^* \in F(T)$; in similar argument as that Theorem 14 we can show that $x^* \in S^* \cap \mathrm{MEP}(F) \cap F(T)$.

From (100)–(102), it follows that, for any $z \in S^* \cap \mathrm{MEP}(F) \cap F(T)$,

$$\begin{aligned}&\langle (I-f)x_{n_k}, x_{n_k} - z\rangle\\ &\quad= \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle v_{n_k}, x_{n_k} - z\rangle - \frac{1}{\alpha_{n_k}}\\ &\qquad\times\langle w_{n_k} - P_C[w_{n_k}], P_C[w_{n_k-1}] - z\rangle\\ &\qquad-\frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle x_{n_k} - Sx_{n_k}, x_{n_k} - z\rangle\\ &\qquad-\frac{(1-\alpha_{n_k})(1-\beta_{n_k})}{\alpha_{n_k}}\langle x_{n_k} - z_{n_k}, x_{n_k} - z\rangle\\ &\qquad-\frac{1}{\alpha_{n_k}}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle y_{n_k} - V_i y_{n_k}, x_{n_k} - z\rangle\\ &\quad\leq \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle v_{n_k}, x_{n_k} - z\rangle\\ &\qquad+\frac{1}{\alpha_{n_k}}\|w_{n_k} - P_C[w_{n_k}]\|\\ &\qquad\times\|w_{n_k-1} - w_{n_k}\| - \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\\ &\qquad\times\langle x_{n_k} - Sx_{n_k}, x_{n_k} - z\rangle\end{aligned}$$

$$+\frac{\left(1-\beta_{n_k}\right)}{\beta_{n_k}}\frac{\beta_{n_k}}{\alpha_{n_k}}\left\|x_{n_k}-z_{n_k}\right\|$$
$$\times\left\|x_{n_k}-z\right\|+\frac{\left(1-\beta_{n_k}\right)}{\beta_{n_k}}\frac{\beta_{n_k}}{\alpha_{n_k}}$$
$$\times\sum_{i=1}^{n_k}\left(\alpha_{i-1}-\alpha_i\right)\left\|\left(I-V_i\right)y_{n_k}\right\|\left\|x_{n_k}-z_{n_k}\right\|$$
$$-\frac{\beta_{n_k}}{\alpha_{n_k}}\sum_{i=1}^{n_k}\left(\alpha_{i-1}-\alpha_i\right)\left\langle\left(I-V_i\right)y_{n_k},x_{n_k}-Sx_{n_k}\right\rangle. \tag{105}$$

Since $\lim_{n\to\infty}\|x_n-z_n\|=0$, $v_n\to 0$, $(I-V_i)y_n\to 0$, and $\|w_{n-1}-w_n\|/\alpha_n\to 0$; letting $k\to\infty$ in (105), we obtain

$$\left\langle(I-f)x^*,x^*-z\right\rangle\le-\tau\left\langle x^*-Sx^*,x^*-z\right\rangle; \tag{106}$$

that is,

$$\left\langle\frac{1}{\tau}(I-f)x^*+(I-S)x^*,z-x^*\right\rangle\ge 0. \tag{107}$$

In the following, we show that (90) has unique solution. Assume that x' is another solution. Then, we have

$$\begin{aligned}\left\langle(I-f)x',x'-x^*\right\rangle&\le-\tau\left\langle x'-Sx',x'-x^*\right\rangle,\\ \left\langle(I-f)x^*,x^*-x'\right\rangle&\le-\tau\left\langle x^*-Sx^*,x^*-x'\right\rangle.\end{aligned} \tag{108}$$

Adding (108), we get

$$\begin{aligned}(1-\rho)\left\|x'-x^*\right\|^2&\le\left\langle(I-f)x'-(I-f)x^*,x'-x^*\right\rangle\\&\le-\tau\langle(I-S)x'-(I-S)x^*,x'-x^*\rangle\le 0.\end{aligned} \tag{109}$$

Then, $x'=x^*$. Since (90) has unique solution, it follows that $w_w(x_n)=\{x^*\}$. Since every weak cluster point of $\{x_n\}$ is also a strong cluster point, we conclude that $\{x_n\}\to x^*$. This completes the proof. □

4. Applications

In this section, we obtain the following results by using a special case of the proposed method. The first result can be viewed as extension and improvement of the method of Gu et al. [11] for finding the approximate element of the common set of solutions of a generalized equilibrium problem and a hierarchical fixed point problem in a real Hilbert space.

Corollary 16. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $D: C\to H$ be η-inverse strongly monotone mappings, respectively. Let $F: C\times C\to\mathbb{R}$ be a bifunction satisfying the assumptions (i)–(iv) of Lemma 4, $S: C\to H$ a nonexpansive mapping, and $\{T_i\}_{i=1}^{\infty}: C\to C$ a countable family of k_i-strict pseudocontraction mappings such that $F(T)\cap \mathrm{MEP}(F)\neq\emptyset$, where $F(T)=\cap_{i=1}^{\infty}F(T_i)$. Let f be a ρ-contraction mapping. For a given $x_0\in C$ arbitrarily, let the iterative sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be generated by*

$$\begin{aligned}&F(u_n,y)+\langle Dx_n,y-u_n\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge 0,\quad\forall y\in C;\\&y_n=\beta_nSx_n+(1-\beta_n)u_n;\\&x_{n+1}=P_C\left[\alpha_nf(x_n)+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)T_iy_n\right],\quad\forall n\ge 0,\end{aligned} \tag{110}$$

where $\alpha_0=1$, $\{\alpha_n\}$ is a strictly decreasing sequence in $(0,1)$, and $\{\beta_n\}$ is a sequence in $(0,1)$ satisfying the following conditions:

(a) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$,
(b) $\lim_{n\to\infty}(\beta_n/\alpha_n)=\tau\in(0,\infty)$,
(c) $\sum_{n=1}^{\infty}(\alpha_{n-1}-\alpha_n)<\infty$ *and* $\sum_{n=1}^{\infty}|\beta_{n-1}-\beta_n|<\infty$,
(d) $\lim_{n\to\infty}((1/\mu)(|r_n-r_{n-1}|+|\alpha_{n-1}-\alpha_n|+|\beta_{n-1}-\beta_n|)/(\alpha_n\beta_n))=0$,
(e) *there exists a constant $K>0$ such that* $(1/\alpha_n)|(1/\beta_n)-(1/\beta_{n-1})|\le K$,
(f) $\liminf_{n\to\infty}r_n>0$ *and* $\sum_{n=1}^{\infty}|r_{n-1}-r_n|<\infty$.

Then, sequence $\{x_n\}$ generated by Algorithm (110) converges strongly to $x^*\in\mathrm{MEP}(F)\cap F(T)$, which is the unique solution of the variational inequality

$$\left\langle\frac{1}{\tau}(I-f)x^*+(I-S)x^*,x-x^*\right\rangle\ge 0,\quad\forall x\in\mathrm{MEP}(F)\cap F(T). \tag{111}$$

Proof. Putting $B_1=B_2=0$ and $k_i=0$, for all $i\ge 1$ in Theorem 15, then conclusion of Corollary 16 is obtained. □

The following result can be viewed as extension and improvement of the method of Yao et al. [31] for finding the approximate element of the common set of solutions of a generalized equilibrium problem and a hierarchical fixed point problem in a real Hilbert space.

Corollary 17. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $D: C\to H$ be η-inverse strongly monotone mappings, respectively. Let $F: C\times C\to\mathbb{R}$ be a bifunction satisfying the assumptions (i)–(iv) of Lemma 4, $S: C\to H$ a nonexpansive mapping, and $T: C\to C$ a countable family of k-strict pseudocontraction mappings such that $F(T)\cap\mathrm{MEP}(F)\neq\emptyset$. Let f be a ρ-contraction mapping. For a given $x_0\in C$ arbitrarily, let the iterative sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be generated by*

$$\begin{aligned}&F(u_n,y)+\langle Dx_n,y-u_n\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge 0,\quad\forall y\in C;\\&y_n=\beta_nSx_n+(1-\beta_n)u_n;\\&x_{n+1}=P_C\left[\alpha_nf(x_n)+(1-\alpha_n)Ty_n\right],\quad\forall n\ge 0,\end{aligned} \tag{112}$$

where $\alpha_0 = 1$, $\{\alpha_n\}$ *is a strictly decreasing sequence in* $(0, 1)$ *and* $\{\beta_n\}$ *is a sequence in* $(0, 1)$ *satisfying the following conditions:*

(a) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(b) $\lim_{n\to\infty}(\beta_n/\alpha_n) = \tau \in (0, \infty)$,

(c) $\sum_{n=1}^{\infty}(\alpha_{n-1} - \alpha_n) < \infty$ *and* $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$,

(d) $\lim_{n\to\infty}((1/\mu)|r_n - r_{n-1}| + |\alpha_{n-1} - \alpha_n| + |\beta_{n-1} - \beta_n|)/\alpha_n\beta_n = 0$,

(e) *there exists a constant* $K > 0$ *such that* $(1/\alpha_n)|(1/\beta_n) - (1/\beta_{n-1})| \leq K$,

(f) $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty} |r_{n-1} - r_n| < \infty$.

Then, sequence $\{x_n\}$ *generated by Algorithm* (112) *converges strongly to* $x^* \in \mathrm{MEP}(F) \cap F(T)$, *which is the unique solution of the variational inequality*

$$\left\langle \frac{1}{\tau}(I - f)x^* + (I - S)x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in \mathrm{MEP}(F) \cap F(T). \tag{113}$$

Proof. Putting $B_1 = B_2 = 0$, $k_i = 0$, and $T_i = T$ for all $i \geq 1$ in Theorem 15, then conclusion of Corollary 17 is obtained. □

5. Conclusions

In this paper, we suggest and analyze an iterative method for finding the approximate element of the common set of solutions of (1), (12), and (15) for a strictly pseudocontraction mapping in real Hilbert space, which can be viewed as a refinement and improvement of some existing methods for solving a system of variational inequality problem, a mixed equilibrium problem, and a hierarchical fixed point problem. It is easy to verify that Algorithm 11 includes some existing methods (e.g., [6, 11, 17, 31]) as special cases. Therefore, the new algorithm is expected to be widely applicable.

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