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Research Article

A Characterization of Semilinear Dense Range Operators and Applications

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We characterize a broad class of semilinear dense range operators $G_H : W \to Z$ given by the following formula, $G_H w = Gw + H(w), w \in W$, where Z, W are Hilbert spaces, $G \in L(W, Z)$, and $H : W \to Z$ is a suitable nonlinear operator. First, we give a necessary and sufficient condition for the linear operator G to have dense range. Second, under some condition on the nonlinear term H, we prove the following statement: If $\overline{\text{Rang}(G)} = Z$, then $\overline{\text{Rang}(G_H)} = Z$ and for all $z \in Z$ there exists a sequence $\{w_\alpha \in Z : 0 < \alpha \leq 1\}$ given by $w_\alpha = G^*(\alpha I + GG^*)^{-1}(z - H(w_\alpha))$, such that $\lim_{\alpha \to 0^+}\{Gu_\alpha + H(u_\alpha)\} = z$. Finally, we apply this result to prove the approximate controllability of the following semilinear evolution equation: $z' = Az + Bu(t) + F(t, z, u(t)), z \in Z, u \in U, t > 0$, where Z, U are Hilbert spaces, $A : D(A) \subset Z \to Z$ is the infinitesimal generator of strongly continuous compact semigroup $\{T(t)\}_{t \geq 0}$ in $Z, B \in L(U, Z)$, the control function u belongs to $L^2(0, \tau; U)$, and $F : [0, \tau] \times Z \times U \to Z$ is a suitable function. As a particular case we consider the controlled semilinear heat equation.

1. Introduction

It is well known from functional analysis that continuous linear surjective operators form an open set in the space of such operators; that is to say, if a surjective linear continuous operator is added to a linear continuous operator with a small enough norm, the resulting operator is still surjective; moreover, if a linear continuous surjective operator is perturbed by a nonlinear Lipschitz operator with a Lipschitz constant small enough, then the resulting operator is still surjective. This result is not true anymore for continuous linear operators that only have dense range; for instance, if a dense range continuous linear operator is perturbed by another linear operator with norm infinitely small, the resulting operator may not have dense range; in other words, the property of having dense range is not robust enough to be surjective. However, in this paper we proved the following statement: if a continuous linear operator with dense range is perturbed by a compact nonlinear operator with bounded range, then the resulting operator also has dense range. This result can have an unlimited number of applications, not only in the study of control theory for semilinear evolution equations, but it can also be used to find the approximate solution of functional equations in Hilbert spaces giving a formula for the error of this approximation, which is very important from the standpoint of numerical analysis. In addition, it is well known that approximate controllability is much more natural and common than exact controlabilidad, since most of the mechanical processes are diffusive, which implies that these systems can never be exactly controllable; and many years have passed to present a general result on semilinear operators with dense range to facilitate the study of the approximate controlabilidad for a large class of semilinear evolution equations whose dynamics are given by compact semigroups. However, in this work, by way of illustration, we only show how this result can be applied to study the approximate controllability of control systems governed by the semilinear heat equation.

Specifically, in this paper we characterize a broad class of semilinear dense range operators.

$G_H : W \to Z$ given by the following formula:

$$G_H w = Gw + H(w), \quad w \in W, \tag{1}$$

where Z, W are Hilbert spaces, $G : W \rightarrow Z$ is a bounded linear operator (continuous and linear), and $H : W \rightarrow Z$ is a suitable nonlinear operator. First, we give a necessary and sufficient condition for the linear operator G to have dense range ($\overline{\text{Rang}(G)} = Z$). Second, we prove the following statement: If $\overline{\text{Rang}(G)} = Z$ and H is smooth enough and $\overline{\text{Rang}(H)}$ is compact, then $\overline{\text{Rang}(G_H)} = Z$ and for all $z \in Z$ there exists a sequence $\{w_\alpha \in W : 0 < \alpha \leq 1\}$ given by

$$w_\alpha = G^*(\alpha I + GG^*)^{-1}(z - H(w_\alpha)), \tag{2}$$

such that

$$\lim_{\alpha \to 0^+} \{Gw_\alpha + H(w_\alpha)\} = z, \tag{3}$$

and the error of this approximation $E_\alpha z$ is given by

$$E_\alpha z = \alpha(\alpha I + GG^*)^{-1}(z - H(w_\alpha)). \tag{4}$$

This result can be viewed as a generalization of the work done in [1–8].

We apply these results to prove the approximate controllability of the following semilinear evolution equation:

$$\begin{gathered} z' = Az + Bu(t) + F(t, z, u(t)), \\ z \in Z,\ u \in U,\ t > 0, \end{gathered} \tag{5}$$

where Z, U are Hilbert spaces, $A : D(A) \subset Z \rightarrow Z$ is the infinitesimal generator of strongly continuous compact semigroup $\{T(t)\}_{t\geq 0}$ in $Z, B \in L(U, Z)$, the control function u belongs to $L^2(0, \tau; U)$, and $F : [0, \tau] \times Z \times U \rightarrow Z$ is a smooth enough function.

Remark 1 (see [2–4]). The function F is smooth enough if

(a) the mild solutions $z(u) = z_u$ of (5) are unique,

(b) the mild solutions $z(u) = z_u$ depends continuously on u,

(c) and if F is a Lipschitz function, then $z(u) = z_u$, as a function of u, is also a Lipchitz function.

As an application we consider the following example of controlled semilinear heat equation.

Example 2 (the interior controllability of the *nD* heat equation). The semilinear heat equation was studied in [8] where the authors prove the interior controllability of the following control system:

$$\begin{aligned} z_t(t, x) &= \Delta z(t, x) + 1_\omega u(t, x) \\ &\quad + f(t, z, u(t, x)) \quad \text{in } (0, \tau] \times \Omega, \\ z &= 0, \quad \text{on } (0, \tau) \times \partial\Omega, \\ z(0, x) &= z_0(x), \quad x \in \Omega, \end{aligned} \tag{6}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$, $z_0 \in L^2(\Omega)$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, the distributed control u belongs to $\in L^2([0, \tau]; L^2(\Omega))$, and the nonlinear function $f : [0, \tau] \times \mathfrak{R} \times \mathfrak{R} \rightarrow \mathfrak{R}$ is smooth enough and there are constants $a, c \in \mathfrak{R}$, with $c \neq -1$, such that

$$\sup_{(t,z,u)\in q_\tau} |f(t, z, u) - az - cu| < \infty, \tag{7}$$

where $q_\tau = [0, \tau] \times \mathfrak{R} \times \mathfrak{R}$.

We note that the interior approximate controllability of the linear heat equation,

$$\begin{gathered} z_t(t, x) = \Delta z(t, x) + 1_\omega u(t, x) \quad \text{in } (0, \tau] \times \Omega, \\ z = 0, \quad \text{on } (0, \tau) \times \partial\Omega, \\ z(0, x) = z_0(x), \quad x \in \Omega, \end{gathered} \tag{8}$$

has been study by several authors, particularly by [9], and in a general fashion in [10].

The approximate controllability of the heat equation under nonlinear perturbation $f(z)$ independents of t and u variables,

$$\begin{gathered} z_t = \Delta z + 1_\omega u(t, x) + f(z) \quad \text{in } (0, \tau] \times \Omega, \\ z = 0, \quad \text{in } (0, \tau) \times \partial\Omega, \\ z(0, x) = z_0(x), \quad x \in \Omega, \end{gathered} \tag{9}$$

has been studied by several authors, particularly in [11–13], depending on conditions imposed to the nonlinear term $f(z)$. For instance, in [12, 13] the approximate controllability of the system (9) is proved if $f(z)$ is sublinear at infinity; that is,

$$|f(z)| \leq d|z| + e. \tag{10}$$

Also, in the above reference, they mentioned that when f is superlinear at the infinity, the approximate controllability of the system (9) fails.

Our result can be applied also to the semilinear Ornstein-Uhlenbeck equation, the Laguerre equation, and the Jacobi equation. Specifically, in [8], the following well-known example of reaction diffusion equations is studied.

Example 3 (see [14, 15]).

(1) The interior controllability of the semilinear Ornstein-Uhlenbeck equation

$$\begin{aligned} z_t &= \sum_{i=1}^{d} \left[\frac{1}{2} \frac{\partial^2 z}{\partial x_i^2} - x_i \frac{\partial z}{\partial x_i} \right] + 1_\omega u(t, x) \\ &\quad + f(t, z, u) \quad t > 0,\ x \in \mathfrak{R}^d, \end{aligned} \tag{11}$$

where $u \in L^2(0, \tau; L^2(\mathfrak{R}^d, \mu))$, $\mu(x) = (1/\pi^{d/2})\prod_{i=1}^{d} e^{-|x_i|^2} dx$ is the Gaussian measure in $\mathfrak{R}^d$, ω is an open nonempty subset of $\mathfrak{R}^d$, and the nonlinear function

$f : [0,\tau] \times \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ is smooth enough and there are constants $a, c \in \mathfrak{R}$, with $c \neq -1$, such that

$$\sup_{(t,z,u)\in q_\tau} |f(t,z,u) - az - cu| < \infty, \tag{12}$$

where $q_\tau = [0,\tau] \times \mathfrak{R} \times \mathfrak{R}$.

(2) The interior controllability of the semilinear Laguerre equation

$$\begin{aligned} z_t = \sum_{i=1}^{d} \Bigg[x_i \frac{\partial^2 z}{\partial x_i^2} & \\ &+ (\alpha_i + 1 - x_i) \frac{\partial z}{\partial x_i} \Bigg] + 1_\omega u(t,x) \\ &+ f(t,z,u), \quad t > 0,\ x \in \mathfrak{R}_+^d, \end{aligned} \tag{13}$$

where $u \in L^2(0,\tau; L^2(\mathfrak{R}_+^d, \mu_\alpha))$, $\mu_\alpha(x) = \prod_{i=1}^d (x_i^{\alpha_i} e^{-x_i} / \Gamma(\alpha_i + 1))dx$ is the Gamma measure in $\mathfrak{R}_+^d$, ω is an open nonempty subset of $\mathfrak{R}_+^d$, and nonlinear function $f : [0,\tau] \times \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ is smooth enough and there are constant $a, c \in \mathfrak{R}$, with $c \neq -1$, such that

$$\sup_{(t,z,u)\in q_\tau} |f(t,z,u) - az - cu| < \infty, \tag{14}$$

where $q_\tau = [0,\tau] \times \mathfrak{R} \times \mathfrak{R}$.

(3) The interior controllability of the semilinear Jacobi equation

$$\begin{aligned} z_t = \sum_{i=1}^{d} \Bigg[\left(1 - x_i^2\right) \frac{\partial^2 z}{\partial x_i^2} & \\ &+ (\beta_i - \alpha_i - (\alpha_i + \beta_i + 2) x_i) \frac{\partial z}{\partial x_i} \Bigg] \\ &+ 1_\omega u(t,x) + f(t,z,u), \end{aligned} \tag{15}$$

where $t > 0, x \in [-1,1]^d$, $u \in L^2(0,\tau; L^2([-1,1]^d, \mu_{\alpha,\beta}))$, $\mu_{\alpha,\beta}(x) = \prod_{i=1}^d (1-x_i)^{\alpha_i}(1+x_i)^{\beta_i} dx$ is the Jacobi measure in $[-1,1]^d$, ω is an open nonempty subset of $[-1,1]^d$, and the nonlinear function $f : [0,\tau] \times \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ is smooth enough and there are constants $a, c \in \mathfrak{R}$, with $c \neq -1$, such that

$$\sup_{(t,z,u)\in q_\tau} |f(t,z,u) - az - cu| < \infty, \tag{16}$$

where $q_\tau = [0,\tau] \times \mathfrak{R} \times \mathfrak{R}$.

2. Dense Range Linear Operators

In this section we shall present a characterization of dense range bounded linear operators. To this end, we denote by $L(W, Z)$ the space of linear and bounded operators mapping W to Z, endowed with the uniform convergence norm, and we will use the following lemma from [16] in Hilbert space.

Lemma 4. *Let $G^* \in L(Z, W)$ be the adjoint operator of $G \in L(W, Z)$. Then the following statements hold:*

(i) $\mathrm{Rang}(G) = Z \Leftrightarrow \exists \gamma > 0$ *such that*

$$\|G^* z\|_W \geq \gamma \|z\|_Z, \quad z \in Z, \tag{17}$$

(ii) $\overline{\mathrm{Rang}(G)} = Z \Leftrightarrow \mathrm{Ker}(G^*) = \{0\}$.

The following lemma follows from Lemma 4 (ii).

Lemma 5 (see [1, 7, 8, 16–22]). *The following statements are equivalent:*

(a) $\overline{\mathrm{Rang}(G)} = Z$,

(b) $\mathrm{Ker}(G^*) = \{0\}$,

(c) $\langle GG^* z, z \rangle > 0$, $z \neq 0$ *in* Z,

(d) $\lim_{\alpha \to 0^+} \alpha(\alpha I + GG^*)^{-1} z = 0$,

(e) *for all* $z \in Z$ *we have* $G w_\alpha = z - \alpha(\alpha I + GG^*)^{-1} z$, *where*

$$w_\alpha = G^* \left(\alpha I + GG^*\right)^{-1} z, \quad \alpha \in (0,1]. \tag{18}$$

So, $\lim_{\alpha \to 0} G w_\alpha = z$ *and the error* $E_\alpha z$ *of this approximation is given by the formula*

$$E_\alpha z = \alpha \left(\alpha I + GG^*\right)^{-1} z, \quad \alpha \in (0,1]. \tag{19}$$

Remark 6. Lemma 5 implies that the family of linear operators $\Gamma_\alpha : Z \to W$, defined for $0 < \alpha \leq 1$ by

$$\Gamma_\alpha z = G^* \left(\alpha I + GG^*\right)^{-1} z, \tag{20}$$

is an approximate inverse for the right of the operator G, in the sense that

$$\lim_{\alpha \to 0} G\Gamma_\alpha = I \tag{21}$$

in the strong topology.

Proposition 7. *If the* $\overline{\mathrm{Rang}(G)} = Z$, *then*

$$\sup_{\alpha > 0} \left\| \alpha \left(\alpha I + GG^*\right)^{-1} \right\| \leq 1. \tag{22}$$

Proof. If $\overline{\mathrm{Rang}(G)} = Z$, then from Lemma 4(ii) we have that

$$\langle GG^* z, z \rangle > 0, \quad z \neq 0. \tag{23}$$

Therefore,

$$\langle (GG^* + \alpha I) z, z \rangle \geq \alpha \|z\|^2, \quad z \neq 0,\ \alpha \in (0,1]. \tag{24}$$

Then, using the Cauchy Schwartz inequality, we obtain

$$\left\| (GG^* + \alpha I) z \right\| \geq \alpha \|z\|, \quad z \neq 0,\ \alpha \in (0,1], \tag{25}$$

which is equivalents to

$$\alpha \left\| (GG^* + \alpha I)^{-1} z \right\| \leq \|z\|, \quad z \neq 0,\ \alpha \in (0,1]. \tag{26}$$

Consequently,

$$\sup_{\alpha > 0} \left\| \alpha \left(\alpha I + GG^*\right)^{-1} \right\| \leq 1. \tag{27}$$

□

Proposition 8. *If for some* $\beta \in (0,1]$ *one has that*

$$\left\|\beta(\beta I + GG^*)^{-1}\right\| < 1, \tag{28}$$

then

$$\operatorname{Rang}(G) = Z. \tag{29}$$

Proof. Suppose that $\|\beta(\beta I + GG^*)^{-1}\| < 1$. Then, from the following identity:

$$GG^* = \beta I + GG^* - \beta I, \tag{30}$$

we get that

$$GG^*(\beta I + GG^*)^{-1} = I - \beta(\beta I + GG^*)^{-1}. \tag{31}$$

Since $\|\beta(\beta I + GG^*)^{-1}\| < 1$, we obtain that $GG^*(\beta I + GG^*)^{-1}$ is a homeomorphism. Consequently, $\operatorname{Rang}(GG^*(\beta I + GG^*)^{-1}) = Z$, which implies that $\operatorname{Rang}(G) = Z$. □

Corollary 9. *If* $\overline{\operatorname{Rang}(G)} = Z$ *and* $\operatorname{Rang}(G) \neq Z$, *then*

$$\left\|\alpha(\alpha I + GG^*)^{-1}\right\| = 1, \quad \forall \alpha \in (0,1]. \tag{32}$$

Moreover,

$$\lim_{\alpha\to 0^+}\left\|(\alpha I + GG^*)^{-1}\right\| = \infty. \tag{33}$$

3. Dense Range Semilinear Operators

In this section we shall look for conditions under which the semilinear operator
$G_H : W \to Z$, given by

$$G_H w = Gw + H(w), \quad w \in W, \tag{34}$$

has dense range.

Theorem 10. *If* $\overline{\operatorname{Rang}(G)} = Z$, *H is continuous, and* $\overline{\operatorname{Rang}(H)}$ *is compact, then* $\overline{\operatorname{Rang}(G_H)} = Z$, *and for all* $z \in Z$ *there exists a sequence* $\{w_\alpha \in Z : 0 < \alpha \leq 1\}$ *given by*

$$w_\alpha = G^*(\alpha I + GG^*)^{-1}(z - H(w_\alpha)), \tag{35}$$

such that

$$\lim_{\alpha\to 0^+}\{Gw_\alpha + H(w_\alpha)\} = z, \tag{36}$$

and the error of this approximation $E_\alpha z$ *is given by*

$$E_\alpha z = \alpha(\alpha I + GG^*)^{-1}(z - H(w_\alpha)). \tag{37}$$

Proof. For each $z \in Z$ fixed we shall consider the following family of nonlinear operators $K_\alpha : W \to W$ given by

$$\begin{aligned} K_\alpha(w) &= \Gamma_\alpha(z - H(w)) \\ &= G^*(\alpha I + GG^*)^{-1}(z - H(w)), \quad (0 < \alpha \leq 1). \end{aligned} \tag{38}$$

First, we shall prove that for all $\alpha \in (0,1]$ the operator K_α has a fix point w_α. In fact, since H is a continuous function, the set $\overline{\operatorname{Rang}(H)}$ is compact, and G is a linear bounded operator, then there exists a constant $M > 0$ such that

$$\|K_\alpha(w)\| \leq \|\Gamma_\alpha\|(\|z\| + M), \quad \forall w \in W. \tag{39}$$

Therefore, the operator K_α maps the ball $B_r(0) \subset W$ of center zero and radio $r \geq \|\Gamma_\alpha\|(\|z\| + M)$ into itself. Hence, applying the Schauder fixed point theorem, we get that the operator K_α has a fixed point $w_\alpha \in B_r(0) \subset W$.

Since $\overline{\operatorname{Rang}(H)}$ is compact, without loss of generality, we can assume that the sequence $H(w_\alpha)$ converges to $y \in Z$ as $\alpha \to 0$. So, if we consider

$$w_\alpha = \Gamma_\alpha(z - H(w_\alpha)) = G^*(\alpha I + GG^*)^{-1}(z - H(w_\alpha)), \tag{40}$$

then,

$$\begin{aligned} Gw_\alpha &= G\Gamma_\alpha(z - H(w_\alpha)) = GG^*(\alpha I + GG^*)^{-1}(z - H(w_\alpha)) \\ &= (\alpha I + GG^* - \alpha I)(\alpha I + GG^*)^{-1}(z - H(w_\alpha)) \\ &= z - H(w_\alpha) - \alpha(\alpha I + GG^*)^{-1}(z - H(w_\alpha)). \end{aligned} \tag{41}$$

Hence,

$$Gw_\alpha + H(w_\alpha) = z - \alpha(\alpha I + GG^*)^{-1}(z - H(w_\alpha)). \tag{42}$$

To conclude the proof of this theorem, it is enough to prove that

$$\lim_{\alpha\to 0}\left\{-\alpha(\alpha I + GG^*)^{-1}(z - H(w_\alpha))\right\} = 0. \tag{43}$$

From Lemma 5(d) we get that

$$\begin{aligned} &\lim_{\alpha\to 0}\left\{-\alpha(\alpha I + GG^*)^{-1}(z - H(w_\alpha))\right\} \\ &\quad = -\lim_{\alpha\to 0}\left\{-\alpha(\alpha I + GG^*)^{-1}H(w_\alpha)\right\} \\ &\quad = \lim_{\alpha\to 0}\alpha(\alpha I + GG^*)^{-1}(H(w_\alpha) - y + y) \\ &\quad = \lim_{\alpha\to 0} -\alpha(\alpha I + GG^*)^{-1}(H(w_\alpha) - y). \end{aligned} \tag{44}$$

On the other hand, from Proposition 7 we get that

$$\left\|\alpha(\alpha I + GG^*)^{-1}(H(w_\alpha) - y)\right\| \leq \left\|(H(w_\alpha) - y)\right\|. \tag{45}$$

Therefore, since $H(w_\alpha)$ converges to y as $\alpha \to 0$, we get that

$$\lim_{\alpha\to 0}\left\{-\alpha(\alpha I + GG^*)^{-1}(H(w_\alpha) - y)\right\} = 0. \tag{46}$$

Consequently,

$$\lim_{\alpha\to 0} G_H(w_\alpha) = z. \tag{47}$$

□

4. Controllability of Nonlinear Evolution Equations

In this section we shall apply the foregoing results to characterize the approximate controllability of the semilinear evolution equation

$$z' = Az + Bu(t) + F(t, z, u(t)), \quad z \in Z,\ u \in U,\ t > 0, \tag{48}$$

where Z, U are Hilbert spaces, $A : D(A) \subset Z \to Z$ is the infinitesimal generator of strongly continuous compact semigroup $\{T(t)\}_{t\geq 0}$ in Z, $B \in L(U, Z)$, the control function u belongs to $L^2(0, \tau; U)$, and $F : [0, \tau] \times Z \times U \to Z$ is smooth enough and there are constants $a, c \in \mathfrak{R}$ such that

$$\sup_{(t,z,u)\in Z_\tau} \left\| F(t, z, u) - az - cB_1 u \right\|_Z < \infty, \tag{49}$$

where $Z_\tau = [0, \tau] \times Z \times U$ and $B_1 : U \to Z$ is a linear and bounded operator.

We observe that the controllability of semilinear systems has been studied by several authors, particularly interesting is the work done by [18–26].

Definition 11 (exact controllability). The system (48) is said to be exactly controllable on $[0, \tau]$ if for every $z_0, z_1 \in Z$ there exists $u \in L^2(0, \tau; U)$ such that the mild solution $z(t)$ of (48) corresponding to u verifies

$$z(0) = z_0, \qquad z(\tau) = z_1, \tag{50}$$

as shown in Figure 1.

Definition 12 (approximate controllability). The system (48) is said to be approximately controllable on $[0, \tau]$ if for every $z_0, z_1 \in Z$, $\varepsilon > 0$ there exists $u \in L^2(0, \tau; U)$ such that the solution $z(t)$ of (48) corresponding to u verifies

$$z(0) = z_0, \qquad \left\| z(\tau) - z_1 \right\| < \varepsilon, \tag{51}$$

as shown in Figure 2.

Definition 13 (controllability to trajectories). The system (48) is said to be controllable to trajectories on $[0, \tau]$ if for every $z_0, \hat{z}_0 \in Z$ and $\hat{u} \in L^2(0, \tau; U)$ there exists $u \in L^2(0, \tau; U)$ such that the mild solution $z(t)$ of (48) corresponding to u verifies:

$$z(\tau, z_0, u) = z(\tau, \hat{z}_0, \hat{u}), \tag{52}$$

as shown in Figure 3.

Definition 14 (null controllability). The system (48) is said to be null controllable on $[0, \tau]$ if for every $z_0 \in Z$ there exists $u \in L^2(0, \tau; U)$ such that the mild solution $z(t)$ of (48) corresponding to u verifies:

$$z(0) = z_0, \qquad z(\tau) = 0, \tag{53}$$

as shown in Figure 4.

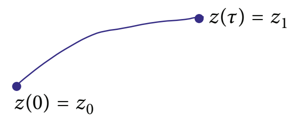

Figure 1

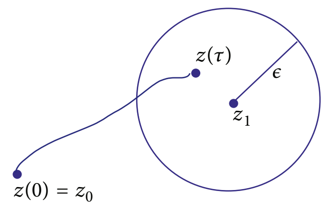

Figure 2

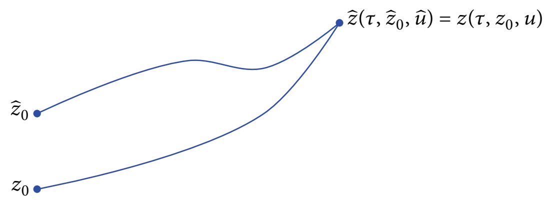

Figure 3

Remark 15. It is clear that exact controllability of the system (48) implies approximate controllability, null controllability, and controllability to trajectories of the system. But, it is well known [27] that due to the diffusion effect or the compactness of the semigroup generated by $-\Delta$, the heat equation can never be exactly controllable. We observe also that the linear case controllability to trajectories and null controllability are equivalent. Nevertheless, the approximate controllability and the null controllability are in general independent. Therefore, in this paper we will concentrated only on the study of the approximate controllability of the system (48).

Now, we shall describe the strategy of this work:

First, we characterize the approximate controllability of the auxiliary linear system

$$z' = Az + Bu(t) + az + cB_1 u(t), \quad t \in [0, \tau]. \tag{54}$$

After that, we write the system (48) in the form

$$z' = Az + Bu(t) + az + cB_1 u(t) + G(t, z, u), \quad t \in [0, \tau], \tag{55}$$

where $G(t, z, u) = F(t, z, u) - az - cB_1 u$ is a smooth enough and bounded function.

Finally, the approximate controllability of the system (55) follows from the controllability of (54), the compactness of the semigroup generated by the operator A, the uniform

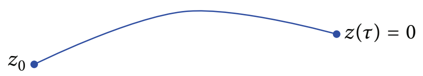

Figure 4

boundedness of the nonlinear term G, and applying Schauder fixed point theorem.

Remark 16. If $c \neq 1$ and $B = B_1$, then the system $z' = Az + Bu(t)$ is approximately controllable if and only if the system (55) is approximately controllable.

4.1. The Linear System. First, we shall characterize the approximate controllability of the linear system (54), and to this end, for all $z_0 \in Z$ and $u \in L^2(0,\tau;U)$ the initial value problem

$$\begin{gathered} z' = Az + Bu(t) + az + cB_1 u(t), \quad t > 0 \\ z(0) = z_0, \end{gathered} \tag{56}$$

admits only one mild solution given by

$$\begin{aligned} z(t) &= e^{at}T(t)z_0 \\ &\quad + \int_0^t e^{a(t-s)}T(t-s)(B + cB_1)u(s)\,ds, \quad t \in [0,\tau]. \end{aligned} \tag{57}$$

Definition 17. For the system (54) we define the following concept: the controllability map (for $\tau > 0$) $G_a : L^2(0,\tau;U) \to Z$ is given by

$$G_a u = \int_0^\tau e^{as}T(s)(B + cB_1)u(s)\,ds, \tag{58}$$

whose adjoint operator $G_a^* : Z \to L^2(0,\tau;Z)$ is

$$(G_a^* z)(s) = (B^* + cB_1^*)e^{as}T^*(s)z, \quad \forall s \in [0,\tau], \forall z \in Z. \tag{59}$$

The following lemma follows from Lemma 5.

Lemma 18. *Equation* (54) *is approximately controllable on* $[0,\tau]$ *if and only if one of the following statements holds:*

(a) $\overline{\mathrm{Rang}(G_a)} = Z$,

(b) $\mathrm{Ker}(G_a^*) = \{0\}$,

(c) $\langle G_a G_a^* z, z\rangle > 0$, $z \neq 0$ *in* Z,

(d) $\lim_{\alpha \to 0^+} \alpha(\alpha I + G_a G_a^*)^{-1} z = 0$,

(e) $(B^* + cB_1^*)e^{at}T^*(t)z = 0$, $\forall t \in [0,\tau]$, $\Rightarrow z = 0$,

(f) *for all* $z \in Z$ *one has* $G_a u_\alpha = z - \alpha(\alpha I + G_a G_a^*)^{-1} z$, *where*

$$u_\alpha = G_a^*(\alpha I + G_a G_a^*)^{-1} z, \quad \alpha \in (0,1]. \tag{60}$$

So, $\lim_{\alpha \to 0} G_a u_\alpha = z$ *and the error* $E_\alpha z$ *of this approximation is given by the formula*

$$E_\alpha z = \alpha(\alpha I + G_a G_a^*)^{-1} z, \quad \alpha \in (0,1]. \tag{61}$$

Remark 19. Lemma 5 implies that the family of linear operators $\Gamma_\alpha : Z \to L^2(0,\tau;U)$, defined for $0 < \alpha \leq 1$ by

$$\begin{aligned} \Gamma_\alpha z &= (B^* + cB_1^*)e^{a(\tau-\cdot)}T^*(\tau-\cdot)(\alpha I + G_a G_a^*)^{-1} z \\ &= G_a^*(\alpha I + G_a G_a^*)^{-1} z, \end{aligned} \tag{62}$$

is an approximate inverse for the right of the operator G_a, in the sense that

$$\lim_{\alpha \to 0} G_a \Gamma_\alpha = I \tag{63}$$

in the strong topology.

4.2. The Semilinear System. Now, we are ready to characterize the approximate controllability of the semilinear system (48), which is equivalent to proof of the approximate controllability of the system (55). To this end, we notice that, for all $z_0 \in Z$ and $u \in L^2(0,\tau;U)$ the initial value problem

$$\begin{gathered} z' = Az + Bu + az + cB_1 u + G(t,z,u), \quad z \in Z,\ t \geq 0, \\ z(0) = z_0 \end{gathered} \tag{64}$$

admits only one mild solution given by

$$\begin{aligned} z_u(t) &= e^{at}T(t)z_0 + \int_0^t e^{a(t-s)}T(t-s)(B + cB_1)u(s)\,ds \\ &\quad + \int_0^t e^{a(t-s)}T(t-s)G(s, z_u(s), u(s))\,ds, \quad t \in [0,\tau]. \end{aligned} \tag{65}$$

Definition 20. For the system (55) we define the following concept: the nonlinear controllability map (for $\tau > 0$) $G_g : L^2(0,\tau;U) \to Z$ is given by

$$\begin{aligned} G_g u &= \int_0^\tau e^{a(\tau-s)}T((\tau-s))(B + cB_1)u(s)\,ds \\ &\quad + \int_0^\tau e^{a(\tau-s)}T((\tau-s))G(s, z_u(s), u(s))\,ds \\ &= G_a(u) + H(u), \end{aligned} \tag{66}$$

where $H : L^2(0,\tau;U) \to Z$ is the nonlinear operator given by

$$\begin{aligned} H(u) = \int_0^\tau e^{a(\tau-s)}T((\tau-s))G(s, z_u(s), u(s))\,ds, \\ u \in L^2(0,\tau;U). \end{aligned} \tag{67}$$

The following lemma is trivial.

Lemma 21. *Equation* (55) *is approximately controllable on* $[0,\tau]$ *if and only if* $\overline{\text{Rang}(G_g)} = Z$.

Definition 22. The following equation will be called the controllability equations associated to the nonlinear equation (55)

$$\begin{aligned} u_\alpha &= \Gamma_\alpha (z - H(u_\alpha)) \\ &= G_a^* (\alpha I + G_a G_a^*)^{-1} (z - H(u_\alpha)), \quad (0 < \alpha \le 1). \end{aligned} \tag{68}$$

Now, we are ready to present a result on the approximate controllability of the semilinear evolutions equation (48).

Theorem 23. *If the linear system* (54) *is approximately controllable, then system* (55) *is approximately controllable on* $[0,\tau]$. *Moreover, a sequence of controls steering the system* (55) *from initial state* z_0 *to an* ϵ *neighborhood of the final state* z_1 *at time* $\tau > 0$ *is given by the formula*

$$\begin{aligned} u_\alpha(t) = {} & (B^* + cB_1^*) e^{a(\tau - t)} T^* (\tau - t) \\ & \times (\alpha I + G_a G_a^*)^{-1} (z_1 - e^{a\tau} T(\tau) z_0 - H(u_\alpha)), \end{aligned} \tag{69}$$

and the error of this approximation E_α *is given by*

$$E_\alpha = \alpha (\alpha I + G_a G_a^*)^{-1} (z_1 - e^{a\tau} T(\tau) z_0 - H(u_\alpha)). \tag{70}$$

Proof. From Theorem 10, it is enough to prove that the function H given by (103) is continuous and $\overline{\text{Rang}(H)}$ is a compact set, which follows from the compactness of the semigroup $\{T(t)\}_{t\ge0}$, the smoothness and the boundedness of the nonlinear term G (see [8, 27]).

So, putting $z = z_1 - e^{a\tau}T(\tau)z_0$ and using (65), we obtain the desired result

$$\begin{aligned} z_1 = \lim_{\alpha \to 0^+} \Big\{ & T(\tau) z_0 + \int_0^\tau T(\tau - s) B u_\alpha(s)\, ds \\ & + \int_0^\tau T(\tau - s) F(s, z_{u_\alpha}(s), u_\alpha(s))\, ds \Big\}. \end{aligned} \tag{71}$$

□

5. Application to the Nonlinear Heat Equation

As an application of this result we shall prove the controllability of the semilinear *nD* heat equation (6). To this end, we shall use the following strategy:

first, we prove that the auxiliary linear system

$$\begin{aligned} z_t(t,x) &= \Delta z(t,x) + 1_\omega u(t,x) \\ &\quad + az + cu(t,x) \quad \text{in } (0,\tau] \times \Omega, \\ z &= 0, \quad \text{on } (0,\tau) \times \partial\Omega, \\ z(0,x) &= z_0(x), \quad x \in \Omega, \end{aligned} \tag{72}$$

is approximately controllable.

After that, we write the system(6) as follows:

$$\begin{aligned} z_t(t,x) &= \Delta z(t,x) + 1_\omega u(t,x) + az \\ &\quad + cu(t,x) + g(t,z,u) \quad \text{in } (0,\tau] \times \Omega, \\ z &= 0, \quad \text{on } (0,\tau) \times \partial\Omega, \\ z(0,x) &= z_0(x), \quad x \in \Omega, \end{aligned} \tag{73}$$

where $g(t,z,u) = f(t,z,u) - az - cu$ is a smooth and bounded function.

Then to prove the controllability of the linear equation (72), we use the classical Unique Continuation Principle for Elliptic Equations (see [28]) and the following results.

Lemma 24 (see Lemma 3.14 from [16, page 62]). *Let* $\{\alpha_j\}_{j\ge1}$ *and* $\{\beta_{i,j} : i = 1,2,\ldots,m\}_{j\ge1}$ *be two sequences of real numbers such that:* $\alpha_1 > \alpha_2 > \alpha_3 \cdots$. *Then*

$$\sum_{j=1}^{\infty} e^{\alpha_j t} \beta_{i,j} = 0, \quad \forall t \in [0, t_1], \; i = 1,2,\ldots,m, \tag{74}$$

iff

$$\beta_{i,j} = 0, \quad i = 1,2,\ldots,m; \; j = 1,2,\ldots,\infty. \tag{75}$$

Finally, the approximate controllability of the system (73) *follows from the controllability of* (72), *the compactness of the semigroup generated by the Laplacean operator* Δ, *and the uniform boundedness of the nonlinear term* g *by applying Theorem 23.*

5.1. Abstract Formulation of the Problem. In this part we choose a Hilbert space where system (6) can be written as an abstract differential equation; to this end, we consider the following notations.

Let us consider the Hilbert space $Z = L^2(\Omega)$ and $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_j \to \infty$ the eigenvalues of $-\Delta$, each one with finite multiplicity γ_j equal to the dimension of the corresponding eigenspace. Then we have the following well-known properties.

(i) There exists a complete orthonormal set $\{\phi_{j,k}\}$ of eigenvectors of $A = -\Delta$.

(ii) For all $z \in D(A)$ we have

$$Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle \xi, \phi_{j,k} \rangle \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j z, \tag{76}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in Z and

$$E_n z = \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}. \tag{77}$$

So, $\{E_j\}$ is a family of complete orthogonal projections in Z and $z = \sum_{j=1}^{\infty} E_j z, z \in H$.

(iii) $-A$ generates a compact analytic semigroup $\{T(t)\}$ given by

$$T(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j z. \tag{78}$$

Consequently, systems (6), (72), and (73) can be written, respectively, as an abstract differential equations in Z:

$$z' = -Az + B_\omega u + f^e(t,z,u), \quad z \in Z,\ t \geq 0, \tag{79}$$

$$z' = -Az + B_\omega u + az + cu, \quad z \in Z,\ t \geq 0, \tag{80}$$

$$z' = -Az + B_\omega u + az + cu + g^e(t,z,u), \quad z \in Z,\ t \geq 0, \tag{81}$$

where $u \in L^2([0,\tau];U)$, $U = Z$, $B_\omega : U \to Z$, $B_\omega u = 1_\omega u$ is a bounded linear operator, $f^e : [0,\tau] \times Z \times U \to Z$ is defined by $f^e(t,z,u)(x) = f(t,z(x),u(x)), \forall x \in \Omega$, and $g^e(t,z,u) = f^e(t,z,u) - az - cu$.

On the other hand, the hypothesis (7) implies that

$$\sup_{(t,z,u)\in Z_\tau} \left\| f^e(t,z,u) - az - cu \right\|_Z < \infty, \tag{82}$$

where $Z_\tau = [0,\tau] \times Z \times U$. Therefore, $g^e : [0,\tau] \times Z \times U \to Z$ is bounded and smooth enough.

5.2. The Linear Heat Equation. In this part we shall prove the interior controllability of the linear system (80). To this end, we notice that for all $z_0 \in Z$ and $u \in L^2(0,\tau;U)$ the initial value problem,

$$\begin{aligned} z' &= -Az + B_\omega u(t) + az(t) + cu(t), \quad z \in Z, \\ z(0) &= z_0, \end{aligned} \tag{83}$$

admits only one mild solution given by

$$\begin{aligned} z(t) &= e^{at} T(t) z_0 \\ &\quad + \int_0^t e^{a(t-s)} T(t-s) \left(B_\omega + cI\right) u(s)\, ds, \quad t \in [0,\tau]. \end{aligned} \tag{84}$$

Definition 25. For the system (80) we define the following concept: the controllability map (for $\tau > 0$) $G_a : L^2(0,\tau;U) \to Z$ is given by

$$G_a u = \int_0^\tau e^{as} T(s) \left(B_\omega + cI\right) u(s)\, ds, \tag{85}$$

whose adjoint operator $G_a^* : Z \to L^2(0,\tau;Z)$ is given by

$$\left(G_a^* z\right)(s) = \left(B_\omega^* + cI\right) e^{as} T^*(s) z, \quad \forall s \in [0,\tau],\ \forall z \in Z. \tag{86}$$

As a consequence of Lemma 18 and (101) one can prove the following result.

Lemma 26. *Equation* (80) *is approximately controllable on* $[0,\tau]$ *if and only if one of the following statements holds:*

(a) $\overline{\mathrm{Rang}(G_a)} = Z$,

(b) $\mathrm{Ker}(G_a^*) = \{0\}$,

(c) $\langle G_a G_a^* z, z\rangle > 0$, $z \neq 0$ *in* Z,

(d) $\lim_{\alpha \to 0^+} \alpha(\alpha I + G_a G_a^*)^{-1} z = 0$,

(e) $(B_\omega^* + aI)e^{at} T^*(t) z = 0$, $\forall t \in [0,\tau]$, $\Rightarrow z = 0$,

(f) *for all* $z \in Z$ *one has* $G u_\alpha = z - \alpha(\alpha I + G_a G_a^*)^{-1} z$, *where*

$$u_\alpha = G_a^* \left(\alpha I + G_a G_a^*\right)^{-1} z, \quad \alpha \in (0,1]. \tag{87}$$

So, $\lim_{\alpha \to 0} G_a u_\alpha = z$ *and the error* $E_\alpha z$ *of this approximation is given by*

$$E_\alpha z = \alpha\left(\alpha I + G_a G_a^*\right)^{-1} z, \quad \alpha \in (0,1]. \tag{88}$$

Theorem 27. *The system* (80) *is approximately controllable on* $[0,\tau]$. *Moreover, a sequence of controls steering the system* (80) *from initial state* z_0 *to an* ϵ *neighborhood of the final state* z_1 *at time* $\tau > 0$ *is given by*

$$\begin{aligned} u_\alpha(t) = &\left(B_\omega^* + cI\right) e^{at} T^*(\tau - t) \\ &\times \left(\alpha I + G_a G_a^*\right)^{-1} \left(z_1 - T(\tau) z_0\right), \end{aligned} \tag{89}$$

and the error of this approximation E_α *is given by*

$$E_\alpha = \alpha\left(\alpha I + G_a G_a^*\right)^{-1} \left(z_1 - T(\tau) z_0\right). \tag{90}$$

Proof. It is enough to show that the restriction $G_{a,\omega} = G_a|_{L^2(0,\tau;L^2(\omega))}$ of G_a to the space $L^2(0,\tau;L^2(\omega))$ has range dense; that is, $\overline{\mathrm{Rang}(G_{a,\omega})} = Z$ or $\mathrm{Ker}(G_{a,\omega}^*) = \{0\}$. Consequently, $G_{a,\omega} : L^2(0,\tau;L^2(\omega)) \to Z$ takes the following form:

$$G_{a,\omega} u = \int_0^\tau e^{as} T(s) (1 + cI) B_\omega u(s)\, ds, \tag{91}$$

whose adjoint operator $G_{a,\omega}^* : Z \to L^2(0,\tau;L^2(\omega))$ is given by

$$\left(G_{a,\omega} z\right)(s) = (1 + c) B_\omega^* e^{as} T^*(s) z, \quad \forall s \in [0,\tau],\ \forall z \in Z. \tag{92}$$

To this end, we observe that $B_\omega = B_\omega^*$ and $T^*(t) = T(t)$. Suppose that

$$(1 + c) B_\omega^* e^{at} T^*(t) z = 0, \quad \forall t \in [0,\tau]. \tag{93}$$

Then, since $1 + c \neq 0$, this is equivalent to

$$B_\omega^* T^*(t) z = 0, \quad \forall t \in [0,\tau]. \tag{94}$$

On the other hand,

$$B_\omega^* T^*(t) z = \sum_{j=1}^{\infty} e^{-\lambda_j t} B_\omega^* E_j z = \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} \left\langle z, \phi_{j,k} \right\rangle 1_\omega \phi_{j,k} = 0,$$

$$\Longleftrightarrow \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} \left\langle z, \phi_{j,k} \right\rangle 1_\omega \phi_{j,k}(x) = 0, \quad \forall x \in \omega. \tag{95}$$

Hence, from Lemma 24, we obtain that

$$E_j z(x) = \sum_{k=1}^{\gamma_j} \left\langle z, \phi_{j,k} \right\rangle \phi_{j,k}(x) = 0, \quad \forall x \in \omega,\ j = 1, 2, 3, \ldots. \tag{96}$$

Now, putting $f(x) = \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}(x)$, $\forall x \in \Omega$, we obtain that

$$\begin{gathered} \left(\Delta + \lambda_j I\right) f \equiv 0 \quad \text{in } \Omega, \\ f(x) = 0 \quad \forall x \in \omega. \end{gathered} \tag{97}$$

Then, from the classical Unique Continuation Principle for Elliptic Equations (see [28]), it follows that $f(x) = 0$, $\forall x \in \Omega$. So,

$$\sum_{k=1}^{\gamma_j} \left\langle z, \phi_{j,k} \right\rangle \phi_{j,k}(x) = 0, \quad \forall x \in \Omega. \tag{98}$$

On the other hand, $\{\phi_{j,k}\}$ is a complete orthonormal set in $Z = L^2(\Omega)$, which implies that $\langle z, \phi_{j,k} \rangle = 0$. Hence, $z = 0$. So, $\overline{\text{Rang}(G_{a,\omega})} = Z$, and consequently $\overline{\text{Rang}(G_a)} = Z$. Hence, the system (80) is approximately controllable on $[0, \tau]$, and the remainder of the proof follows from Lemma 26. □

5.3. The Semilinear Heat Equation. In this part we shall prove the interior controllability of the semilinear *nD* heat equation given by (6), which is equivalent to the proof of the approximate controllability of the system (81). To this end, for all $z_0 \in Z$ and $u \in L^2(0, \tau; U)$ the initial value problem,

$$\begin{gathered} z' = -Az + B_\omega u + az + cu + g^e(t, z, u), \quad z \in Z, t \geq 0 \\ z(0) = z_0, \end{gathered} \tag{99}$$

admits only one mild solution given by

$$\begin{aligned} z_u(t) = e^{at} T(t) z_0 &+ \int_0^t e^{a(t-s)} T(t-s) \left(B_\omega + cI\right) u(s)\, ds \\ &+ \int_0^t e^{a(t-s)} T(t-s) g^e\left(s, z_u(s), (s)\right) ds, \quad t \in [0, \tau]. \end{aligned} \tag{100}$$

Definition 28. For the system (81) we define the following concept: the nonlinear controllability map (for $\tau > 0$) $G_g : L^2(0, \tau; U) \to Z$ is given by

$$G_g u = \int_0^\tau e^{a(\tau-s)} T(\tau - s) \left(B_\omega + cI\right) u(s)\, ds \tag{101}$$

$$\begin{aligned} &+ \int_0^\tau e^{a(\tau-s)} T(\tau - s) g^e\left(s, z_u(s), (s)\right) ds \\ &= G_a(u) + H(u), \end{aligned} \tag{102}$$

where $H : L^2(0, \tau; U) \to Z$ is the nonlinear operator given by

$$H(u) = \int_0^\tau e^{a(\tau-s)} T(\tau - s) g^e\left(s, z_u(s), (s)\right) ds, \quad u \in L^2(0, \tau; U). \tag{103}$$

The following lemma is trivial.

Lemma 29. *Equation* (81) *is approximately controllable on* $[0, \tau]$ *if and only if* $\overline{\text{Rang}(G_g)} = Z$.

Definition 30. The following equation will be called the controllability equations associated to the nonlinear equation (81):

$$u_\alpha = \Gamma_\alpha\left(z - H(u_\alpha)\right) = G_a^* \left(\alpha I + G_a G_a^*\right)^{-1} \left(z - H(u_\alpha)\right), \quad (0 < \alpha \leq 1). \tag{104}$$

Now, we are ready to present a result on the interior approximate controllability of the semilinear *nD* heat equation (6).

Theorem 31. *The system* (81) *is approximately controllable on* $[0, \tau]$. *Moreover, a sequence of controls steering the system* (81) *from initial state* z_0 *to an* ϵ *neighborhood of the final state* z_1 *at time* $\tau > 0$ *is given by*

$$\begin{aligned} u_\alpha(t) = &\left(B_\omega^* + cI\right) e^{a(\tau-t)} T^*(\tau - t) \left(\alpha I + G_a G_a^*\right)^{-1} \\ &\times \left(z_1 - T(\tau) z_0 - H(u_\alpha)\right), \end{aligned} \tag{105}$$

and the error of this approximation E_α *is given by*

$$E_\alpha = \alpha \left(\alpha I + G_a G_a^*\right)^{-1} \left(z_1 - T(\tau) z_0 - H(u_\alpha)\right). \tag{106}$$

6. Conclusion

We believe that these results can be applied to a broad class of reaction diffusion equation like the following well-known systems of partial differential equations.

Example 32. The thermoelastic plate equation

$$\begin{aligned} &w_{tt} + \Delta^2 w + \alpha \Delta w \\ &\quad = 1_\omega u_1(t,x) + f_1(t,w,w_t,u), \quad \text{in } (0,\tau)\times\Omega, \\ &\theta_t - \beta\Delta\theta - \alpha\Delta w_t \\ &\quad = 1_\omega u_2(t,x) + f_2(t,w,w_t,u), \quad \text{in } (0,\tau)\times\Omega, \\ &\theta = w = \Delta w = 0, \quad \text{on } (0,\tau)\times\partial\Omega, \end{aligned} \tag{107}$$

where $\alpha \neq 0$, $\beta > 0$, Ω is a sufficiently regular bounded domain in $\mathfrak{R}^3$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, the distributed control $u_i \in L^2([0,\tau]; L^2(\Omega))$, $i = 1, 2$, w, θ denote the vertical deflection and the temperature of the plate, respectively, and the nonlinear terms $f_i(t,z,u)$, $i = 1,2$, are smooth enough and there are constants $a_i, c_i \in \mathfrak{R}$, with $c_i \neq -1$, $i = 1,2$, such that

$$\sup_{(t,w,v,u)\in q_\tau} \left| f_i(t,w,v,u) - a_i w - c_i u \right| < \infty, \quad i = 1,2, \tag{108}$$

where $q_\tau = [0,\tau] \times \mathfrak{R} \times \mathfrak{R} \times \mathfrak{R}$.

Example 33. The equation modelling the damped flexible beam:

$$\begin{aligned} \frac{\partial^2 z}{\partial^2 t} &= -\frac{\partial^4 z}{\partial^4 x} + 2\alpha\frac{\partial^3 z}{\partial t \partial^2 x} + 1_\omega u(t,x) \\ &\quad + f(t,z,z_t,u) \quad t \geq 0,\ 0 \leq x \leq 1, \\ z(t,1) &= z(t,0) = \frac{\partial^2 z}{\partial^2 x}(0,t) \\ &= \frac{\partial^2 z}{\partial^2 x}(1,t) = 0, \\ z(0,x) = \phi_0(x), &\quad \frac{\partial z}{\partial t}(0,x) = \psi_0(x), \quad 0 \leq x \leq 1, \end{aligned} \tag{109}$$

where $\alpha > 0$, $u \in L^2([0,r]; L^2[0,1])$, ω is an open nonempty subset of $[0,1]$, $\phi_0, \psi_0 \in L^2[0,1]$, and nonlinear function $f : [0,\tau] \times \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ is smooth enough and there are constant $a, c \in \mathfrak{R}$, with $c \neq -1$, such that

$$\sup_{(t,z,v,u)\in q_\tau} \left| f(t,z,v,u) - az - cu \right| < \infty, \tag{110}$$

where $q_\tau = [0,\tau] \times \mathfrak{R} \times \mathfrak{R} \times \mathfrak{R}$.

Example 34. The strongly damped wave equation with Dirichlet boundary conditions:

$$\begin{aligned} &\frac{\partial^2 w}{\partial^2 t} + \eta(-\Delta)^{1/2}\frac{\partial w}{\partial t} + \gamma(-\Delta) w \\ &\quad = 1_\omega u(t,x) + f(t,w,w_t,u), \quad t \geq 0,\ x \in \Omega, \\ &\quad w(t,x) = 0, \quad t \geq 0,\ x \in \partial\Omega, \\ &w(0,x) = \phi_0(x), \quad \frac{\partial z}{\partial t}(0,x) = \psi_0(x), \quad x \in \Omega, \end{aligned} \tag{111}$$

where Ω is a sufficiently smooth bounded domain in $\mathfrak{R}^N$, $u \in L^2([0,r]; L^2(\Omega))$, ω is an open nonempty subset of Ω, $\phi_0, \psi_0 \in L^2(\Omega)$, and nonlinear function $f : [0,\tau] \times \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ is smooth enough and there are constants $a, c \in \mathfrak{R}$, with $c \neq -1$, such that

$$\sup_{(t,w,v,u)\in q_\tau} \left| f(t,w,v,u) - aw - cu \right| < \infty, \tag{112}$$

where $q_\tau = [0,\tau] \times \mathfrak{R} \times \mathfrak{R} \times \mathfrak{R}$.

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