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Research Article

Jensen's Inequality for Generalized Peng's g-Expectations and Its Applications

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We study Jensen's inequality for generalized Peng's g-expectations and give four equivalent conditions on Jensen's inequality for generalized Peng's g-expectations without the assumption that the generator g is continuous with respect to t. This result includes and extends some existing results. Furthermore, we give some applications of Jensen's inequality for generalized Peng's g-expectations.

1. Introduction

By Pardoux and Peng [1], we know that there exists a unique adapted and square integrable solution to a backward stochastic differential equation (BSDE for short) of the type

$$y_t = \xi + \int_t^T g(s, y_s, z_s)\, ds - \int_t^T z_s \cdot dW_s, \quad t \in [0, T], \quad (1)$$

provided that the function g is Lipschitz in both variables y and z, and ξ and $(g(t,0,0))_{t\in[0,T]}$ are square integrable. g is said to be the generator of BSDE (1). We denote the unique adapted and square integrable solution of BSDE (1) by $(y_t^{(T,g,\xi)}, z_t^{(T,g,\xi)})_{t\in[0,T]}$.

Based on such a BSDE, Peng [2] introduced the notion of g-expectation. He proved that the g-expectation preserves many of properties of the classical mathematical expectation, but not the linearity property, and thus the g-expectation is a type of nonlinear mathematical expectation. Indeed, g-expectation is a kind of nonlinear expectation, which can be considered as a nonlinear extension of the well-known Girsanov transformations. The original motivation for studying g-expectation comes from the theory of expected utility. Since the notion of g-expectation was introduced, many properties of g-expectation have been investigated by many researchers. In 1997, Peng [3] introduced the notions of conditional g-expectation and g-martingale. Later, Briand et al. [4] studied Jensen's inequality for g-expectations and gave a counter example and a proposition to indicate that even for a linear function, Jensen's inequality might fail for some g-expectations. This yields a natural question: under which conditions on g in the g-expectation does Jensen's inequality hold for any convex function? Under the assumptions that g does not depend on y and is convex, Chen et al. [5, 6] studied Jensen's inequality for g-expectations and gave a necessary and sufficient condition on g under which Jensen's inequality holds for convex functions. Provided that g only does not depend on y, Jiang [7] gave another necessary and sufficient condition on g under which Jensen's inequality holds for convex functions. It was an improved result in comparison with the result that Chen et al. yielded. Later, this result was improved by Hu [8] and Jiang [9] showing that, in fact, g must be independent of y. But these results need the assumption that the generator g is continuous with respect to t.

In this paper, without the assumption that the generator g is continuous with respect to t, we study Jensen's inequality for generalized Peng's g-expectations and give four equivalent conditions on Jensen's inequality for generalized Peng's g-expectations, which generalize the known results on Jensen's inequality for g-expectations in Chen et al. [5, 6], Jiang [7, 9], and Hu [8]. Furthermore, we give some applications of Jensen's inequality for generalized Peng's g-expectations.

This paper is organized as follows: in Section 2, we introduce some notations, assumptions, notions, and lemmas which will be useful in this paper; in Section 3, we give our main results including the proofs and applications.

2. Preliminaries

Firstly, let us list some notations, assumptions, notions, lemmas, and propositions that are used in this paper. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $(W_t)_{t\geq 0}$ be a d-dimensional standard Brownian motion with respect to filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by Brownian motion and all P-null subsets, that is,

$$\mathcal{F}_t = \sigma\{W_s; s \leq t\} \vee \mathcal{N}, \tag{2}$$

where $\mathcal{N}$ is the set of all P-null subsets. Fix a real number $T > 0$. For any positive integer n and $z \in R^n$, $|z|$ denotes its Euclidean norm.

We define the following usual spaces of processes (random variables):

(i) Consider $L^p(\Omega, \mathcal{F}_T, P) = \{\xi : \xi$ is $\mathcal{F}_T$-measurable random variable such that $E[|\xi|^p] < \infty, p \geq 1\}$;

(ii) Consider $\mathcal{L}(\Omega, \mathcal{F}_T, P) = \bigcup_{p>1} L^p(\Omega, \mathcal{F}_T, P)$;

(iii) Consider $\mathcal{S}^p_{\mathcal{F}}(0, T; P; R) = \{V : V$ is a continuous process with $E[\sup_{0\leq t\leq T}|V_t|^p] < \infty, p \geq 1\}$;

(iv) Consider $\mathcal{S}_{\mathcal{F}}(0, T; P; R) = \bigcup_{p>1} \mathcal{S}^p_{\mathcal{F}}(0, T; P; R)$;

(v) Consider $\mathcal{L}^p_{\mathcal{F}}(0, T; P; R^n) = \{V : V$ is a progressively measurable process with $E[(\int_0^T |V_s|^2 ds)^{p/2}] < \infty, p \geq 1\}$;

(vi) Consider $\mathcal{L}_{\mathcal{F}}(0, T; P; R^n) = \bigcup_{p>1} L^p_{\mathcal{F}}(0, T; P; R^n)$.

Suppose the generator $g(\omega, t, y, z) : \Omega \times [0, T] \times R \times R^d \mapsto R$ satisfies the following assumptions:

(A.1) there exists a constant $\mu > 0$, such that P-a.s., we have: $\forall t \in [0, T]$, $\forall y_1, y_2 \in R$, $z_1, z_2 \in R^d$, $|g(t, y_1, z_1) - g(t, y_2, z_2)| \leq \mu(|y_1 - y_2| + |z_1 - z_2|)$;

(A.2) P-a.s., $\forall(t, y) \in [0, T] \times R$, $g(t, y, 0) \equiv 0$.

The following lemma is a special case of Theorem 4.2 in Briand et al. [10].

Lemma 1. *Suppose g satisfies (A.1) and (A.2). Then for each given $\xi \in L^p(\Omega, \mathcal{F}_T, P)$, where $1 < p < 2$, the BSDE (1) has a unique pair of adapted processes $(y_t^{(T,g,\xi)}, z_t^{(T,g,\xi)})_{t\in[0,T]} \in \mathcal{S}^p_{\mathcal{F}}(0, T; P; R) \times l^p_{\mathcal{F}}(0, T; P; R^d)$.*

From Lemma 1, we have the following.

Remark 2. Suppose g satisfies (A.1) and (A.2). Then for each given $\xi \in \mathcal{L}(\Omega, \mathcal{F}_T, P)$, the BSDE (1) has a unique pair of adapted processes $(y_t^{(T,g,\xi)}, z_t^{(T,g,\xi)})_{t\in[0,T]} \in \mathcal{S}_{\mathcal{F}}(0, T; P; R) \times \mathcal{L}_{\mathcal{F}}(0, T; P; R^d)$.

Now, we introduce the notions of generalized Peng's g-expectation and generalized conditional Peng's g-expectation.

Definition 3 (generalized Peng's g-expectation [11]). Suppose g satisfies (A.1) and (A.2). For any $\xi \in \mathcal{L}(\Omega, \mathcal{F}_T, P)$, let $(y_t^{(T,g,\xi)}, z_t^{(T,g,\xi)})_{t\in[0,T]}$ be the solution of BSDE (1). Consider the mapping $\mathcal{E}_g[\cdot] : \mathcal{L}(\Omega, \mathcal{F}_T, P) \mapsto R$, denoted by $\mathcal{E}_g[\xi] = y_0^{(T,g,\xi)}$. One calls $\mathcal{E}_g[\xi]$ the generalized Peng's g-expectation of ξ.

Definition 4 (generalized Peng's conditional g-expectation [11]). Suppose g satisfies (A.1) and (A.2). The generalized Peng's conditional g-expectation of ξ with respect to $\mathcal{F}_t$ is defined by

$$\mathcal{E}_g[\xi \mid \mathcal{F}_t] = y_t^{(T,g,\xi)}, \quad t \in [0, T]. \tag{3}$$

Then, let us list some basic properties of generalized Peng's g-expectation.

Proposition 5 (see [11]). *Consider $\mathcal{E}_g[\xi \mid \mathcal{F}_t]$ is the unique random variable η in $\mathcal{L}(\Omega, \mathcal{F}_t, P)$ such that*

$$\mathcal{E}_g[1_A \xi] = \mathcal{E}_g[1_A \eta], \quad \forall A \in \mathcal{F}_t. \tag{4}$$

Proposition 6 (see [11]). *Suppose g satisfies (A.1) and (A.2). If g does not depend on y, that is, $g(\omega, t, z) : \Omega \times [0, T] \times R^d \mapsto R$, then*

$$\mathcal{E}_g[\xi + \eta \mid \mathcal{F}_t] = \mathcal{E}_g[\xi \mid \mathcal{F}_t] + \eta, \quad \forall \xi \in \mathcal{L}(\Omega, \mathcal{F}_T, P), \quad \forall \eta \in \mathcal{L}(\Omega, \mathcal{F}_t, P). \tag{5}$$

Proposition 7 (see [11]). *Suppose g satisfies (A.1) and (A.2). For $\xi, \eta_n \in L^p(\Omega, \mathcal{F}_T, P)$, where $n = 1, 2, \ldots$ and $p > 1$, if $E[|\xi - \eta_n|^p \mid \mathcal{F}_t] \to 0$, a.s., $t \in [0, T]$, then*

$$\lim_{n\to\infty} \mathcal{E}_g[\eta_n \mid \mathcal{F}_t] = \mathcal{E}_g[\xi \mid \mathcal{F}_t], \quad a.s., t \in [0, T]. \tag{6}$$

Applying Proposition 7, one can immediately obtain the following.

Remark 8. (i) For any $\xi \in \mathcal{L}(\Omega, \mathcal{F}_T, P)$, let $\xi^n = (\xi \wedge n) \vee (-n)$, $n = 1, 2, \ldots$, then $\lim_{n\to\infty} \mathcal{E}_g[\xi^n \mid \mathcal{F}_t] = \mathcal{E}_g[\xi \mid \mathcal{F}_t]$, a.s., $\forall t \in [0, T]$.

(ii) For any $\xi_n \in \mathcal{L}(\Omega, \mathcal{F}_T, P)$, if $\lim_{n\to\infty} \xi_n = \xi$ a.s. and $|\xi_n| \leq \eta$ a.s. with $\eta \in \mathcal{L}(\Omega, \mathcal{F}_T, P)$, then $\lim_{n\to\infty} \mathcal{E}_g[\xi^n \mid \mathcal{F}_t] = \mathcal{E}_g[\xi \mid \mathcal{F}_t]$, a.s., $\forall t \in [0, T]$.

Lemma 9. *Suppose g satisfies (A.1) and (A.2). Let $\{A_i\}_{i=1}^m$ be a $\mathcal{F}_t$-measurable partition of Ω (i.e., $A_i \in \mathcal{F}_t$, $A_i \bigcap A_j = \emptyset$ if $i \neq j$ and $\bigcup_{i=1}^m A_i = \Omega$), where $t \leq T$. Then for each $X_i \in \mathcal{L}(\Omega, \mathcal{F}_T, P)$, $i = 1, \ldots, m$, one has*

$$\sum_{i=1}^m 1_{A_i} \mathcal{E}_g[X_i \mid \mathcal{F}_t] = \mathcal{E}_g\left[\sum_{i=1}^m 1_{A_i} X_i \mid \mathcal{F}_t\right] \quad a.s. \tag{7}$$

Proof. We consider the following BSDEs:

$$\begin{aligned}\mathscr{E}_g[X_i \mid \mathscr{F}_t] &= X_i + \int_t^T g\left(s, \mathscr{E}_g[X_i \mid \mathscr{F}_s], z_s^{(T,g,X_i)}\right) ds \\ &\quad - \int_t^T z_s^{(T,g,X_i)} \cdot dW_s, \quad i = 1, \ldots, m,\end{aligned} \tag{8}$$

$$\begin{aligned}\mathscr{E}_g\left[\sum_{i=1}^m 1_{A_i} X_i \mid \mathscr{F}_t\right] &= \sum_{i=1}^m 1_{A_i} X_i + \int_t^T g\left(s, \mathscr{E}_g\left[\sum_{i=1}^m 1_{A_i} X_i \mid \mathscr{F}_s\right], z_s^{(T,g,\sum_{i=1}^m 1_{A_i} X_i)}\right) ds \\ &\quad - \int_t^T z_s^{(T,g,\sum_{i=1}^m 1_{A_i} X_i)} \cdot dW_s.\end{aligned} \tag{9}$$

By the fact that $\sum_{i=1}^m 1_{A_i} g(s, \mathscr{E}_g[X_i \mid \mathscr{F}_s], z_s^{(T,g,X_i)}) = g(s, \sum_{i=1}^m 1_{A_i} \mathscr{E}_g[X_i \mid \mathscr{F}_s], \sum_{i=1}^m 1_{A_i} z_s^{(T,g,X_i)}), t \leq s \leq T$ and from (8), we have

$$\begin{aligned}\sum_{i=1}^m 1_{A_i} \mathscr{E}_g[X_i \mid \mathscr{F}_t] &= \sum_{i=1}^m 1_{A_i} X_i \\ &\quad + \int_t^T g\left(s, \sum_{i=1}^m 1_{A_i} \mathscr{E}_g[X_i \mid \mathscr{F}_s], \sum_{i=1}^m 1_{A_i} z_s^{(T,g,X_i)}\right) ds \\ &\quad - \int_t^T \sum_{i=1}^m 1_{A_i} z_s^{(T,g,X_i)} \cdot dW_s.\end{aligned} \tag{10}$$

Comparing this with (9), it follows that $\sum_{i=1}^m 1_{A_i} \mathscr{E}_g[X_i \mid \mathscr{F}_t] = \mathscr{E}_g[\sum_{i=1}^m 1_{A_i} X_i \mid \mathscr{F}_t]$ a.s. The proof of Lemma 9 is complete. □

Proposition 10. *Suppose g satisfies (A.1) and (A.2). Then the following two statements are equivalent:*

(i) *consider* $\forall (X,k) \in \mathscr{L}(\Omega, \mathscr{F}_T, P) \times R$, $\mathscr{E}_g[X+k \mid \mathscr{F}_t] = \mathscr{E}_g[X \mid \mathscr{F}_t] + k$ *a.s.,*

(ii) *consider* $\forall (X,\eta) \in \mathscr{L}(\Omega, \mathscr{F}_T, P) \times \mathscr{L}(\Omega, \mathscr{F}_t, P)$, $\mathscr{E}_g[X+\eta \mid \mathscr{F}_t] = \mathscr{E}_g[X \mid \mathscr{F}_t] + \eta$ *a.s.*

Proof. It is obvious that (ii) implies (i). We only need to prove that (i) implies (ii). Suppose (i) holds. Let $\{A_i\}_{i=1}^m$ be a $\mathscr{F}_t$-measurable partition of Ω and let $\lambda_i \in R$ ($i = 1, 2, \ldots, m$). From Lemma 9 and (i), we deduce that for each $X \in \mathscr{L}(\Omega, \mathscr{F}_T, P)$,

$$\begin{aligned}\mathscr{E}_g\left[X + \sum_{i=1}^m \lambda_i 1_{A_i} \mid \mathscr{F}_t\right] &= \mathscr{E}_g\left[\sum_{i=1}^m 1_{A_i}(X + \lambda_i) \mid \mathscr{F}_t\right] \\ &= \sum_{i=1}^m 1_{A_i} \mathscr{E}_g[X + \lambda_i \mid \mathscr{F}_t] \\ &= \sum_{i=1}^m 1_{A_i}\left(\mathscr{E}_g[X \mid \mathscr{F}_t] + \lambda_i\right) \\ &= \mathscr{E}_g[X \mid \mathscr{F}_t] + \sum_{i=1}^m \lambda_i 1_{A_i} \quad \text{a.s.}\end{aligned} \tag{11}$$

In other words, for any $X \in \mathscr{L}(\Omega, \mathscr{F}_T, P)$ and any simple function $\eta \in \mathscr{L}(\Omega, \mathscr{F}_t, P)$,

$$\mathscr{E}_g[X + \eta \mid \mathscr{F}_t] = \mathscr{E}_g[X \mid \mathscr{F}_t] + \eta \quad \text{a.s.} \tag{12}$$

Let

$$\begin{aligned}\eta_n :=& \sum_{i=0}^{n2^n-1} \frac{i}{2^n} 1_{\{(i/2^n) \leq \eta < ((i+1)/2^n)\}} + n 1_{\{\eta \geq n\}} \\ &+ \sum_{i=0}^{n2^n-1} \frac{-i}{2^n} 1_{\{-((i+1)/2^n) \leq \eta < -(i/2^n)\}} \\ &+ (-n) 1_{\{\eta < -n\}}, \quad n = 1, 2, \ldots.\end{aligned} \tag{13}$$

Obviously, for each n, η_n is a simple function in $\mathscr{L}(\Omega, \mathscr{F}_t, P)$. From (12), we have

$$\mathscr{E}_g[X + \eta_n \mid \mathscr{F}_t] = \mathscr{E}_g[X \mid \mathscr{F}_t] + \eta_n \quad \text{a.s.} \tag{14}$$

On the other hand, $\lim_{n\to\infty}(X + \eta_n) = X + \eta$, $|X + \eta_n| \leq |X| + |\eta|$. Thus, from Remark 8 (ii), it follows that (ii) is true. The proof of Proposition 10 is complete. □

3. Main Results and Applications

Definition 11. Let $g: \Omega \times [0,T] \times R \times R^d \mapsto R$. The function g is said to be superhomogeneous if for each $(y,z) \in R \times R^d$ and any real number λ, then $g(t, \lambda y, \lambda z) \geq \lambda g(t, y, z)$, $dP \times dt$ a.s. The function g is said to be positively homogeneous if for each $(y,z) \in R \times R^d$ and any real number $\lambda \geq 0$, then $g(t, \lambda y, \lambda z) = \lambda g(t, y, z)$, $dP \times dt$ a.s.

Before we give our main results, let us see an example.

Example 12. Fix $T = 1$ and $d = 1$. Let $\xi = f(W_1)$, where $f(x) = \exp((x^2/2p_1) - x) 1_{(x \geq p_1)}$, $1 < p_1 < 2$.

Obviously, f is an increasing function. We can easily get

$$\begin{aligned}E\left[|\xi|^{p_1}\right] &= \int_{p_1}^{\infty} \exp\left(\frac{x^2}{2} - p_1 x\right) \frac{1}{\sqrt{2\pi}} e^{-(1/2)x^2} dx \\ &= \frac{1}{\sqrt{2\pi} p_1} e^{-p_1^2} < \infty, \quad E\left[|\xi|^{p}\right] = \infty, \quad \forall p > p_1.\end{aligned} \tag{15}$$

Hence, $\xi \in \mathscr{L}(\Omega, \mathscr{F}_1, P)$, but $\xi \notin L^2(\Omega, \mathscr{F}_1, P)$.

Let $\xi^n = \xi \wedge n$, $n = 1, 2, \ldots$. Clearly, for each n, $\xi^n \in L^2(\Omega, \mathcal{F}, P)$. For simplicity, we will write $\mathcal{E}^{\mu}[\cdot] \equiv \mathcal{E}_g[\cdot]$ for $g = \mu|z|$. From Theorem 1 in Chen and Kulperger's [12], we know that $\mathcal{E}^{\mu}[\xi^n] = E_Q[\xi^n]$, where $dQ/dP = e^{-(1/2)\mu^2 + \mu W_1}$.

By Remark 8(i), we have $\mathcal{E}^{\mu}[\xi^n] \to \mathcal{E}^{\mu}[\xi]$, as $n \to \infty$. On the other hand, applying Hölder's inequality and noting that $E[e^{-(1/2)\mu^2 + \mu W_1}] = 1$ and $E[e^{-(1/2)\mu^2 q^2 + \mu q W_1}] = 1$, we obtain

$$\begin{aligned} E_Q[\xi] &\le \left(E[|\xi|^{p_1}]\right)^{1/p_1} \left(E\left[\left(\frac{dQ}{dP}\right)^q\right]\right)^{1/q} \\ &\le e^{(1/2)(q-1)\mu^2} \left(E\left[|\xi|^{p_1}\right]\right)^{1/p_1} < \infty, \end{aligned} \tag{16}$$

where $(1/p_1) + (1/q) = 1$. It then follows from the monotonic convergence theorem that

$$E_Q[\xi^n] \longrightarrow E_Q[\xi], \quad \text{as } n \longrightarrow \infty. \tag{17}$$

Thus

$$\mathcal{E}^{\mu}[\xi] = E_Q[\xi]. \tag{18}$$

Let $\varphi(x) = (x-k)^+$, where $k \in R$. Obviously, $\varphi(x)$ is a convex and increasing function. From this, we know that $\varphi \circ f$ is an increasing function. In a similar manner of the above, we can deduce that

$$\mathcal{E}^{\mu}[\varphi(\xi)] = E_Q[\varphi(\xi)]. \tag{19}$$

From (18), (19), and the classical Jensen's inequality, we have

$$\varphi(\mathcal{E}^{\mu}[\xi]) = \varphi(E_Q[\xi]) \le E_Q[\varphi(\xi)] = \mathcal{E}^{\mu}[\varphi(\xi)]. \tag{20}$$

This problem yields a natural question: in general, under which conditions on g do generalized Peng's g-expectations satisfy Jensen's inequality for convex functions?

The following theorem will answer this question.

Theorem 13. *Let g satisfy (A.1) and (A.2). Then the following four statements are equivalent.*

(i) *Jensen's inequality for generalized Peng's g-expectation $\mathcal{E}_g[\cdot \mid \mathcal{F}_t]$ holds in general, that is, for each convex function $\varphi(x): R \mapsto R$ and each $\xi \in \mathcal{L}(\Omega, \mathcal{F}_T, P)$, if $\varphi(\xi) \in \mathcal{L}(\Omega, \mathcal{F}_T, P)$, then one has*

$$\mathcal{E}_g[\varphi(\xi)\mathcal{F}_t] \ge \varphi\left(\mathcal{E}_g[\xi \mid \mathcal{F}_t]\right) \quad \text{a.s.}; \tag{21}$$

(ii) *consider $\forall(\xi, a, b) \in L^2(\Omega, \mathcal{F}_T, P) \times R \times R$, $\mathcal{E}_g[a\xi + b] \ge a\mathcal{E}_g[\xi] + b$;*

(iii) *consider $\forall(\xi, a, b) \in L^2(\Omega, \mathcal{F}_T, P) \times R \times R$, $\mathcal{E}_g[a\xi + b \mid \mathcal{F}_t] \ge a\mathcal{E}_g[\xi \mid \mathcal{F}_t] + b$ a.s.;*

(iv) *consider g is independent of y, superhomogeneous, and positively homogeneous with respect to z.*

Proof. (i)⇒(ii) is obvious.

(ii)⇒(iii): let $\eta = \xi + b$. By (ii), we have

$$\mathcal{E}_g[\eta - b] \ge \mathcal{E}_g[\eta] - b, \tag{22}$$

That is,

$$\mathcal{E}_g[\xi] + b \ge \mathcal{E}_g[\xi + b]. \tag{23}$$

Thus, for each $(\xi, b) \in L^2(\Omega, \mathcal{F}_T, P) \times R$,

$$\mathcal{E}_g[\xi + b] = \mathcal{E}_g[\xi] + b. \tag{24}$$

For each $(X, t, k) \in L^2(\Omega, \mathcal{F}_T, P) \times [0, T] \times R$, by (24), we know that for each $A \in \mathcal{F}_t$,

$$\begin{aligned} \mathcal{E}_g[1_A(X+k)] &= \mathcal{E}_g[1_A X + 1_A k - k] + k \\ &= \mathcal{E}_g[1_A X + 1_{A^C}(-k)] + k \\ &= \mathcal{E}_g\left[\mathcal{E}_g[1_A X + 1_{A^C}(-k) \mid \mathcal{F}_t]\right] + k \\ &= \mathcal{E}_g\left[1_A \mathcal{E}_g[X \mid \mathcal{F}_t] + 1_{A^C}(-k)\right] + k \\ &= \mathcal{E}_g\left[1_A \mathcal{E}_g[X \mid \mathcal{F}_t] + 1_{A^C}(-k) + k\right] \\ &= \mathcal{E}_g\left[1_A\left(\mathcal{E}_g[X \mid \mathcal{F}_t] + k\right)\right]. \end{aligned} \tag{25}$$

Thus,

$$\mathcal{E}_g[X + k \mid \mathcal{F}_t] = \mathcal{E}_g[X \mid \mathcal{F}_t] + k \quad \text{a.s.}, \ \forall t \in [0, T]. \tag{26}$$

On the other hand, for each $\lambda \neq 0$, define

$$\mathcal{E}^{\lambda}[\cdot \mid \mathcal{F}_t] = \frac{\mathcal{E}_g[\lambda \cdot \mid \mathcal{F}_t]}{\lambda}, \quad \forall t \in [0, T]. \tag{27}$$

It is easy to check that $\mathcal{E}_g[\cdot \mid \mathcal{F}_t]$ and $\mathcal{E}^{\lambda}[\cdot \mid \mathcal{F}_t]$ are two $\mathcal{F}$-expectations on $L^2(\Omega, \mathcal{F}_T, P)$ (the notion of $\mathcal{F}$-expectation can be seen in [13]). From (ii), we have if $\lambda > 0$, for each $\xi \in L^2(\Omega, \mathcal{F}_T, P)$

$$\mathcal{E}^{\lambda}[\xi] \ge \mathcal{E}_g[\xi]. \tag{28}$$

Hence, by Lemma 4.5 in [13], we have

$$\mathcal{E}^{\lambda}[\xi \mid \mathcal{F}_t] \ge \mathcal{E}_g[\xi \mid \mathcal{F}_t] \quad \text{a.s.}, \ \forall t \in [0, T]. \tag{29}$$

Similarly, if $\lambda < 0$, for each $\xi \in L^2(\Omega, \mathcal{F}_T, P)$

$$\mathcal{E}^{\lambda}[\xi] \le \mathcal{E}_g[\xi]. \tag{30}$$

Hence, by Lemma 4.5 in [13] again, we have

$$\mathcal{E}^{\lambda}[\xi \mid \mathcal{F}_t] \le \mathcal{E}_g[\xi \mid \mathcal{F}_t] \quad \text{a.s.}, \ \forall t \in [0, T]. \tag{31}$$

Thus from (29) and (31), we have $\forall(\xi, \lambda) \in L^2(\Omega, \mathcal{F}_T, P) \times R$,

$$\mathcal{E}_g[\lambda\xi \mid \mathcal{F}_t] \ge \lambda\mathcal{E}_g[\xi \mid \mathcal{F}_t] \quad \text{a.s.}, \ \forall t \in [0, T]. \tag{32}$$

From (26) and (32), we have

$$\forall (\xi, a, b) \in L^2(\Omega, \mathcal{F}_T, P) \times R \times R,$$
$$\mathcal{E}_g[a\xi + b \mid \mathcal{F}_t] \geq a\mathcal{E}_g[\xi \mid \mathcal{F}_t] + b \quad \text{a.s.}, \quad \forall t \in [0, T]. \tag{33}$$

(iii)⇒(iv): Firstly, we prove that g is independent of y. From (iii), we can obtain that for each $(\xi, y) \in L^2(\Omega, \mathcal{F}_T, P) \times R$,

$$\mathcal{E}_g[\xi - y \mid \mathcal{F}_t] = \mathcal{E}_g[\xi \mid \mathcal{F}_t] - y, \quad \text{a.s.}, \ \forall t \in [0, T]. \tag{34}$$

For each $(t, y, z) \in [0, T] \times R \times R^d$, let $Y_\cdot^{t,y,z}$ be the solution of the following SDE defined on $[t, T]$:

$$Y_s^{t,y,z} = y - \int_t^s g(r, Y_r^{t,y,z}, z)\,dr + z \cdot (W_s - W_t). \tag{35}$$

From (34), we have

$$Y_r^{t,y,z} - y = \mathcal{E}_g[Y_s^{t,y,z} \mid \mathcal{F}_r] - y = \mathcal{E}_g[Y_s^{t,y,z} - y \mid \mathcal{F}_r], \quad t \leq r \leq s \leq T. \tag{36}$$

Let $Y_s = Y_s^{t,y,z} - y$, $s \in [t, T]$ and Z be the corresponding part of Itô's integrand. It then follows that

$$\begin{aligned} Y_s &= -\int_t^s g(r, Y_r^{t,y,z}, z)\,dr + \int_t^s z \cdot dW_r \\ &= -\int_t^s g(r, Y_r, Z_r)\,\mathrm{d}r + \int_t^s Z_r \cdot dW_r. \end{aligned} \tag{37}$$

Thus, $Z_r \equiv z$ and

$$g(r, Y_r, z) = g(r, Y_r^{t,y,z} - y, z) = g(r, Y_r^{t,y,z}, z). \tag{38}$$

Then, we can apply Lemma 4.4 in Peng [14] to obtain that for each $(y, z) \in R \times R^d$,

$$g(t, y, z) = g(t, 0, z), \quad dP \times dt \ \text{a.s.} \tag{39}$$

Namely, g is independent of y.

Now we prove that g is superhomogeneous with respect to z. From (iii), we can obtain that for each $(\xi, \lambda) \in L^2(\Omega, \mathcal{F}_T, P) \times R$,

$$\lambda\mathcal{E}_g[\xi \mid \mathcal{F}_t] \leq \mathcal{E}_g[\lambda\xi \mid \mathcal{F}_t], \quad \text{a.s.}, \ \forall t \in [0, T]. \tag{40}$$

For each $(t, z) \in [0, T] \times R^d$, let $Y_\cdot^{t,z}$ be the solution of the following SDE defined on $[t, T]$:

$$Y_s^{t,z} = -\int_t^s g(r, z)\,dr + z \cdot (W_s - W_t). \tag{41}$$

From (40), we have

$$\mathcal{E}_g[\lambda Y_s^{t,z} \mid \mathcal{F}_r] \geq \lambda\mathcal{E}_g[Y_s^{t,z} \mid \mathcal{F}_r] = \lambda Y_r^{t,z}, \quad t \leq r \leq s \leq T. \tag{42}$$

Thus, $(\lambda Y_s^{t,z})_{s\in[t,T]}$ is an $\mathcal{E}_g$-submartingale. From the decomposition theorem of $\mathcal{E}_g$-supermartingale (see [15]), it follows that there exists an increasing process $(A_s)_{s\in[t,T]}$ such that

$$\lambda Y_s^{t,z} = -\int_t^s g(r, Z_r)\,dr + A_s - A_t + \int_t^s Z_r \cdot dW_r, \quad s \in [t, T]. \tag{43}$$

This with $\lambda Y_s^{t,z} = -\int_t^s \lambda g(r, z)\mathrm{d}r + \int_t^s \lambda z \cdot dW_r$ yields $Z_r \equiv \lambda z$ and

$$\lambda g(r, z) \leq g(r, \lambda z), \quad dP \times dt \ \text{a.s.} \tag{44}$$

At last, we prove that g is positively homogeneous with respect to z. From (iii), we can obtain that for each fixed $\lambda > 0$ and $\xi \in L^2(\Omega, \mathcal{F}_T, P)$,

$$\frac{1}{\lambda}\mathcal{E}_g[\lambda\xi \mid \mathcal{F}_t] \leq \mathcal{E}_g[\xi \mid \mathcal{F}_t], \quad \text{a.s.}, \ \forall t \in [0, T], \tag{45}$$

that is,

$$\mathcal{E}_g[\lambda\xi \mid \mathcal{F}_t] \leq \lambda\mathcal{E}_g[\xi \mid \mathcal{F}_t], \quad \text{a.s.}, \ \forall t \in [0, T]. \tag{46}$$

Thus, we have

$$\mathcal{E}_g[\lambda\xi \mid \mathcal{F}_t] = \lambda\mathcal{E}_g[\xi \mid \mathcal{F}_t], \quad \text{a.s.}, \ \forall t \in [0, T]. \tag{47}$$

Obviously, if $\lambda = 0$, (47) still holds. Thus, for each $\lambda \geq 0$,

$$\mathcal{E}_g[\lambda\xi \mid \mathcal{F}_t] = \lambda\mathcal{E}_g[\xi \mid \mathcal{F}_t], \quad \text{a.s.}, \ \forall t \in [0, T]. \tag{48}$$

For each $(t, z) \in [0, T] \times R^d$, let $Y_\cdot^{t,z}$ be the solution of SDE (34). From (48), for each $\lambda \geq 0$, we have

$$\mathcal{E}_g[\lambda Y_s^{t,z} \mid \mathcal{F}_r] = \lambda\mathcal{E}_g[Y_s^{t,z} \mid \mathcal{F}_r] = \lambda Y_r^{t,z}, \quad t \leq r \leq s \leq T. \tag{49}$$

This implies that there exists a process $Z_\cdot^{t,z,\lambda}$ such that

$$\lambda Y_s^{t,z} = -\int_t^s g(r, Z_r^{t,z,\lambda})\,dr + \int_t^s Z_r^{t,z,\lambda} \cdot dW_r, \quad s \in [t, T]. \tag{50}$$

Comparing this with $\lambda Y_s^{t,z} = -\int_t^s \lambda g(r, z)dr + \int_t^s \lambda z \cdot dW_r$, it follows that $Z_r^{t,z,\lambda} \equiv \lambda z$ and

$$\lambda g(r, z) = g(r, \lambda z), \quad dP \times dt \ \text{a.s.} \tag{51}$$

(iv)⇒(iii): By comparison theorem (for example, we can see [3]), it is easy to obtain (iii).

(iii)⇒(i): Suppose (iii) holds. From (iii) and by Remark 8 (i), we have

$$\forall (X, k) \in \mathcal{L}(\Omega, \mathcal{F}_T, P) \times R,$$
$$\mathcal{E}_g[X + k \mid \mathcal{F}_t] = \mathcal{E}_g[X \mid \mathcal{F}_t] + k \quad \text{a.s.}, \tag{52}$$

$$\forall (X, \lambda) \in \mathcal{L}(\Omega, \mathcal{F}_T, P) \times R,$$
$$\mathcal{E}_g[\lambda X \mid \mathcal{F}_t] \geq \lambda\mathcal{E}_g[X \mid \mathcal{F}_t] \quad \text{a.s.} \tag{53}$$

From (53), we can deduce that for each bounded variable $\zeta \in \mathcal{F}_t$,

$$\forall X \in \mathcal{L}(\Omega, \mathcal{F}_T, P), \quad \mathcal{E}_g[\zeta X \mid \mathcal{F}_t] \ge \zeta \mathcal{E}_g[X \mid \mathcal{F}_t] \quad \text{a.s.} \tag{54}$$

In fact, let $\{A_i\}_{i=1}^m$ be a $\mathcal{F}_t$-measurable partition of Ω and let $\lambda_i \in R$ $(i = 1, 2, \ldots, m)$. By (53), we have

$$\begin{aligned}\mathcal{E}_g\left[\sum_{i=1}^m \lambda_i 1_{A_i} X \mid \mathcal{F}_t\right] &= \sum_{i=1}^m 1_{A_i} \mathcal{E}_g[\lambda_i X \mid \mathcal{F}_t] \\ &\ge \sum_{i=1}^m 1_{A_i} \lambda_i \mathcal{E}_g[X \mid \mathcal{F}_t] \quad \text{a.s.}\end{aligned} \tag{55}$$

In other words, for each $X \in \mathcal{L}(\Omega, \mathcal{F}_T, P)$ and each simple function $\zeta \in \mathcal{L}(\Omega, \mathcal{F}_t, P)$,

$$\mathcal{E}_g[\zeta X \mid \mathcal{F}_t] \ge \zeta \mathcal{E}_g[X \mid \mathcal{F}_t] \quad \text{a.s.} \tag{56}$$

Thus, thanks to Remark 8(ii), it follows that (54) is true.

The main idea of the following proof is derived from [7]. Given $\xi \in \mathcal{L}(\Omega, \mathcal{F}_T, P)$ and convex function φ such that $\varphi(\xi) \in \mathcal{L}(\Omega, \mathcal{F}_T, P)$, we set $\eta_t = \varphi'_-(\mathcal{E}_g[\xi \mid \mathcal{F}_t])$. Then η_t is $\mathcal{F}_t$-measurable. Since φ is convex, we have

$$\varphi(x) - \varphi(y) \ge \varphi'_-(y)(x - y), \quad \forall x, y \in R. \tag{57}$$

Take $x = \xi$, $y = \mathcal{E}_g[\xi \mid \mathcal{F}_t]$. Then we have

$$\varphi(\xi) - \varphi\left(\mathcal{E}_g[\xi \mid \mathcal{F}_t]\right) \ge \eta_t\left(\xi - \mathcal{E}_g[\xi \mid \mathcal{F}_t]\right) \quad \text{a.s.} \tag{58}$$

For each $n \in N$, we define

$$\Omega_{t,n} := \left\{\left|\mathcal{E}_g[\xi \mid \mathcal{F}_t]\right| + |\eta_t| + \left|\varphi\left(\mathcal{E}_g[\xi \mid \mathcal{F}_t]\right)\right| \le n\right\}, \tag{59}$$

so we have

$$\begin{aligned}&\mathcal{E}_g\left[1_{\Omega_{t,n}}\varphi(\xi) \mid \mathcal{F}_t\right] \\ &\quad\ge \mathcal{E}_g\left[1_{\Omega_{t,n}}\varphi\left(\mathcal{E}_g[\xi \mid \mathcal{F}_t]\right) - 1_{\Omega_{t,n}}\eta_t\mathcal{E}_g[\xi \mid \mathcal{F}_t]\right. \\ &\quad\quad \left. + 1_{\Omega_{t,n}}\eta_t\xi \mid \mathcal{F}_t\right] \quad \text{a.s.}\end{aligned} \tag{60}$$

By the definition of $1_{\Omega_{t,n}}$, we know

$$1_{\Omega_{t,n}}\varphi\left(\mathcal{E}_g[\xi \mid \mathcal{F}_t]\right) - 1_{\Omega_{t,n}}\eta_t\mathcal{E}_g[\xi \mid \mathcal{F}_t] \in \mathcal{L}(\Omega, \mathcal{F}_t, P). \tag{61}$$

Thus, in view of (52) and from Proposition 10, we can get

$$\begin{aligned}\mathcal{E}_g\left[1_{\Omega_{t,n}}\varphi(\xi) \mid \mathcal{F}_t\right] &\ge 1_{\Omega_{t,n}}\varphi\left(\mathcal{E}_g[\xi \mid \mathcal{F}_t]\right) \\ &\quad - 1_{\Omega_{t,n}}\eta_t\mathcal{E}_g[\xi \mid \mathcal{F}_t] \\ &\quad + \mathcal{E}_g\left[1_{\Omega_{t,n}}\eta_t\xi \mid \mathcal{F}_t\right] \quad \text{a.s.}\end{aligned} \tag{62}$$

Moreover, from (54), considering that $1_{\Omega_{t,n}}\eta_t \in \mathcal{F}_t$ and is bounded by n, we can get

$$\mathcal{E}_g\left[1_{\Omega_{t,n}}\eta_t\xi \mid \mathcal{F}_t\right] \ge 1_{\Omega_{t,n}}\eta_t\mathcal{E}_g[\xi \mid \mathcal{F}_t] \quad \text{a.s.} \tag{63}$$

Hence, we can deduce that for each $n \in N$,

$$\mathcal{E}_g\left[1_{\Omega_{t,n}}\varphi(\xi) \mid \mathcal{F}_t\right] \ge 1_{\Omega_{t,n}}\varphi\left(\mathcal{E}_g[\xi \mid \mathcal{F}_t]\right) \quad \text{a.s.} \tag{64}$$

Finally, thanks to Remark 8 (ii) again, we can get

$$\mathcal{E}_g[\varphi(\xi) \mid \mathcal{F}_t] \ge \varphi\left(\mathcal{E}_g[\xi \mid \mathcal{F}_t]\right) \quad \text{a.s.} \tag{65}$$

Hence, Jensen's inequality for $\mathcal{E}_g[\cdot \mid \mathcal{F}_t]$ holds in general. The proof of Theorem 13 is complete. □

Example 14. Suppose H is a bounded, convex, and closed subset of R^d and D = the set of R^d-valued continuous processes $(v_t)_{t\in[0,T]}$ such that for each t, $v_t \in H$ a.s.. Define the probability measure Q^v by

$$\frac{dQ^v}{dP} = e^{-(1/2)\int_0^T |v_s|^2 ds + \int_0^T v_s \cdot dW_s}. \tag{66}$$

Thus, for any convex function φ,

$$\begin{aligned}\varphi\left(\operatorname*{ess\,sup}_{v\in D} E_{Q^v}[\xi \mid \mathcal{F}_t]\right) &\le \operatorname*{ess\,sup}_{v\in D} E_{Q^v}[\varphi(\xi) \mid \mathcal{F}_t], \\ &\quad \text{a.s.}, \forall t \in [0, T],\end{aligned} \tag{67}$$

whenever $\xi, \varphi(\xi) \in \mathcal{L}(\Omega, \mathcal{F}_T, P)$.

Proof. Let $g(t, z) = \operatorname{ess\,sup}_{v\in D} v_t \cdot z$. Obviously, $g(t, z)$ is superhomogeneous and positively homogeneous with respect to z. and satisfies (A.1) and (A.2).

From El Karoui and Quenez [16], we have $\operatorname{ess\,sup}_{v\in D} E_{Q^v}[\xi \mid \mathcal{F}_t] = \mathcal{E}_g[\xi \mid \mathcal{F}_t]$, a.s., $\forall \xi \in L^2(\Omega, \mathcal{F}_T, P)$. Now we prove $\operatorname{ess\,sup}_{v\in D} E_{Q^v}[\xi \mid \mathcal{F}_t] = \mathcal{E}_g[\xi \mid \mathcal{F}_t]$, a.s., $\forall \xi \in \mathcal{L}(\Omega, \mathcal{F}_T, P)$. Indeed, for any $\xi \in \mathcal{L}(\Omega, \mathcal{F}_T, P)$, there exists $1 < p < 2$ such that $\xi \in L^p(\Omega, \mathcal{F}_T, P)$. Let $\xi^n = (\xi \wedge n) \vee (-n)$, $n = 1, 2, \ldots$. Clearly, for each n, $\xi^n \in L^2(\Omega, \mathcal{F}_T, P)$, then

$$\operatorname*{ess\,sup}_{v\in D} E_{Q^v}[\xi^n \mid \mathcal{F}_t] = \mathcal{E}_g[\xi^n \mid \mathcal{F}_t], \quad \text{a.s.} \tag{68}$$

Since

$$\begin{aligned}&\operatorname*{ess\,sup}_{v\in D} E_{Q^v}[\xi^n \mid \mathcal{F}_t] \\ &\quad= \operatorname*{ess\,sup}_{v\in D}\left(E_{Q^v}[\xi^n - \xi \mid \mathcal{F}_t] + E_{Q^v}[\xi \mid \mathcal{F}_t]\right) \\ &\quad\le \operatorname*{ess\,sup}_{v\in D} E_{Q^v}[\xi^n - \xi \mid \mathcal{F}_t] + \operatorname*{ess\,sup}_{v\in D} E_{Q^v}[\xi \mid \mathcal{F}_t],\end{aligned} \tag{69}$$

we have

$$\begin{aligned}&\mathcal{E}_g[\xi^n \mid \mathcal{F}_t] - \operatorname*{ess\,sup}_{v\in D} E_{Q^v}[\xi \mid \mathcal{F}_t] \\ &\quad\le \operatorname*{ess\,sup}_{v\in D} E_{Q^v}[\xi^n - \xi \mid \mathcal{F}_t].\end{aligned} \tag{70}$$

With an approach similar to the one above, we can get easily that

$$\begin{aligned}&\mathcal{E}_g[\xi^n \mid \mathcal{F}_t] - \operatorname*{ess\,sup}_{v\in D} E_{Q^v}[\xi \mid \mathcal{F}_t] \\ &\quad\ge \operatorname*{ess\,inf}_{v\in D} E_{Q^v}[\xi^n - \xi \mid \mathcal{F}_t].\end{aligned} \tag{71}$$

Combining (42) with (43), we have

$$\begin{aligned}&\left|\mathscr{E}_g[\xi^n \mid \mathscr{F}_t] - \operatorname{ess\,sup}_{v\in D} E_{Q^v}[\xi \mid \mathscr{F}_t]\right| \\ &\le \left(\left|\operatorname{ess\,inf}_{v\in D} E_{Q^v}[\xi^n - \xi \mid \mathscr{F}_t]\right| \vee \left|\operatorname{ess\,sup}_{v\in D} E_{Q^v}[\xi^n - \xi \mid \mathscr{F}_t]\right|\right) \\ &\le \operatorname{ess\,sup}_{v\in D} E_{Q^v}[|\xi^n - \xi| \mid \mathscr{F}_t].\end{aligned} \tag{72}$$

By Hölder's inequality and noting that $(e^{-(1/2)\int_0^t |v_s|^2 ds + \int_0^t v_s \cdot dW_s})_{t\in[0,T]}$ and $(e^{-(1/2)\int_0^t |qv_s|^2 ds + \int_0^t qv_s \cdot dW_s})_{t\in[0,T]}$ are both martingales with respect to $(\mathscr{F}_t)_{t\in[0,T]}$, we can obtain

$$\begin{aligned}&E_{Q^v}[|\xi^n - \xi| \mid \mathscr{F}_t] \\ &= \frac{E[|\xi^n - \xi| (dQ^v/dP) \mid \mathscr{F}_t]}{E[(dQ^v/dP) \mid \mathscr{F}_t]} \\ &\le \frac{\left(E[|\xi^n - \xi|^p \mid \mathscr{F}_t]\right)^{1/p}\left(E[(dQ^v/dP)^q \mid \mathscr{F}_t]\right)^{1/q}}{E[(dQ^v/dP) \mid \mathscr{F}_t]} \\ &\le e^{(1/2)(q-1)\mu^2 T}\left(E[|\xi^n - \xi|^p \mid \mathscr{F}_t]\right)^{1/p},\end{aligned} \tag{73}$$

where $(1/p) + (1/q) = 1$. It then follows from Lebesgue's dominated convergence theorem that

$$\operatorname{ess\,sup}_{v\in D} E_{Q^v}[|\xi^n - \xi| \mid \mathscr{F}_t] \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{74}$$

Hence,

$$\left|\mathscr{E}_g[\xi^n \mid \mathscr{F}_t] - \operatorname{ess\,sup}_{v\in D} E_{Q^v}[\xi \mid \mathscr{F}_t]\right| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{75}$$

On the other hand, from Remark 8(i), we have

$$\mathscr{E}_g[\xi^n \mid \mathscr{F}_t] \longrightarrow \mathscr{E}_g[\xi \mid \mathscr{F}_t], \quad \text{as } n \longrightarrow \infty. \tag{76}$$

Thus,

$$\mathscr{E}_g[\xi \mid \mathscr{F}_t] = \operatorname{ess\,sup}_{v\in D} E_{Q^v}[\xi \mid \mathscr{F}_t], \quad \text{a.s.}, \forall \xi \in \mathscr{L}(\Omega, \mathscr{F}_T, P). \tag{77}$$

Applying Theorem 13, we have

$$\varphi\left(\operatorname{ess\,sup}_{v\in D} E_{Q^v}[\xi \mid \mathscr{F}_t]\right) \le \operatorname{ess\,sup}_{v\in D} E_{Q^v}[\varphi(\xi) \mid \mathscr{F}_t], \quad \text{a.s.} \tag{78}$$

□

Definition 15. Suppose g satisfies (A.1) and (A.2). A process $(X_t)_{t\in[0,T]}$ satisfying that for each t, $X_t \in \mathscr{L}(\Omega, \mathscr{F}_t, P)$ is called a generalized Peng's g-martingale (resp., generalized Peng's g-supermartingale, generalized Peng's g-submartingale), if for any t, s satisfying $t \le s \le T$,

$$\mathscr{E}_g[X_s \mid \mathscr{F}_t] = X_t \quad (\text{resp.} \le X_t, \ge X_t), \text{a.s.} \tag{79}$$

Applying Theorem 13, immediately we have the following.

Theorem 16. *Suppose g is independent of y, superhomogeneous and positively homogeneous with respect to z and satisfies (A.1) and (A.2). If $(X_t)_{t\in[0,T]}$ is a generalized Peng's g-martingale and φ is a convex function such that $\varphi(X_t) \in \mathscr{L}(\Omega, \mathscr{F}_t, P)$, then $(\varphi(X_t))_{t\in[0,T]}$ is a generalized Peng's g-submartingale.*

Remark 17. Suppose g is independent of y, superhomogeneous and positively homogeneous with respect to z and satisfies (A.1) and (A.2). Similarly, we can get the following.

(i) If $(X_t)_{t\in[0,T]}$ is a generalized Peng's g-submartingale and φ is a convex and increasing function such that $\varphi(X_t) \in \mathscr{L}(\Omega, \mathscr{F}_t, P)$, then $(\varphi(X_t))_{t\in[0,T]}$ is a generalized Peng's g-submartingale.

(ii) If $(X_t)_{t\in[0,T]}$ is a generalized Peng's g-supermartingale and φ is a convex and decreasing function such that $\varphi(X_t) \in \mathscr{L}(\Omega, \mathscr{F}_t, P)$, then $(\varphi(X_t))_{t\in[0,T]}$ is a generalized Peng's g-submartingale.

Example 18. (i) Let $g = \mu|z|$ and $\varphi(x) = (x-a)^+$ where $a \in R$. Obviously, g satisfies the assumptions of Remark 17 and φ is a convex and increasing function. By Remark 17 (i), we have the following: suppose $(X_t)_{t\in[0,T]}$ is a $\mathscr{E}^\mu$-submartingale, then $((X_t - a)^+)_{t\in[0,T]}$ is a $\mathscr{E}^\mu$-submartingale.

(ii) Let $g = \mu|z|$ and $\varphi(x) = (x - b)^-$ where $b \in R$. With the similar argument, we have the following: suppose $(Y_t)_{t\in[0,T]}$ is a $\mathscr{E}^\mu$-supermartingale, then $((Y_t - b)^-)_{t\in[0,T]}$ is a $\mathscr{E}^\mu$-submartingale.

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References

[1] E. Pardoux and S. Peng, "Adapted solution of a backward stochastic differential equation," *Systems & Control Letters*, vol. 14, no. 1, pp. 55–61, 1990.

[2] S. Peng, "A generalized dynamic programming principle and Hamilton-Jacobi-Bellman equation," *Stochastics and Stochastics Reports*, vol. 38, no. 2, pp. 119–134, 1992.

[3] S. Peng, "Backward SDE and related g-expectation," in *Backward Stochastic Differential Equations*, N. El Karoui and L. Mazliak, Eds., vol. 364 of *Pitman Research Notes in Mathematics Series*, pp. 141–159, Longman, Harlow, UK, 1997.

[4] Ph. Briand, F. Coquet, Y. Hu, J. Mémin, and S. Peng, "A converse comparison theorem for BSDEs and related properties of g-expectation," *Electronic Communications in Probability*, vol. 5, pp. 101–117, 2000.

[5] Z. Chen, R. Kulperger, and L. Jiang, "Jensen's inequality for g-expectation. I," *Comptes Rendus Mathématique*, vol. 337, no. 11, pp. 725–730, 2003.

[6] Z. Chen, R. Kulperger, and L. Jiang, "Jensen's inequality for g-expectation. II," *Comptes Rendus Mathématique*, vol. 337, no. 12, pp. 797–800, 2003.

[7] L. Jiang, "Representation theorems for generators of backward stochastic differential equations and their applications," *Stochastic Processes and their Applications*, vol. 115, no. 12, pp. 1883–1903, 2005.

[8] Y. Hu, "On Jensen's inequality for g-expectation and for nonlinear expectation," *Archiv der Mathematik*, vol. 85, no. 6, pp. 572–680, 2005.

[9] L. Jiang, "Jensen's inequality for backward stochastic differential equations," *Chinese Annals of Mathematics Series B*, vol. 27, no. 5, pp. 553–564, 2006.

[10] Ph. Briand, B. Delyon, Y. Hu, E. Pardoux, and L. Stoica, "L^p solutions of backward stochastic differential equations," *Stochastic Processes and their Applications*, vol. 108, no. 1, pp. 109–129, 2003.

[11] F. Hu and Z. Chen, "Generalized Peng's g-expectations and related properties," *Statistics & Probability Letters*, vol. 80, no. 3-4, pp. 191–195, 2010.

[12] Z. Chen and R. Kulperger, "Minimax pricing and Choquet pricing," *Insurance: Mathematics and Economics*, vol. 38, no. 3, pp. 518–528, 2006.

[13] F. Coquet, Y. Hu, J. Mémin, and S. Peng, "Filtration-consistent nonlinear expectations and related g-expectations," *Probability Theory and Related Fields*, vol. 123, no. 1, pp. 1–27, 2002.

[14] S. Peng, "Modelling derivatives pricing mechanisms with their generating functions," http://arxiv.org/abs/math/0605599, 2006.

[15] S. Peng, "Monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob-Meyer's type," *Probability Theory and Related Fields*, vol. 113, no. 4, pp. 473–499, 1999.

[16] N. El Karoui and M.-C. Quenez, "Dynamic programming and pricing of contingent claims in an incomplete market," *SIAM Journal on Control and Optimization*, vol. 33, no. 1, pp. 29–66, 1995.