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Research Article

A Multiplier Theorem for Herz-Type Hardy Spaces Associated with the Dunkl Transform

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The main purpose of this paper is to establish a Hörmander multiplier theorem for Herz-type Hardy spaces associated with the Dunkl transform.

1. Introduction

Let $T_m(f)$ be a multiplier operator defined in terms of Fourier transforms by $T_m(f) = \mathscr{F}^{-1}(m\mathscr{F}(f))$ for suitable functions f. The multiplier theorem of Hörmander [1] gives a sufficient condition on m for the operator T_m to be bounded on $L^p(\mathbb{R}^n)$ whenever $1 < p < \infty$, namely, that m is a bounded C^ℓ-function on $\mathbb{R}^n \setminus \{0\}$ satisfying the Hörmander condition $M(2,\ell)$ as follows:

$$\left(\int_R^{2R} \left|m^{(s)}(\xi)\right|^2 d\xi\right)^{1/2} \leq CR^{(n+1)/2-s}, \quad \forall R > 0, \tag{1}$$

where ℓ is the least integer greater than $n/2$ and $s = 0, 1, \ldots, \ell$. In [2], the authors proved that if m satisfies the Hörmander condition with $\ell > n(1/p - 1/2)$, then T_m is bounded on the Hardy spaces $H^p(\mathbb{R}^n)$ with $0 < p \leq 1$.

In [3], the authors considered the following multiplier operator which is associated with the Dunkl transform:

$$T_m^\alpha(f) = \mathscr{F}_\alpha^{-1}\left(m\mathscr{F}_\alpha(f)\right), \tag{2}$$

where $\mathscr{F}_\alpha$ designs the Dunkl transform and using Hörmander's technique proved the following theorem.

Theorem 1. *Let ℓ be the least integer greater than $\alpha + 1$ and let m be a bounded C^ℓ-function on $\mathbb{R} \setminus \{0\}$ which satisfies the Hörmander condition $M_\alpha(2,\ell)$ as follows:*

$$\left(\int_R^{2R} \left|m^{(s)}(\xi)\right|^2 d\mu_\alpha(\xi)\right)^{1/2} \leq CR^{\alpha+1-s}, \quad \forall R > 0, \tag{3}$$

where C is a constant independent of R and $s = 0, 1, \ldots, \ell$. Then, the multiplier operator associated with the Dunkl transform can be extended to a bounded operator from $L^p(\mu_\alpha)$ into itself for $1 < p < \infty$, where $L^p(\mu_\alpha)$ is the Lebesgue space on $\mathbb{R}$ with respect to the following measure:

$$\mu_\alpha(x) = \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^{-1}|x|^{2\alpha+1}, \quad \left(\alpha > -\frac{1}{2}\right). \tag{4}$$

The Hardy spaces associated with Herz spaces can be regarded as the local version at the origin of the classical Hardy spaces H^p and they are good substitutes for H^p when we study the boundedness of nontranslation invariant operators. To establish the boundedness of operators in hardy-type spaces on $\mathbb{R}^n$, one usually appeals to the atomic decomposition characterization of these spaces. In [4, 5], the authors studied the Herz-type Hardy spaces $H\dot{K}_{\alpha,2}^{\beta,p}$ for the Dunkl operator in one-dimension and gave an atomic decomposition characterization of these spaces. The aim of this work is to prove the following Hörmander multiplier theorem on the spaces $H\dot{K}_{\alpha,2}^{\beta,p}$.

Theorem 2. *Let $0 < p \leq 1$, $\beta = (1/p) - (1/2)$, and ℓ be an integer greater than $2(\alpha+1)\beta$. If m satisfies the Hörmander condition $M_\alpha(2,\ell)$, then the operator T_m^α is bounded on $H\dot{K}_{\alpha,2}^{\beta,p}$.*

The paper is organized as follows. In Section 2, we recall some results about harmonic analysis and Herz-type Hardy spaces associated with the Dunkl operator on $\mathbb{R}$. In Section 3, we give the proof of the main result of this work. Then, as an application, we obtain the boundedness of the generalized Hilbert transform on $H\dot{K}_{\alpha,2}^{\beta,p}$.

Throughout this paper, let $S(\mathbb{R})$ be the usual Schwartz space and let $\mathscr{E}(\mathbb{R})$ be the space of C^{∞}-functions on $\mathbb{R}$. We always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We use the shorter notation $\|f\|_{p,\alpha}$ instead of $\|f\|_{L^p(\mu_\alpha)}$.

2. Preliminaries

In this section, we recapitulate some results about harmonic analysis on Dunkl hypergroups and the Herz-type Hardy space and its atomic decomposition which will be used later. For details, the reader is referred to [6–8].

Let $\alpha > -1/2$. We consider the differential-difference operator introduced in [9] as follows:

$$\Lambda_\alpha(f)(x) = \frac{df}{dx}(x) + \frac{2\alpha+1}{x} \cdot \frac{f(x)-f(-x)}{2}, \quad f \in \mathscr{E}(\mathbb{R}), \tag{5}$$

and call it the *Dunkl operator*.

For $\lambda \in \mathbb{C}$, the following initial value problem:

$$\Lambda_\alpha(f)(x) = \lambda f(x), \quad f(0) = 1, \ x \in \mathbb{R}, \tag{6}$$

has a unique solution $E_\alpha(\lambda\ \cdot)$ (called the *Dunkl kernel*) given by

$$E_\alpha(z) = j_\alpha(iz) + \frac{z}{2(\alpha+1)} j_{\alpha+1}(iz), \quad z \in \mathbb{C}, \tag{7}$$

where j_α is the normalized Bessel function of the first kind (with order α) defined on $\mathbb{C}$ by

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} (-1)^n \frac{(z/2)^{2n}}{n!\,\Gamma(n+\alpha+1)}. \tag{8}$$

The integral representation of E_α is given by

$$E_\alpha(i\lambda x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+(1/2))} \int_{-1}^{1} (1-t)\left(1-t^2\right)^{\alpha-(1/2)} e^{-i\lambda x t}\,dt. \tag{9}$$

From which, we get

$$\left|\partial_x^n E_\alpha(i\lambda x)\right| \le |\lambda|^n, \quad \lambda, x \in \mathbb{R}, \ n \in \mathbb{N}. \tag{10}$$

The *Dunkl transform* $\mathscr{F}_\alpha$, which was introduced by [10] and studied in [11], is defined for $f \in L^1(\mu_\alpha)$ by

$$\mathscr{F}_\alpha(f)(x) = \int_{\mathbb{R}} E_\alpha(-ixy) f(y)\,d\mu_\alpha(y), \quad x \in \mathbb{R}. \tag{11}$$

This transform satisfies the following properties.

(i) For all $f \in L^1(\mu_\alpha)$, we have

$$\|\mathscr{F}_\alpha(f)\|_{\infty,\alpha} \le \|f\|_{1,\alpha}. \tag{12}$$

(ii) For all $f \in L^1(\mu_\alpha)$ such that $\mathscr{F}_\alpha(f) \in L^1(\mu_\alpha)$, we have the following inversion formula:

$$\mathscr{F}_\alpha^{-1}(f)(x) = \mathscr{F}_\alpha(f)(-x), \quad \text{a.e. } x \in \mathbb{R}. \tag{13}$$

(iii) For all $f \in \mathcal{S}(\mathbb{R})$,

$$\mathscr{F}_\alpha(\Lambda_\alpha f)(x) = ix\mathscr{F}_\alpha(f)(x). \tag{14}$$

(iv) $\mathscr{F}_\alpha$ is a topological isomorphism from $\mathcal{S}(\mathbb{R})$ into itself.

(v) $\mathscr{F}_\alpha$ is an isometric isomorphism of $L^2(\mu_\alpha)$, and we have the following Parseval formula:

$$\int_{\mathbb{R}} f(x)\overline{g(x)}\,d\mu_\alpha(x) = \int_{\mathbb{R}} \mathscr{F}_\alpha(f)(x)\overline{\mathscr{F}_\alpha(g)(x)}\,d\mu_\alpha(x), \qquad \|\mathscr{F}_\alpha(f)\|_{2,\alpha} = \|f\|_{2,\alpha}. \tag{15}$$

The following lemma can be proved, similar to Lemma 7.25, page 343, in [12].

Lemma 3. *Let ℓ be the least integer greater than $\alpha+1$. If m satisfies the Hörmander condition $M_\alpha(2,\ell)$, then there is a constant C independent of m, such that if $q=1$ or $s-\ell+\alpha+1 < (\alpha+1)/q \le \alpha+1$, the following inequality holds:*

$$\int_R^{2R} \left|m^{(s)}(\xi)\right|^{2q} d\mu_\alpha(\xi) \le CR^{2(\alpha+1)-2qs}, \quad \forall R > 0. \tag{16}$$

Furthermore, in case $s-\ell+\alpha+1<0$, then $|x|^s|m^{(s)}(x)| \le C$ and $m^{(s)}$ is continuous on $\mathbb{R}\setminus\{0\}$.

Notation. For all $x, y, z \in \mathbb{R}$, we put

$$W_\alpha(x,y,z) = \left[1-\sigma_{x,y,z}+\sigma_{z,x,y}+\sigma_{z,y,x}\right]\Delta_\alpha(|x|,|y|,|z|), \tag{17}$$

where

$$\sigma_{x,y,z} = \begin{cases} \dfrac{x^2+y^2-z^2}{2xy}, & \text{if } x, y \in \mathbb{R}\setminus\{0\}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\Delta_\alpha(|x|,|y|,|z|) = \begin{cases} d_\alpha \dfrac{\left[\left((|x|+|y|)^2-z^2\right)\left(z^2-(|x|-|y|)^2\right)\right]^{\alpha-1/2}}{|xyz|^{2\alpha}}, & \text{if } |z| \in A_{x,y}, \\ 0, & \text{otherwise,} \end{cases}$$

$$d_\alpha = \frac{2^{1-\alpha}(\Gamma(\alpha+1))^2}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}, \qquad A_{x,y} = \left[\big||x|-|y|\big|, |x|+|y|\right]. \tag{18}$$

The Dunkl translation operator τ_x, $x \in \mathbb{R}$ is defined for a continuous function f on $\mathbb{R}$ by

$$\tau_x f(y) = \int_{\mathbb{R}} f(z)\, d\gamma_{x,y}(z), \quad y \in \mathbb{R}, \tag{19}$$

where $\gamma_{x,y}$ is the signed measures given by

$$d\gamma_{x,y}(z) = \begin{cases} W_\alpha(x,y,z)\, d\mu_\alpha(z), & \text{if } x, y \in \mathbb{R} \setminus \{0\}, \\ d\delta_x(z), & \text{if } y = 0, \\ d\delta_y(z), & \text{if } x = 0. \end{cases} \tag{20}$$

The operator τ_x has the following properties.

(i) For $x, y \in \mathbb{R}$ and a continuous function f on $\mathbb{R}$, we have

$$\tau_x(f)(y) = \tau_y(f)(x). \tag{21}$$

(ii) For all $x \in \mathbb{R}$, the operator τ_x can be extended to $L^p(\mu_\alpha)$ $(p \geq 1)$, and for $f \in L^p(\mu_\alpha)$, we have

$$\|\tau_x(f)\|_{p,\alpha} \leq 3\|f\|_{p,\alpha}. \tag{22}$$

(iii) For all $x, \lambda \in \mathbb{R}$ and $f \in L^1(\mu_\alpha)$, we have

$$\mathcal{F}_\alpha(\tau_x(f))(\lambda) = E_\alpha(i\lambda x)\mathcal{F}_\alpha(f)(\lambda). \tag{23}$$

Let $p, q, r \in [1, \infty]$ such that $1/p + 1/q = 1/r + 1$. The convolution product of $f \in L^p(\mu_\alpha)$ and $g \in L^q(\mu_\alpha)$ is defined by

$$f *_\alpha g(x) = \int_{\mathbb{R}} \tau_x(f)(-y)\, g(y)\, d\mu_\alpha(y), \quad \text{a.e. } x, \tag{24}$$

and we have

$$\|f *_\alpha g\|_{r,\alpha} \leq 3\|f\|_{p,\alpha}\|g\|_{q,\alpha}. \tag{25}$$

If $f, g \in L^1(\mu_\alpha)$, then

$$\mathcal{F}_\alpha(f *_\alpha g) = \mathcal{F}_\alpha(f)\mathcal{F}_\alpha(g). \tag{26}$$

Now, let us recall the definition of the Herz-type Hardy space and its atomic decomposition. For $N \in \mathbb{N}$ being sufficiently large, we denote by F_N the subset of $S(\mathbb{R})$ constituted by all those $\phi \in S(\mathbb{R})$ such that $\operatorname{supp}(\phi) \subset [-1, 1]$ and for all $m, n \in \mathbb{N}$ such that $m, n \leq N$, we have

$$\rho_{m,n}(\phi) = \sup_{x \in \mathbb{R}} (1 + |x|)^m \left|\Lambda_\alpha^n \phi(x)\right| \leq 1. \tag{27}$$

Moreover, the system of seminorms $\{\rho_{m,n}\}_{m,n \in \mathbb{N}}$ generates the topology of $S(\mathbb{R})$.

Let $f \in S'(\mathbb{R})$. We define the α-grand maximal function $G_\alpha(f)$ of f by

$$G_\alpha(f)(x) = \sup_{t > 0, \phi \in F_N} \left|\phi_t *_\alpha f(x)\right|, \quad x \in \mathbb{R}, \tag{28}$$

where ϕ_t is the dilation of ϕ given by

$$\phi_t(x) = t^{-2(\alpha+1)}\phi\left(\frac{x}{t}\right), \quad x \in \mathbb{R}. \tag{29}$$

Definition 4. Let $\beta \in \mathbb{R}$, $p \in]0, \infty[$, and $q \in [1, \infty]$.

(i) The homogeneous weighted Herz space $\dot{K}_{\alpha,q}^{\beta,p}$ is the space constituted by all functions $f \in L_{\mathrm{loc}}^q(\mu_\alpha)$, such that

$$\|f\|_{\dot{K}_{\alpha,q}^{\beta,p}} = \left[\sum_{k=-\infty}^{\infty} 2^{2(\alpha+1)\beta k p}\|f\chi_k\|_{q,\alpha}^p\right]^{1/p} < \infty, \tag{30}$$

where χ_k is the characteristic function of $A_k = \{x \in \mathbb{R}/2^{k-1} \leq |x| \leq 2^k\}$.

(ii) The nonhomogeneous weighted Herz space $K_{\alpha,q}^{\beta,p}$ is defined, as usual, by $K_{\alpha,q}^{\beta,p} = L^q(\mu_\alpha) \cap \dot{K}_{\alpha,q}^{\beta,p}$. Moreover, $\|f\|_{K_{q,\alpha}^{\beta,p}} = \|f\|_{q,\alpha} + \|f\|_{\dot{K}_{\alpha,q}^{\beta,p}}$.

Note that $\dot{K}_{\alpha,q}^{0,q} = K_{\alpha,q}^{0,q} = L^q(\mu_\alpha)$.

Definition 5. Let $\beta \in \mathbb{R}$, $p \in]0, \infty]$, and $q \in]1, \infty]$. The Herz-type Hardy space $H\dot{K}_{\alpha,q}^{\beta,p}$ is the space of distributions $f \in S'(\mathbb{R})$ such that $G_\alpha(f) \in \dot{K}_{\alpha,q}^{\beta,p}$. Moreover, we define

$$\|f\|_{H\dot{K}_{\alpha,q}^{\beta,p}} = \|G_\alpha(f)\|_{\dot{K}_{\alpha,q}^{\beta,p}}. \tag{31}$$

In the same way, we define the space $HK_{\alpha,q}^{\beta,p}$ for the nonhomogeneous case.

Definition 6. Let $q \in]1, \infty]$ and $\beta \geq 1 - 1/q$. A measurable function a on $\mathbb{R}$ is called a (central) (β, q, s)-atom if it satisfies the following:

(i) $\operatorname{supp}(a) \subset [-r, r]$, for some $r > 0$,

(ii) $\|a\|_{q,\alpha} \leq r^{-2(\alpha+1)\beta}$,

(iii) $\int_{\mathbb{R}} a(x)x^k d\mu_\alpha(x) = 0$, $k = 0, 1, \ldots, s$, where $s = [2(\alpha+1)(\beta - 1 + 1/q)]$ and $[\cdot]$ denotes the integer part function.

The following theorem is shown in [4].

Theorem 7. *Let $0 < p \leq 1 < q \leq \infty$ and $\beta \geq 1 - 1/q$. Then, $f \in H\dot{K}_{\alpha,q}^{\beta,p}$ if and only if, for all $j \in \mathbb{N} \setminus \{0\}$, there exist a (β, q, s)-atom a_j and $\lambda_j \in \mathbb{C}$, such that $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$ and $f = \sum_{j=1}^{\infty} \lambda_j a_j$. Moreover,*

$$\|f\|_{H\dot{K}_{\alpha,q}^{\beta,p}} \sim \inf\left(\sum_{j=1}^{\infty}\left|\lambda_j\right|^p\right)^{1/p}, \tag{32}$$

where the infimum is taking over all atomic decompositions of f.

In the sequel, fix $q = 2$ and $\beta = 1/p - 1/2$.

Definition 8. For $0 < p \leq 1$. Set $s \geq [2(\alpha+1)(1/p-1)]$, $\varepsilon > s/2(\alpha+1)$, $a = 1 - \dfrac{1}{p} + \varepsilon$, and $b = 1/2 + \varepsilon$. A central (p, s, ε)-molecule is a function $M \in L^2(\mu_\alpha)$ satisfying the following:

(i) $M(x)|x|^{2(\alpha+1)b} \in L^2(\mu_\alpha)$,

(ii) $\|M\|_{2,\alpha}^{a/b} \|M(x)|x|^{2(\alpha+1)b}\|_{2,\alpha}^{1-a/b} \equiv N(M) < \infty$,

(iii) $\int_{\mathbb{R}} M(x) x^k d\mu_\alpha(x) = 0$, $k = 0, 1, \ldots, s$.

Proposition 9. *Let (p, s, ε) be the triple cited in the previous definition. Every central (p, s, ε)-molecule M belongs to $H\dot{K}_{\alpha,2}^{\beta,p}$ and $\|M\|_{H\dot{K}_{\alpha,2}^{\beta,p}} \leq CN(M)$, where the constant C is independent of M.*

Proof. Let M be a central (p, s, ε)-molecule and suppose that $\|M\|_{H\dot{K}_{\alpha,2}^{\beta,p}} = 1$. In the general case, letting $\widetilde{M} = M/\|M\|_{H\dot{K}_{\alpha,2}^{\beta,p}}$, we have $\|\widetilde{M}\|_{H\dot{K}_{\alpha,2}^{\beta,p}} = 1$.

Let $E_0 = \{|x| \leq 1\}$, $E_k = \{2^{k-1} < |x| \leq 2^k\}$, and $M_k = M\chi_k$, $k = 1, 2, 3, \ldots$, where χ_k is the characteristic function of E_k. For each k, there exists a unique polynomial Q_k, of degree at most s, such that if $P_k = Q_k \chi_k$; then

$$\int_{\mathbb{R}} (M_k - P_k) x^j d\mu_\alpha(x) = 0, \quad j = 0, 1, \ldots, s. \tag{33}$$

Using some ideas in [2], we can show that each $(M_k - P_k)$ is a multiple of a central $(\beta, 2, s)$-atom with a sequence of coefficients in l^p. We also show that the sum $\sum_{k=0}^{+\infty} P_k$ can be written as an infinite linear combination of central (β, ∞, s)-atom with a sequence of coefficients in l^p. Since a (β, ∞, s)-atom is also $(\beta, 2, s)$-atom, hence,

$$M = \sum_{k=0}^{+\infty} M_k = \sum_{k=0}^{+\infty} (M_k - P_k) + \sum_{k=0}^{+\infty} P_k = \sum_{i=0}^{+\infty} \lambda_i a_i, \tag{34}$$

where a_i is a central $(\beta, 2, s)$-atom and $\sum_{k=0}^{+\infty} |\lambda_i|^p < \infty$. It follows from Theorem 7 that $M \in H\dot{K}_{\alpha,2}^{\beta,p}$ and $\|M\|_{H\dot{K}_{\alpha,2}^{\beta,p}} \leq CN(M)$. □

The following Lemma plays an important role in the proof of the main result of this work.

Lemma 10. *Let a be a $(\beta, 2, s)$-atom. For all integer $0 \leq k \leq s$ and every $1 \leq u \leq \infty$, there exists a constant C independent of a, such that*

$$\begin{gathered}
\text{(i)} \left|(\mathcal{F}_\alpha(a))^{(k)}(y)\right| \leq C|y|^{s+1-k} \|a\|_{2,\alpha}^{A}, \\
A = 1 - \frac{1}{\beta}\left(\frac{1}{2} + \frac{s+1}{2(\alpha+1)}\right) \quad y \in \mathbb{R}, \\
\text{(ii)} \left\|\left((\mathcal{F}_\alpha(a))^{(k)}(y)\right)^2\right\|_{u',\alpha} \leq C\|a\|_{2,\alpha}^{2-(1/\beta)((k/\alpha+1)+(1/u))}, \\
\frac{1}{u} + \frac{1}{u'} = 1 \quad y \in \mathbb{R}.
\end{gathered} \tag{35}$$

Proof. (i) Let a be a $(\beta, 2, s)$-atom. Consider that $r > 0$ such that $\operatorname{supp}(a) \subset [-r, r]$ and that $\|a\|_{2,\alpha} \leq r^{-2(\alpha+1)\beta}$. From (9), (iii) of Definition (19), and the estimate for the remainder in Taylors' formula, it follows that

$$\begin{aligned}
&(\mathcal{F}_\alpha(a))^{(k)}(y) \\
&\quad = C_\alpha \int_{-1}^{1} \left((1-t)\left(1-t^2\right)^{\alpha-1/2} t^k \right. \\
&\qquad \times \int_{-r}^{r} (ix)^k \left[\exp(ixyt) - \sum_{n=0}^{s-k} \frac{(ixyt)^n}{n!}\right] \\
&\qquad \left. \times a(x) \, d\mu_\alpha(x) \right) dt \\
&\quad \leq C|y|^{s+1-k} \int_{-r}^{r} |x|^{s+1} |a(x)| \, d\mu_\alpha(x) \\
&\quad \leq C|y|^{s+1-k} \|a\|_{2,\alpha} \left[\int_{-r}^{r} |x|^{2(s+1)} d\mu_\alpha(x)\right]^{1/2} \\
&\quad \leq C|y|^{s+1-k} \|a\|_{2,\alpha} r^{s+\alpha+2}.
\end{aligned} \tag{36}$$

From (ii) of Definition (19), we obtain

$$\begin{gathered}
\left|(\mathcal{F}_\alpha(a))^{(k)}(y)\right| \leq C|y|^{s+1-k} \|a\|_{2,\alpha}^{A}, \\
A = 1 - \frac{1}{\beta}\left(\frac{1}{2} + \frac{s+1}{2(\alpha+1)}\right).
\end{gathered} \tag{37}$$

(ii) For $u = 1$,

$$(\mathcal{F}_\alpha(a))^{(k)}(y) = \int_{-r}^{r} \partial_y^k E_\alpha(-iyx) \, a(x) \, d\mu_\alpha(x). \tag{38}$$

Using (10), we get the following for all $y \in \mathbb{R}$:

$$\begin{aligned}
\left|(\mathcal{F}_\alpha(a))^{(k)}(y)\right| &\leq C \int_{-r}^{r} |x|^k |a(x)| \, d\mu_\alpha(x) \\
&\leq C\left(\int_{-r}^{r} |a(x)|^2 d\mu_\alpha(x)\right)^{1/2} \\
&\quad \times \left(\int_{-r}^{r} |x|^{2k} d\mu_\alpha(x)\right)^{1/2} \\
&\leq C\|a\|_{2,\alpha} r^{k+\alpha+1}.
\end{aligned} \tag{39}$$

From (ii) of Definition (19), we obtain the following for all $y \in \mathbb{R}$:

$$\left|(\mathcal{F}_\alpha(a))^{(k)}(y)\right|^2 \leq C\|a\|_{2,\alpha}^{2-(1/\beta)(k/(\alpha+1)+1)}. \tag{40}$$

For $u = \infty$,

$$\begin{aligned}
\int_{\mathbb{R}} \left|(\mathcal{F}_\alpha(a))^{(k)}(x)\right|^2 d\mu_\alpha(x) &\leq C \int_{-r}^{r} |x|^{2k} |a(x)|^2 d\mu_\alpha(x) \\
&\leq Cr^{2k} \|a\|_{2,\alpha}^2 \\
&\leq C\|a\|_{2,\alpha}^{2-(1/\beta)(k/(\alpha+1))}.
\end{aligned} \tag{41}$$

For $1 < u < \infty$,

$$\begin{aligned}
&\int_{\mathbb{R}} \left|(\mathcal{F}_\alpha(a))^{(k)}(y)\right|^{2u'} d\mu_\alpha(x) \\
&= \int_{\mathbb{R}} \left|(\mathcal{F}_\alpha(a))^{(k)}(y)\right|^{2} \left|(\mathcal{F}_\alpha(a))^{(k)}(y)\right|^{2u'-2} d\mu_\alpha(x) \\
&\le C\|a\|_{2,\alpha}^{(u'-1)(2-(1/\beta)(k/(\alpha+1)))} \int_{\mathbb{R}} \left|(\mathcal{F}_\alpha(a))^{(k)}(x)\right|^{2} d\mu_\alpha(x) \\
&\le C\|a\|_{2,\alpha}^{u'(2-(1/\beta)(k/(\alpha+1)+(1/u')))}.
\end{aligned} \tag{42}$$

Finally, we get the following for all $y \in \mathbb{R}$:

$$\left\|((\mathcal{F}_\alpha(a))^{(k)}(y))^2\right\|_{u',\alpha} \le C\|a\|_{2,\alpha}^{2-(1/\beta)(k/(\alpha+1)+(1/u))}. \tag{43}$$

□

3. Proof of Theorem 2

Let $0 < p \le 1$ and ℓ be an integer greater than $2(\alpha+1)\beta$. Set $s = [2(\alpha+1)(1/p-1)]$, $\epsilon = \ell/2(\alpha+1)-(1/2)$, $a = 1-(1/p)+\epsilon$, and $b = \epsilon+(1/2)$.

We have $\ell - 1 \ge s$; then, according to Proposition 9 to prove Theorem 2 it suffices to prove that, for any $(\beta, 2, \ell)$-atom f, $T_m^\alpha f$ is a central (p, s, ϵ)-molecule with $N(T_m^\alpha f) < C$ for some constant C independent of f. In other words, we need to check that

$$\begin{aligned}
&\text{(i) } T_m^\alpha f(\xi), T_m^\alpha f(\xi)|\xi|^\ell \in L^2(\mu_\alpha), \\
&\text{(ii) } \|T_m^\alpha f\|_{2,\alpha}^{a/b} \left\|T_m^\alpha f(\xi)|\xi|^\ell\right\|_{2,\alpha}^{1-a/b} \equiv N(T_m^\alpha f) < C, \\
&\text{(iii) } \int_{\mathbb{R}} T_m^\alpha f(\xi)\xi^j d\mu_\alpha(\xi) = 0 \quad \forall j = 0, 1, \ldots, s.
\end{aligned} \tag{44}$$

Firstly, we prove (i) and (ii).

m satisfies the Hörmander condition $M_\alpha(2, \ell)$; then, by Theorem 1, there exists a constant C independent of f, such that

$$\|T_m^\alpha f\|_{2,\alpha} \le C\|f\|_{2,\alpha}. \tag{45}$$

From (14) and (13), we have

$$\Lambda_\alpha^\ell(\mathcal{F}_\alpha(T_m^\alpha f))(\xi) = \mathcal{F}_\alpha\left((-ix)^\ell T_m^\alpha f(x)\right)(\xi). \tag{46}$$

Then, by Plancherel theorem to estimate $\|T_m^\alpha f(\xi)|\xi|^\ell\|_{2,\alpha}$, it suffices to estimate $\|\Lambda_\alpha^\ell(m\mathcal{F}_\alpha(f))\|_{2,\alpha}$, which turns out to prove that

$$\left\|\Lambda_\alpha^\ell(m\mathcal{F}_\alpha(f))\right\|_{2,\alpha} \le C\|f\|_{2,\alpha}^{1-(\ell/2(\alpha+1)\beta)}. \tag{47}$$

By induction, we have

$$\begin{aligned}
\Lambda_\alpha^\ell(m\mathcal{F}_\alpha(f))(\xi) &= \sum_{r=0}^{l} a_r \xi^{r-\ell}(m\mathcal{F}_\alpha(f))^{(r)}(\xi) \\
&\quad + \sum_{r=0}^{l} b_r \xi^{r-\ell}(m\mathcal{F}_\alpha(f))^{(r)}(-\xi),
\end{aligned} \tag{48}$$

where a_r and b_r are constants.

But, using Leibniz formula, we have the following for $r \in \{0, 1, \ldots, \ell\}$:

$$(m\mathcal{F}_\alpha(f))^{(r)}(\xi) = \sum_{k=0}^{r} C_r^k (\mathcal{F}_\alpha(f))^{(k)}(\xi)(m)^{(r-k)}(\xi). \tag{49}$$

So, to establish (47), it suffices to claim that

$$\begin{aligned}
&\left\|\xi^{r-\ell}(\mathcal{F}_\alpha(f))^{(k)}(\xi)(m)^{(r-k)}(\xi)\right\|_{2,\alpha} \\
&\quad \le C\|f\|_{2,\alpha}^{1-(\ell/2(\alpha+1)\beta)} \quad \text{for all integers } 0 \le k \le r \le \ell.
\end{aligned} \tag{50}$$

For the case $k = \ell$, we use Lemma 10 (ii) with $u = \infty$ and Lemma 3 to get the following:

$$\begin{aligned}
\left\|\xi^{r-\ell}(\mathcal{F}_\alpha(f))^{(k)}(\xi)(m)^{(r-k)}(\xi)\right\|_{2,\alpha} &\le C\left\|(\mathcal{F}_\alpha(f))^{(k)}\right\|_{2,\alpha} \\
&\le C\|f\|_{2,\alpha}^{1-(\ell/2(\alpha+1)\beta)}.
\end{aligned} \tag{51}$$

For $0 \le k < \ell$, we have

$$\begin{aligned}
&\left\|\xi^{r-\ell}(\mathcal{F}_\alpha(f))^{(k)}(\xi)(m)^{(r-k)}(\xi)\right\|_{2,\alpha}^2 \\
&= \sum_{j\in\mathbb{Z}} \int_{2^j<|\xi|<2^{j+1}} |\xi|^{2(r-\ell)} \left|(\mathcal{F}_\alpha(f))^{(k)}(\xi)\right|^2 \\
&\quad \times \left|(m)^{(r-k)}(\xi)\right|^2 d\mu_\alpha(\xi)
\end{aligned} \tag{52}$$

$$= S_1 + S_2,$$

where

$$\begin{aligned}
S_1 &= \sum_{j=-\infty}^{j_0} \int_{2^j<|\xi|<2^{j+1}} |\xi|^{2(r-\ell)} \left|(\mathcal{F}_\alpha(f))^{(k)}(\xi)\right|^2 \\
&\quad \times \left|(m)^{(r-k)}(\xi)\right|^2 d\mu_\alpha(\xi), \\
S_2 &= \sum_{j=j_0+1}^{+\infty} \int_{2^j<|\xi|<2^{j+1}} |\xi|^{2(r-\ell)} \left|(\mathcal{F}_\alpha(f))^{(k)}(\xi)\right|^2 \\
&\quad \times \left|(m)^{(r-k)}(\xi)\right|^2 d\mu_\alpha(\xi),
\end{aligned} \tag{53}$$

and j_0 is the integer, such that

$$2^{2(\alpha+1)j_0} < \|f\|_{2,\alpha}^{1/\beta} \le 2^{2(\alpha+1)(j_0+1)}. \tag{54}$$

Firstly, we estimate S_1.

Using (i) of Lemma 10 and the fact that m satisfies the Hörmander condition $M_\alpha(2, \ell)$, we get

$$\begin{aligned}
&\int_{2^j<|\xi|<2^{j+1}} |\xi|^{2(r-\ell)} \left|(\mathcal{F}_\alpha(f))^{(k)}(\xi)\right|^2 \left|(m)^{(r-k)}(\xi)\right|^2 d\mu_\alpha(\xi) \\
&\quad \le C\|f\|_{2,\alpha}^{2-(1/\beta)(\ell/(\alpha+1)+1)} \\
&\quad \times \int_{2^j<|\xi|<2^{j+1}} |\xi|^{2(r+1-k)} \left|(m)^{(r-k)}(\xi)\right|^2 d\mu_\alpha(\xi) \\
&\quad \le C\|f\|_{2,\alpha}^{2-(1/\beta)(\ell/(\alpha+1)+1)} 2^{2(\alpha+1)j}.
\end{aligned} \tag{55}$$

By (54), we obtain

$$S_1 \le C\|f\|_{2,\alpha}^{2-(1/\beta)(\ell/(\alpha+1)+1)} 2^{2(\alpha+1)j_0} \le C\|f\|_{2,\alpha}^{2-(\ell/(\alpha+1)\beta)}. \tag{56}$$

Now, we estimate S_2. By Holder's inequality, we have

$$\begin{aligned}
&\int_{2^j<|\xi|<2^{j+1}} |\xi|^{2(r-\ell)} \left|(\mathscr{F}_\alpha(f))^{(k)}(\xi)\right|^2 \left|(m)^{(r-k)}(\xi)\right|^2 d\mu_\alpha(\xi) \\
&\le 2^{2j(r-\ell)} \left(\int_{2^j<|\xi|<2^{j+1}} \left|(\mathscr{F}_\alpha(f))^{(k)}(\xi)\right|^{2u'} d\mu_\alpha(\xi)\right)^{1/u'} \\
&\quad\times \left(\int_{2^j<|\xi|<2^{j+1}} \left|(m)^{(r-k)}(\xi)\right|^{2u} d\mu_\alpha(\xi)\right)^{1/u}.
\end{aligned} \tag{57}$$

Using (ii) of Lemmas 10 and 3, we get

$$S_2 \le C\|f\|_{2,\alpha}^{2-(1/\beta)((k/(\alpha+1))+(1/u))} \sum_{j=j_0+1}^{+\infty} \left(2^{(2(\alpha+1)/u)-2(\ell-k)}\right)^j. \tag{58}$$

To guarantee the convergence of this summation, we choose the pair (k,u) as follows:

(a) if $l-k > \alpha+1$, we choose $u=1$;

(b) if $0 < l-k \le \alpha+1$ and $k > \alpha+1$, we choose $u=\infty$;

(c) if $0 < l-k \le \alpha+1$ and $k \le \alpha+1$, we choose $0 < u < \infty$ such that $k > (\alpha+1)(1-(1/u))$.

Furthermore, by (54), we get

$$\begin{aligned}
S_2 &\le C\|f\|_{2,\alpha}^{2-(1/\beta)(k/(\alpha+1)+(1/u))} \left(2^{(2(\alpha+1)/u)-2(\ell-k)}\right)^{j_0+1} \\
&\le C\|f\|_{2,\alpha}^{2-(\ell/(\alpha+1)\beta)}.
\end{aligned} \tag{59}$$

Finally, combining (56) and (59), we obtain (47). (i) and (ii) are hence proved.

To prove (iii), it suffices to prove that $T_m^\alpha f(\xi)\xi^j \in L^1(\mu_\alpha)$ for all integer $0 \le j \le s$ and $\Lambda_\alpha^j(m\mathscr{F}_\alpha(f))(0)=0$: indeed if $T_m^\alpha f(\xi)\xi^j \in L^1(\mu_\alpha)$ according to (14), which we have $\Lambda_\alpha^j(\mathscr{F}_\alpha(T_m^\alpha f))(x) = C\mathscr{F}_\alpha(T_m^\alpha f(\xi)\xi^j)(x)$ is continuous, and hence $\int_{\mathbb{R}} T_m^\alpha f(\xi)\xi^j d\mu_\alpha(\xi) = C\Lambda_\alpha^j(m\mathscr{F}_\alpha(f))(0)$.

Now, we check $T_m^\alpha f(\xi)\xi^j \in L^1(\mu_\alpha)$. We write $\int_{\mathbb{R}} |T_m^\alpha f(\xi)\xi^j| d\mu_\alpha(\xi) = I_1 + I_2$, where

$$\begin{aligned}
I_1 &= \int_{|\xi|\le 1} \left|T_m^\alpha f(\xi)\xi^j\right| d\mu_\alpha(\xi), \\
I_2 &= \int_{|\xi|>1} \left|T_m^\alpha f(\xi)\xi^j\right| d\mu_\alpha(\xi).
\end{aligned} \tag{60}$$

Using the fact that $T_m^\alpha f \in L^2(\mu_\alpha)$ and Schwarz's inequality, we get

$$\begin{aligned}
I_1 &\le \int_{|\xi|\le 1} \left|T_m^\alpha f(\xi)\right| d\mu_\alpha(\xi) \\
&\le \left(\int_{|\xi|\le 1} \left|T_m^\alpha f(\xi)\right|^2 d\mu_\alpha(\xi)\right)^{1/2} \left(\int_{|\xi|\le 1} d\mu_\alpha(\xi)\right)^{1/2} \\
&\le C\|T_m^\alpha f\|_{2,\alpha} \le \infty.
\end{aligned} \tag{61}$$

For $0 \le j \le s$, we have

$$\begin{aligned}
I_2 &\le \int_{|\xi|>1} \left|T_m^\alpha f(\xi)\xi^s\right| d\mu_\alpha(\xi) \\
&\le \left(\int_{|\xi|>1} \left|T_m^\alpha f(\xi)\right|^2 \left|\xi^{2\ell}\right| d\mu_\alpha(\xi)\right)^{1/2} \\
&\quad\times \left(\int_{|\xi|>1} \left|\xi^{2(s-\ell)}\right| d\mu_\alpha(\xi)\right)^{1/2} \\
&= \left\|T_m^\alpha f(\xi)|\xi|^l\right\|_{2,\alpha} \left(\int_{|\xi|>1} \left|\xi^{2(s-\ell)}\right| d\mu_\alpha(\xi)\right)^{1/2}.
\end{aligned} \tag{62}$$

Using the fact that $s-\ell < \alpha+1$, we get $I_2 \le C$.

Finally, we check

$$\Lambda_\alpha^j(m\mathscr{F}_\alpha(f))(0) = 0, \quad 0 \le j \le s. \tag{63}$$

We have

$$\begin{aligned}
\Lambda_\alpha^j(m\mathscr{F}_\alpha(f))(h) &= \sum_{r=0}^{j} a_r h^{r-j} (m\mathscr{F}_\alpha(f))^{(r)}(h) \\
&\quad+ \sum_{r=0}^{j} b_r h^{r-j} (m\mathscr{F}_\alpha(f))^{(r)}(-h),
\end{aligned} \tag{64}$$

where a_r and b_r are constants. Then, to prove (63), it suffices to prove that

$$\begin{aligned}
&\lim_{h\to 0} \left|h^{r-j} m^{(r-k)}(h) (\mathscr{F}_\alpha(f))^{(k)}(h)\right| = 0, \\
&\text{for all integers } 0 \le k \le r \le j \le s.
\end{aligned} \tag{65}$$

By (i) of Lemma 10, we have

$$\begin{aligned}
&\left|h^{r-j} m^{(r-k)}(h) (\mathscr{F}_\alpha(f))^{(k)}(h)\right| \\
&\le C|h|^{s+1-j} |h|^{r-k} \left|m^{(r-k)}(h)\right| \|f\|_{2,\alpha}^A.
\end{aligned} \tag{66}$$

According to Lemma 3, we have $|h|^{r-k}|m^{(r-k)}(h)| \le C$; indeed $2(r-k-\ell)+\alpha+1 < 0$; then, we obtain

$$\lim_{h\to 0} \left|h^{r-j} m^{(r-k)}(h) (\mathscr{F}_\alpha(f))^{(k)}(h)\right| \le C \lim_{h\to 0} |h|^{s+1-j} = 0, \tag{67}$$

where (63) is hence proved. This finishes the proof of Theorem 2.

Corollary 11. *Let $0 < p \le 1$. Then, the generalized Hilbert transform H_α defined by*

$$H_\alpha(f) = \frac{\Gamma(\alpha+(3/2))}{\sqrt{\pi}\Gamma(\alpha+1)} \lim_{\varepsilon\to 0} \int_{|y|>\varepsilon} \frac{\tau_x(f)(-y)}{y} dy, \tag{68}$$

where τ_x is given by (19)*, is bounded on $H\dot{K}_{\alpha,2}^{\beta,p}$.*

Proof. From Proposition 3.6 in [3], the generalized Hilbert transform H_α is a multiplier operator T_m^α with $m(\xi) = -\operatorname{sign}(\xi)$; then the proof of the corollary follows from Theorem 2. □

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