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Research Article

Smoothing Techniques and Augmented Lagrangian Method for Recourse Problem of Two-Stage Stochastic Linear Programming

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The augmented Lagrangian method can be used for solving recourse problems and obtaining their normal solution in solving two-stage stochastic linear programming problems. The augmented Lagrangian objective function of a stochastic linear problem is not twice differentiable which precludes the use of a Newton method. In this paper, we apply the smoothing techniques and a fast Newton-Armijo algorithm for solving an unconstrained smooth reformulation of this problem. Computational results and comparisons are given to show the effectiveness and speed of the algorithm.

1. Introduction

In stochastic programming, some data are random variables with specific possibility distribution [1], which was first introduced by the designer of linear programming problems, Dantzig, in [2].

In this paper, we consider the following two-stage stochastic linear program (slp) with recourse which involves the calculation of an expectation over a discrete set of scenarios:

$$\min_{x\in X} f(x) = c^T x + \phi(x), \qquad X = \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\}, \tag{1}$$

where

$$\phi(x) = E(Q(x,\omega)) = \sum_{i=1}^{N} Q(x,\omega^i)\,\rho(\omega^i) \tag{2}$$

and E shows the expectation of function $Q(x,\omega)$ which depend on the random variable ω. The function Q is defined as follows:

$$Q(x,\omega) = \min_{y\in\mathbb{R}^{n_2}} \left\{ q(\omega)^T y \mid W(\omega)^T y \geq h(\omega) - T(\omega)x \right\}, \tag{3}$$

where $A \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$. Also, in the problem (3) vector of coefficients $q(\cdot) \in \mathbb{R}^{n_2}$, matrix of coefficients $W^T(\cdot) \in \mathbb{R}^{m_2\times n_2}$, demand vector $h(\cdot) \in \mathbb{R}^{m_2}$, and matrix $T(\cdot) \in \mathbb{R}^{m_2\times n}$ depend on the random vector ω with support space Ω. The problems (1) and (3) are called master and recourse problems of stochastic programming, respectively.

We assume that the problem (3) has a solution for each $x \in X$ and $\omega \in \Omega$.

In general, the recourse function $\phi(x)$ is not differentiable everywhere. Therefore, the traditional methods use nonsmooth optimization techniques [3–5]. However, in the last decade, it is proposed smoothing method for recourse function in standard form of recourse problem [6–11]. In this paper, we apply a smooth approximation technique to smooth recourse function that the recourse problem has inequality linear constrained. For more explanation see Section 2. The approximated problem is based on the least two-norm solution of recourse problem. This paper considers the augmented Lagrangian method to obtain least two-norm solution (Section 3). For convenience, Euclidean least two-norm solution of linear programming problem is named normal solution. This effective method contains solving an unconstrained quadratic problem which its objective function is not twice differentiable. To apply a fast Newton method we use the soothing technique and replace plus function by an accurate smooth approximation [12, 13]. In Section 4, the smoothing algorithm and the numerical results are presented. Also, concluding remarks are given in Section 5.

We now describe our notation. Let $a = [a_i]$ be a vector in $\mathbb{R}^n$. By a_+ we mean a vector in $\mathbb{R}^n$ whose ith entry is 0 if $a_i < 0$ and equals a_i if $a_i \geq 0$. By A^T we mean the transpose of matrix A, and $\nabla f(x_0)$ is the gradient of f at x_0. For $x \in \mathbb{R}^n$, $\| x \|$ and $\| x \|_\infty$ denote 2-norm and infinity norm, respectively.

2. Approximation of Recourse Function

As mentioned the objective function of (1) is nondifferentiable. This disadvantage property occurs on the recourse function. In this section, there is an attempt to approximate it to a differentiable function.

Using dual of the problem (3), function $Q(x, \omega)$ can be written as follows:

$$\begin{aligned} Q(x,\omega) = \max_{z\in\mathbb{R}^{m_2}} & (h(\omega) - T(\omega)x)^T z \\ \text{s.t. } & W(\omega) z = q(\omega), \quad z \geq 0. \end{aligned} \tag{4}$$

Unlike the linear recourse function, the quadratic recours function is differentiable. Thus in this paper, the approximation is based on the following quadratic problem with helpful properties:

$$\begin{aligned} Q_\epsilon(x,\omega) = \max_{z\in\mathbb{R}^{m_2}} & (h(\omega) - T(\omega)x)^T z - \frac{\epsilon}{2}\|z\|^2 \\ \text{s.t. } & W(\omega) z = q(\omega), \quad z \geq 0. \end{aligned} \tag{5}$$

The next theorem shows that, for the sufficiently small $\epsilon > 0$, the solution of this problem is the normal solution of the problem (4).

Theorem 1. *For functions $Q(x, \omega)$ and $Q_\epsilon(x, \omega)$ introduced in (4) and (5), the following can be presented:*

(a) $\exists\bar{\epsilon} > 0$ *such that, for each $\epsilon \in (0, \bar{\epsilon}]$, the solution for the problem (5) is the normal solution for the problem (4).*

(b) *For each $\epsilon > 0$, function $Q_\epsilon(x, \omega)$ is differentiable with respect to x.*

(c) *The gradient of function $Q_\epsilon(x, \omega)$ at point x is*

$$\nabla Q_\epsilon(x,\omega) = -T^T(\omega) z_\epsilon^*(x,\omega) \tag{6}$$

in which $z_\epsilon^*(x, \omega)$ is the solution of the problem (5).

Proof. To prove (a), refer to [14, 15].

Also, (b) and (c) can be easily proved considering that function $Q_\epsilon(x, \omega)$ is the conjugate of function

$$p(z) = \begin{cases} \frac{\epsilon}{2}\|z\|^2, & z \in Z, \\ \infty, & z \notin Z, \end{cases} \tag{7}$$

where

$$Z = \left\{z \in \mathbb{R}^{m_2} : W(\omega) z = q(\omega),\ z \geq 0\right\}, \tag{8}$$

and Theorems (26-3) and (23-5) in [16]. □

Using the approximated recourse function $Q_\epsilon(x, \omega)$, we can define a differentiable approximation function to the objective function of (1):

$$f_\epsilon(x) = c^T x + \sum_{i=1}^{N} Q_\epsilon\left(x, \omega^i\right) \rho\left(\omega^i\right). \tag{9}$$

By (6), the gradient of above function exists and is obtained by

$$\begin{aligned} \nabla f_\epsilon(x) &= c + \sum_{i=1}^{N} \nabla Q_\epsilon\left(x, \omega^i\right) \rho\left(\omega^i\right) \\ &= c - \sum_{i=1}^{N} T^T\left(\omega^i\right) z_\epsilon^*\left(x, \omega^i\right) \rho\left(\omega^i\right). \end{aligned} \tag{10}$$

This approximation has paved the way to use the optimization algorithm for master problem (1) in which the objective function is substituted by $f_\epsilon(x)$

$$\min_{x\in X} f_\epsilon(x). \tag{11}$$

In [7], it is considered slp problem with inequality constrained in master problem and equality constrained in recourse problem. Also, in Theorem 2.3 of [7], it is shown that a solution of the approximated problem is a good approximation to a solution of master problem. Here we can express a similar theorem for the problem (1) by using the similar technique in the proof of Theorem 2.3 in [7].

Theorem 2. *Consider the problem (1). Then, for any $x \in X$, there exists an $\bar{\epsilon}(x) > 0$ such that for any $\epsilon \in (0, \bar{\epsilon}(x)]$*

$$\left|f(x) - f_\epsilon(x)\right| \leq \frac{\epsilon}{2} M, \tag{12}$$

where M is defined as follows:

$$M = \max_{i=1,2,\dots,N} \left\|z_\epsilon^*(x, \omega^i)\right\|^2. \tag{13}$$

Let x^* be a solution of (1) and x_ϵ^* a solution of (11). Then, there exists an $\bar{\epsilon} > 0$ such that for any $0 < \epsilon \leq \bar{\epsilon}$

$$\max\left\{f\left(x_\epsilon^*\right) - f\left(x^*\right), f_\epsilon\left(x^*\right) - f_\epsilon\left(x_\epsilon^*\right)\right\} \leq \frac{\epsilon}{2} M. \tag{14}$$

Further, one assumes that f or f_ϵ are strongly convex on X with modulus $\mu > 0$. Then,

$$\left\|x^* - x_\epsilon^*\right\| \leq M \frac{\epsilon}{\mu}. \tag{15}$$

According to Theorem 1, it can be found that for obtaining the gradient of function $f_\epsilon(x)$ in each iteration, we need the normal solution of N linear programming problems (4). In this paper, the augmented Lagrangian method [17] is used for this purpose.

3. Smooth Approximation and Augmented Lagrangian Method

In the augmented Lagrangian method, the unconstrained maximization problem is solved which gives the project of a point on the solution set of the problem (4).

Assume that $\hat{z}$ is an arbitrary vector. Consider the problem of finding the least 2-norm projection $\hat{z}_*$ of $\hat{z}$ on the solution set Z_* of the problem (4)

$$\frac{1}{2}\left\|\hat{z}_* - \hat{z}\right\|^2 = \min_{z \in Z_*} \frac{1}{2}\|z - \hat{z}\|^2,$$
$$Z_* = \left\{z \in \mathbb{R}^{m_2} : W(\omega) z = q(\omega),\ \xi^T z = Q(x, \omega),\ z \geq 0\right\}. \tag{16}$$

In this problem, vector x and random variable ω are constants; therefore, for simplicity, this is assumed to be $\xi = h(\omega) - T(\omega)x$, and function $\widehat{Q}(\xi)$ is defined in a way that $\widehat{Q}(\xi) = Q(x, \omega)$.

Considering that the objective function of the problem (16) is strictly convex, its solution is unique. Let us introduce the Lagrangian function for the problem (16) as follow:

$$\begin{aligned} L(z, p, \beta, \hat{z}, \xi, \omega) = {} & \frac{1}{2}\|z - \hat{z}\|^2 + p^T (W(\omega) z - q(\omega)) \\ & + \beta\left(\xi^T z - \widehat{Q}(\xi)\right), \end{aligned} \tag{17}$$

where $p \in \mathbb{R}^{n_2}$ and $\beta \in \mathbb{R}$ are Lagrangian multipliers and ξ, $\hat{z}$ are constant values. Therefore, the dual problem of (16) becomes

$$\max_{\beta \in \mathbb{R}} \max_{p \in \mathbb{R}^{n_2}} \min_{z \in \mathbb{R}_+^{m_2}} L(z, p, \beta, \hat{z}, \xi, \omega). \tag{18}$$

By solving the inner minimization of the problem (18), duality of the problem (16) is obtained:

$$\max_{\beta \in \mathbb{R}} \max_{p \in \mathbb{R}^{n_2}} \widehat{L}(p, \beta, \hat{z}, \xi), \tag{19}$$

where duality function is

$$\begin{aligned} \widehat{L}(p, \beta, \hat{z}, \xi) = {} & q^T(\omega) p - \frac{1}{2}\left\|\left(\hat{z} + W^T(\omega) p + \beta\xi\right)_+\right\|^2 \\ & + \beta\widehat{Q}(\xi) + \frac{1}{2}\|\hat{z}\|^2. \end{aligned} \tag{20}$$

The following theorem states that if β is sufficiently large, solving the inner maximization of (19) gives the solution of the problem (16).

Theorem 3 (see [17]). *Consider the following maximization problem*

$$\max_{p \in \mathbb{R}^{n_2}} S(p, \beta, \hat{z}, \xi, \omega) \tag{21}$$

in which β, $\hat{z}$, and ξ are constants, and function $S(p, \beta, \hat{z}, \xi)$ is introduced as follows:

$$S(p, \beta, \hat{z}, \xi, \omega) = q^T(\omega) p - \frac{1}{2}\left\|\left(\hat{z} + W^T(\omega) p + \beta\xi\right)_+\right\|^2. \tag{22}$$

Also, assume that the set Z_* is nonempty, and the rank of submatrix W_l of W corresponding to nonzero components of $\hat{z}_*$ is n_2. In such a case, there is β^* which for all $\beta \geq \beta^*$, $\hat{z}_* = (\hat{z} + W^T p(\beta) + \beta\xi)_+$ is the unique and exact solution for the problem (16), where $p(\beta)$ is the point obtained from solving the problem (21).

Also, in special conditions, the solution for the problem (3) can be also obtained and the following theorem expresses this issue.

Theorem 4 (see [17]). *Assume that the solution set Z_* is nonempty. For each $\beta > 0$ and $\hat{z} \in Z_*$, $y_* = p(\beta)/\beta$ is one exact solution for the linear programming problem (3), where $p(\beta)$ is the solution for the problem (21).*

According to the theorems mentioned above, augmented Lagrangian method presents the following iteration process for solving the problem (16):

$$\begin{aligned} p_{k+1} &\in \arg\max_{p \in \mathbb{R}^{n_2}} \left\{q^T(\omega) p - \frac{1}{2}\left\|\left(z_k + W(\omega)^T p + \beta\xi\right)_+\right\|^2\right\}, \\ z_{k+1} &= \left(z_k + W^T(\omega) p_{k+1} + \beta\xi\right)_+, \end{aligned} \tag{23}$$

where z_0 is an arbitrary vector and here we can use zero vector as initial vector for obtaining normal solution of the problem (4).

We note that the problem (23) is a concave problem and its objective function is piecewise quadratic and is not twice differentiable. Applying the smoothing techniques [18, 19] and replacing x_+ by a smooth approximation, we transform this problem to a twice continuously differentiable problem.

Chen and Mangasarian [19] introduced a family of smoothing functions, which is built as follows. Let $\rho : R \to [0, \infty)$ be a piecewise continuous density function satisfying

$$\int_{-\infty}^{+\infty} \rho(s)\, ds = 1, \qquad \int_{-\infty}^{+\infty} |s|\, \rho(s)\, ds < \infty. \tag{24}$$

It is obvious that the derivative of plus function is step function, that is, $(x)_+ = \int_{-\infty}^{x} \delta(t) dt$, where the step function $\delta(x)$ is defined 1 if $x > 0$ and equals 0 if $x \leq 0$. Therefore, a smoothing approximation function of the plus function is defined by

$$\varphi(x, \alpha) = \int_{-\infty}^{x} \psi(t, \alpha)\, dt, \tag{25}$$

where $\psi(x, \alpha)$ is smoothing approximation function of step function and is defined as

$$\psi(x, \alpha) = \int_{-\infty}^{x} \alpha\rho(\alpha t)\, dt. \tag{26}$$

By choosing

$$\rho(s) = \frac{e^{-s}}{(1 + e^{-s})^2}, \tag{27}$$

specific cases of these approaches are obtained as follows:

$$\psi(x,\alpha) = \frac{1}{1+e^{-\alpha x}} \approx \delta(x),$$
$$\varphi(x,\alpha) = x + \frac{1}{\alpha}\log\left(1+e^{-\alpha x}\right) \approx (x)_+. \tag{28}$$

The function φ with a smoothing parameter α is used here to replace the plus function of (22) to obtain a smooth reformulation of function (22):

$$\widehat{S}(p,\beta,\widehat{z},\xi,\beta,\omega,\alpha) := q^T(\omega)p - \left\|\varphi(\widehat{z}+W^T(\omega)p+\beta\xi,\alpha)\right\|^2. \tag{29}$$

Therefore, we have the following iterative process instead of (23) and (28):

$$p_{k+1} \in \arg\max_{p\in\mathbb{R}^{n_2}} \left\{q^T(\omega)p - \left\|\varphi(z_k+W^T(\omega)p+\beta\xi,\alpha)\right\|^2\right\},$$
$$z_{k+1} = \varphi\left(z_k + W^T(\omega)p_{k+1}+\beta\xi,\alpha\right). \tag{30}$$

It can be shown that as the smoothing parameter α approaches infinity any solution of smooth problem (29) approaches the solution of the equivalent problem (22) (see [19]).

We begin with a simple lemma that bounds the square difference between the plus function x_+ and its smooth approximation $\varphi(x,\alpha)$.

Lemma 5 (see [13]). *For $x\in\mathbb{R}$ and $|x|<\varpi$*

$$\varphi^2(x,\alpha) - (x_+)^2 \le \left(\frac{\log(2)}{\alpha}\right)^2 + \frac{2\varpi}{\alpha}\log(2), \tag{31}$$

where $\varphi(x,\alpha)$ is the φ function of (28) with smoothing parameter $\alpha>0$.

Theorem 6. *Consider the problems (21) and*

$$\max_{p\in\mathbb{R}^{n_2}} \widehat{S}(p,\beta,\widehat{z},\xi,\omega,\alpha). \tag{32}$$

Then, for any $p\in\mathbb{R}^{n_2}$ and $\alpha>0$

$$\left|S(p,\beta,\widehat{z},\xi,\omega) - \widehat{S}(p,\beta,\widehat{z},\xi,\omega,\alpha)\right| \le \left(\frac{\log(2)}{\alpha}\right)^2 + 2Mm_2\frac{\log(2)}{\alpha}, \tag{33}$$

where M is defined as follows:

$$M = \max_{1\le i\le m_2}\left|\widehat{z}_i + W_i^T(\omega)p+\beta\xi_i\right|. \tag{34}$$

Let p^* be a solution of (21) and p_α^* a solution of (32). Then

$$\max\left\{S(p^*,\beta,\widehat{z},\xi,\omega) - S(p_\alpha^*,\beta,\widehat{z},\xi,\omega),\ \widehat{S}(p_\alpha^*,\beta,\widehat{z},\xi,\omega,\alpha) - \widehat{S}(p^*,\beta,\widehat{z},\xi,\omega,\alpha)\right\} \le \left(\frac{\log(2)}{\alpha}\right)^2 + 2Mm_2\frac{\log(2)}{\alpha}. \tag{35}$$

Further, one assumes that W^T is a full rank matrix. Then,

$$\left\|p^* - p_\alpha^*\right\| \le \mu\left(\frac{2\log(2)}{\alpha}\right)^2 + 8Mm_2\mu\frac{\log(2)}{\alpha}. \tag{36}$$

Proof. For any $\alpha>0$ and $p\in\mathbb{R}^{n_2}$

$$\varphi\left(\widehat{z}_i + W_i^T(\omega)p+\beta\xi_i,\alpha\right) \ge \left(\widehat{z}_i + W_i^T(\omega)p+\beta\xi_i,\alpha\right)_+. \tag{37}$$

Hence

$$\begin{aligned}
&\left|S(p,\beta,\widehat{z},\xi,\omega) - \widehat{S}(p,\beta,\widehat{z},\xi,\omega,\alpha)\right| \\
&= S(p,\beta,\widehat{z},\xi,\omega) - \widehat{S}(p,\beta,\widehat{z},\xi,\omega,\alpha) \\
&= \left\|\varphi(\widehat{z}+W^T(\omega)p+\beta\xi,\alpha)\right\|^2 - \left\|\left(\widehat{z}+W^T(\omega)p+\beta\xi,\alpha\right)_+\right\|^2 \\
&= \sum_{i=1}^{m_2}\left(\varphi^2\left(\widehat{z}_i+W_i^T(\omega)p+\beta\xi_i,\alpha\right) - \left(\widehat{z}_i+W_i^T(\omega)p+\beta\xi_i,\alpha\right)_+^2\right).
\end{aligned} \tag{38}$$

By using Lemma 5, we get that

$$\begin{aligned}
&\left|S(p,\beta,\widehat{z},\xi,\omega) - \widehat{S}(p,\beta,\widehat{z},\xi,\omega,\alpha)\right| \\
&\le \sum_{i=1}^{m_2}\left(\left(\frac{\log(2)}{\alpha}\right)^2 + 2\left|\widehat{z}_i+W_i^T(\omega)p+\beta\xi_i,\alpha\right|\frac{\log(2)}{\alpha}\right) \\
&\le \left(\frac{\log(2)}{\alpha}\right)^2 + 2\frac{\log(2)}{\alpha} \times \sum_{i=1}^{m_2}\max_{1\le i\le m_2}\left|\widehat{z}_i+W_i^T(\omega)p+\beta\xi_i\right| \\
&= \left(\frac{\log(2)}{\alpha}\right)^2 + 2Mm_2\frac{\log(2)}{\alpha}.
\end{aligned} \tag{39}$$

From above inequality, we have

$$\begin{aligned}
\widehat{S}(p,\beta,\widehat{z},\xi,\omega,\alpha) &\le S(p,\beta,\widehat{z},\xi,\omega) \\
&\le \widehat{S}(p,\beta,\widehat{z},\xi,\omega,\alpha) + \left(\frac{\log(2)}{\alpha}\right)^2 + 2Mm_2\frac{\log(2)}{\alpha}.
\end{aligned} \tag{40}$$

Table 1: Comparative between smooth augmented Lagrangian Newton method (SALN) and CPLEX solver.

N. P	Recourse problem $n_2 \times m_2 \times d$	Solver	$\|Wz - q\|_\infty$	$\|\widehat{Q}(\xi) - \xi^T z\|$	$\|z\|$	Time (second)
P1	$50 \times 50 \times 0.68$	SALN	$9.9135e-011$	$7.2760e-012$	41.0362	0.3292
		CPLEX	$1.0717e-007$	$5.2589e-007$	41.0362	0.3182
P2	$100 \times 105 \times 0.4$	SALN	$1.8622e-010$	$4.3074e-009$	57.1793	0.1275
		CPLEX	$6.5591e-008$	$1.6105e-006$	57.1826	0.1585
P3	$150 \times 150 \times 0.5$	SALN	$3.1559e-010$	$5.6361e-009$	74.9098	0.6593
		CPLEX	$5.9572e-010$	$1.2014e-009$	74.9098	0.1605
P4	$200 \times 200 \times 0.5$	SALN	$5.3819e-010$	$1.0506e-008$	85.6646	0.2530
		CPLEX	$1.1734e-007$	$1.3964e-006$	85.6646	0.1820
P5	$300 \times 300 \times 0.5$	SALN	$9.4178e-010$	$2.7951e-008$	102.4325	2.1356
		CPLEX	$4.1638e-010$	$4.4456e-009$	102.4325	0.1830
P6	$350 \times 350 \times 0.5$	SALN	$1.2787e-009$	$2.6226e-008$	110.4189	3.2102
		CPLEX	$7.7398e-010$	$3.4452e-009$	110.4189	0.2116
P7	$450 \times 500 \times 0.05$	SALN	$1.6564e-010$	$4.7094e-009$	124.1204	0.7807
		CPLEX	$9.0949e-013$	$1.6371e-011$	124.1205	0.2606
P8	$500 \times 550 \times 0.04$	SALN	$1.0425e-010$	$1.0241e-009$	134.3999	0.7567
		CPLEX	$6.1618e-011$	$1.9081e-009$	134.3999	0.2660
P9	$700 \times 800 \times 0.6$	SALN	$4.6139e-009$	$2.3908e-007$	153.9782	7.1364
		CPLEX	$9.1022e-009$	$1.3768e-007$	153.9906	0.8435
P10	$900 \times 1100 \times 0.4$	SALN	$5.3396e-009$	$1.5207e-007$	178.1151	7.9383
		CPLEX	$4.5020e-011$	$2.0373e-010$	178.1163	1.3643
P11	$1500 \times 2000 \times 0.1$	SALN	$1.1289e-008$	$2.0696e-007$	231.5284	13.2493
		CPLEX	$7.5886e-011$	$1.1059e-009$	231.5286	2.1343
P12	$1000 \times 2000 \times 0.01$	SALN	$3.6398e-010$	$7.9162e-009$	198.4905	5.2620
		CPLEX	$6.8212e-013$	$1.1642e-010$	198.4922	0.6752
P13	$1000 \times 5000 \times 0.001$	SALN	$2.7853e-010$	$1.9281e-010$	190.0141	0.2709
		CPLEX	$4.8203e-010$	$1.5425e-009$	190.0142	0.2162
P14	$1000 \times 10000 \times 0.001$	SALN	$1.1221e-010$	$1.4988e-009$	212.2416	0.4867
		CPLEX	$7.9094e-009$	$2.6691e-007$	212.3453	0.3353
P15	$1000 \times 1e5 \times 0.001$	SALN	$9.7702e-010$	$2.3283e-009$	231.7930	3.7511
		CPLEX	$2.7285e-012$	$1.8044e-009$	231.8763	1.2472
P16	$1000 \times 1e6 \times 0.0002$	SALN	$9.9432e-013$	$2.0464e-012$	121.3937	9.0948
		CPLEX	$9.9098e-006$	$1.1089e-004$	121.3940	3.7848
P17	$100 \times 1e6 \times 0.001$	SALN	$3.9563e-010$	$5.6607e-009$	73.8493	6.8537
		CPLEX	$2.0082e-003$	$1.1777e-002$	73.9412	2.5582
P18	$10 \times 1e4 \times 0.001$	SALN	$2.2737e-013$	$7.9581e-013$	19.5735	0.0166
		CPLEX	$2.2737e-013$	$1.1369e-013$	19.5739	0.1386
P19	$10 \times 1e6 \times 0.001$	SALN	$1.2478e-009$	$1.0710e-008$	18.8192	5.6399
		CPLEX	$5.9615e-004$	$6.4811e-005$	18.8863	2.5623
P20	$10 \times 5e6 \times 0.01$	SALN	$2.0425e-008$	$1.1816e-008$	20.8470	28.7339
		CPLEX	$4.9966e+004$	$3.3500e+005$	0.0000	1.9482
P21	$100 \times 1e6 \times 0.01$	SALN	$3.8654e-012$	$7.7875e-012$	42.0698	7.8895
		CPLEX	$1.7994e-004$	$6.2712e-005$	42.0931	8.8324
P22	$100 \times 1e4 \times 0.1$	SALN	$1.1084e-012$	$2.1600e-012$	39.4563	0.1440
		CPLEX	$1.2518e-005$	$1.3174e-005$	39.4563	0.3099
P23	$100 \times 1e5 \times 0.05$	SALN	$1.7053e-012$	$6.3121e-012$	43.5944	1.1379
		CPLEX	$3.9827e-006$	$2.8309e-006$	43.5944	1.5729

Table 1: Continued.

N. P	Recourse problem $n_2 \times m_2 \times d$	Solver	$\|Wz-q\|_\infty$	$\|\widehat{Q}(\xi)-\xi^T z\|$	$\|z\|$	Time (second)
P24	$1000 \times 5e4 \times 0.1$	SALN	$1.3074e-012$	$2.4102e-011$	117.4693	15.1065
		CPLEX	$6.7455e-007$	$6.7379e-006$	117.4699	20.3967
P25	$1000 \times 1e5 \times 0.08$	SALN	$2.5011e-012$	$1.0516e-011$	116.6964	23.2540
		CPLEX	$9.4298e-006$	$5.5861e-004$	116.6964	33.4319

Therefore

$$\begin{aligned}
&S(p^*,\beta,\hat{z},\xi,\omega)-S(p^*_\alpha,\beta,\hat{z},\xi,\omega)\\
&\quad\le \widehat{S}(p^*,\beta,\hat{z},\xi,\omega,\alpha)-\widehat{S}(p^*_\alpha,\beta,\hat{z},\xi,\omega,\alpha)\\
&\qquad+\left(\frac{\log(2)}{\alpha}\right)^2+2Mm_2\frac{\log(2)}{\alpha}\\
&\quad\le\left(\frac{\log(2)}{\alpha}\right)^2+2Mm_2\frac{\log(2)}{\alpha},\\
&\widehat{S}(p^*_\alpha,\beta,\hat{z},\xi,\omega,\alpha)-\widehat{S}(p^*,\beta,\hat{z},\xi,\omega,\alpha)\\
&\quad\le S(p^*_\alpha,\beta,\hat{z},\xi,\omega)-S(p^*,\beta,\hat{z},\xi,\omega)\\
&\qquad+\left(\frac{\log(2)}{\alpha}\right)^2+2Mm_2\frac{\log(2)}{\alpha}\\
&\quad\le\left(\frac{\log(2)}{\alpha}\right)^2+2Mm_2\frac{\log(2)}{\alpha}.
\end{aligned}\tag{41}$$

Suppose that W^T is full rank. Then the Hessian of $\widehat{S}$ is negative definite, and $\widehat{S}$ is strongly concave on bounded sets. By the definition of strong concavity, for any $\gamma \in (0,1)$,

$$\begin{aligned}
&\widehat{S}(\gamma p^*_\alpha+(1-\gamma)p^*,\beta,\hat{z},\xi,\omega,\alpha)-\gamma\widehat{S}(p^*_\alpha,\beta,\hat{z},\xi,\omega,\alpha)\\
&\quad-(1-\gamma)\widehat{S}(p^*,\beta,\hat{z},\xi,\omega,\alpha)\\
&\qquad\ge\frac{1}{2}\mu\gamma(1-\gamma)\|p^*_\alpha-p^*\|^2.
\end{aligned}\tag{42}$$

Let $\gamma = 1/2$, then

$$\begin{aligned}
&\frac{1}{8}\mu\|p^*_\alpha-p^*\|^2\\
&\quad\le\widehat{S}\left(\frac{1}{2}(p^*_\alpha+p^*),\beta,\hat{z},\xi,\omega,\alpha\right)\\
&\qquad-\frac{1}{2}\left(\widehat{S}(p^*_\alpha,\beta,\hat{z},\xi,\omega,\alpha)+\widehat{S}(p^*,\beta,\hat{z},\xi,\omega,\alpha)\right)\\
&\quad\le\widehat{S}(p^*_\alpha,\beta,\hat{z},\xi,\omega,\alpha)\\
&\qquad-\frac{1}{2}\left(\widehat{S}(p^*_\alpha,\beta,\hat{z},\xi,\omega,\alpha)+\widehat{S}(p^*,\beta,\hat{z},\xi,\omega,\alpha)\right)\\
&\quad\le\frac{1}{2}\left(\widehat{S}(p^*_\alpha,\beta,\hat{z},\xi,\omega,\alpha)-\widehat{S}(p^*,\beta,\hat{z},\xi,\omega,\alpha)\right)\\
&\quad\le\frac{1}{2}\left(\frac{\log(2)}{\alpha}\right)^2+Mm_2\frac{\log(2)}{\alpha}.
\end{aligned}\tag{43}$$

□

Considering the advantage of the twice differentiability of the objective function of the problem (32) allows us to use a quadratically convergent Newton algorithm with an Armijo stepsize [20] that makes the algorithm globally convergent.

4. Numerical Results and Algorithm

In each iteration of the process (30), one concave, quadratic, unconstrained maximization problem is solved. For solving it, the fast Newton method can be used.

In the algorithm, the Hessian matrix may be singular, thus we use a modified Newton. The direction in each iteration for solving (30) is obtained through the following relation:

$$\begin{aligned}
d_s&=-\left(\nabla^2_p\widehat{S}(p,\beta,\hat{z},\xi,\omega,\alpha)-\delta I_{n_2}\right)^{-1}\left(\nabla_p\widehat{S}(p,\beta,\hat{z},\xi,\omega,\alpha)\right),\\
p_{s+1}&=p_s+\lambda_s d_s,
\end{aligned}\tag{44}$$

where δ is a small positive number, I_{n_2} is the identity matrix of order n_2, and λ_s is the suitable step length that Armijo algorithm is used for determining it (see Algorithm 1).

The proposed algorithm was applied to solve some recourse problems. Table 1 compares this algorithm with CPLEX v. 12.1 solver for quadratic convex programming problems (5). As is evident from Table 1, most of recourse problems could be solved more successful by the algorithm which is based on smooth augmented Lagrangian Newton method (SALN) than CPLEX package (for illustration see the problems 21–25 in Table 1). This algorithm gives us high accuracy and the solution with minimum norm in suitable time (see last column of Table 1). Also, we can find that CPLEX is better than the algorithm proposed for some recourse problems in which the matrices are approximately square (Ex. line 5–12).

The test generator generates recourse problems. These problems are generated using the MATLAB code show in Algorithm 2.

The algorithm considered for solving several recourse problems was run on a computer with 2.5 dual-core CPU and 4 GB memory in MATLAB 7.8 programming environment. Also, in the generated problems, recourse matrix W is the Sparse matrix $(n_2 \times m_2)$ with the density d. The constants β and δ in the above algorithm in (44) were selected 1 and 10^{-8}, respectively.

In Table 1, the second column indicates the size and density of matrix W, the forth column indicates the feasibility of the primal problem (4), and the next column indicates the error norm function of this problem (the MATLAB code of this paper is available from the authors upon request).

Choose a $z_0 \in R^{m_2}$, $\alpha > 1$, $\iota > 1$, $\epsilon > 0$ be error tolerance and δ is a small positive number.
$i = 0$;
While $\|z_i - z_{i-1}\|_\infty \geq \epsilon$
Choose a $p_0 \in R^{n_2}$ and set $k = 0$.
While $\|\nabla_p \widehat{S}(p_k, \beta, z_i, \xi, \omega, \alpha)\|_\infty \geq \epsilon$
Choose $\lambda_k = \max\{s, s\sigma, s\sigma^2, \ldots\}$ *such that*
$\widehat{S}(p_k, \beta, z_i, \xi, \omega, \alpha) - \widehat{S}(p_k + \lambda_k d_k, \beta, z_i, \xi, \omega, \alpha) \geq -\lambda_i \mu \nabla(\widehat{S}(p_k, \beta, z_i, \xi, \omega, \alpha))^T d_k$,
where,
$d_k = -(\nabla^2 \widehat{S}(p_k, \beta, z_i, \xi, \omega, \alpha,) - \delta I_{n_2})^{-1} \nabla \widehat{S}(p_k, \beta, z_i, \xi, \omega, \alpha)$, $s > 0$ *be a constant*,
$\sigma \in (0, 1)$ *and* $\mu \in (0, 1)$.
Put $p_{k+1} = p_k + \lambda_k d_k$, $k = k + 1$ *and* $\alpha = \iota\alpha$.
end
Set $p_{i+1} = p_{k+1}$, $z_{i+1} = \varphi(z_i + W^T(\omega) p_{i+1} + \beta\xi, \alpha)$ *and* $i = i + 1$.
end

ALGORITHM 1: Newton method with the Armijo rule.

```
%Sgen: Generate random solvable recourse problems:
%Input: m,n,d(ensity); Output: W,q,ξ;
m=input('Enter n2:')
n=input('Enter m2:')
d=input('Enter d:')
pl=inline('(abs(x)+x)/2')
W=sprand(n2,m2,d);W=100*(W-0.5*spones(W));
z=sparse(10*pl(rand(m2,1)));
q=W*z;
y=spdiags((sign(pl(rand(n2,1)-rand(n2,1)))),0,n2,n2)
*5*((rand(n2,1)-rand(n2,1)));
ξ=W'*y-10*spdiags((ones(m2,1)-sign(z)),0,m2,m2)*ones(m2,1));
format short e; nnz(W)/prod(size(W))
```

ALGORITHM 2

5. Conclusion

In this paper, a smooth reformulation process, based on augmented Lagrangian algorithm, was proposed for obtaining the normal solution of recourse problem of a stochastic linear programming. This smooth iterative process allows us to use a quadratically convergent Newton algorithm, which accelerates obtaining the normal solution.

Table 1 shows that the proposed algorithm has appropriate speed in most of the problems. This result, specifically, can be observed in recourse problems with the matrix of coefficients in which the number of constraints is noticeably more than the number of variables. The more challenging is solving the problems which their coefficient matrix is square (the numbers of constraints and variables get closer to each other), and more time is needed by the algorithm for solving the problem.

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