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Research Article

General Decay for the Degenerate Equation with a Memory Condition at the Boundary

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We consider a degenerate equation with a memory condition at the boundary. For a wider class of relaxation functions, we establish a more general decay result, from which the usual exponential and polynomial decay rates are only special cases.

1. Introduction

The main purpose of this paper is to investigate the asymptotic behavior of the solutions of the degenerate equation with a memory condition at the boundary

$$K(x)u'' + \Delta^2 u + f(u) = 0 \quad \text{in } Q = \Omega \times (0,\infty), \tag{1}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_0 \times (0,\infty), \tag{2}$$

$$-u + \int_0^t g_1(t-s)\mathscr{B}_2 u(s)\,ds = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \tag{3}$$

$$\frac{\partial u}{\partial \nu} + \int_0^t g_2(t-s)\mathscr{B}_1 u(s)\,ds = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \tag{4}$$

$$u(0,x) = u_0(x), \qquad u'(0,x) = u_1(x) \text{ in } \Omega, \tag{5}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary Γ and let us assume that Γ, can be divided into two nonnull parts $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ and $K \in C^1(\overline{\Omega})$ and $K(x) \geq 0$ for all $x \in \Omega$ which satisfies some appropriate conditions. ν is the unit outward normal to Γ and $\tau = (-\nu_2, \nu_1)$ the corresponding unit tangent vector. Here, the relaxation functions g_i $(i = 1, 2)$ are positive and nondecreasing, the function $f \in C^1(\mathbb{R})$ and

$$\mathscr{B}_1 u = \Delta u + (1-\mu) B_1 u, \qquad \mathscr{B}_2 u = \frac{\partial \Delta u}{\partial \nu} + (1-\mu)\frac{\partial B_2 u}{\partial \tau},$$
$$B_1 u = 2\nu_1\nu_2 u_{xy} - \nu_1^2 u_{yy} - \nu_2^2 u_{xx},$$
$$B_2 u = (\nu_1^2 - \nu_2^2) u_{xy} + \nu_1\nu_2 (u_{yy} - u_{xx}), \tag{6}$$

and the constant μ, $0 < \mu < 1/2$, represents Poisson's ratio.

From the physical point of view, we know that the memory effect described in integral equations (3) and (4) can be caused by the interaction with another viscoelastic element. In fact, the boundary conditions (3) and (4) mean that Ω is composed of a material which is clamped in a rigid body in Γ_0 and is clamped in a body with viscoelastic properties in the complementary part of its boundary named Γ_1. Problems related to (1)–(5) are interesting not only from the point of view of PDE general theory, but also due to its applications in mechanics.

The existence of global solutions and exponential decay to the degenerate equation with $\partial\Omega = \Gamma_0$ has been investigated by several authors. See Cavalcanti et al. [1] and Menezes et al. [2]. For instance, when $K(x)$ is equal to 1, (1) describes the transverse deflection $u(x,t)$ of beams. There exists a large body of literature regarding viscoelastic problems with the memory term acting in the domain or at the boundary (see [3–17]). Santos et al. [18] studied the asymptotic behavior of the solutions of a nonlinear wave equation of Kirchhoff type with boundary condition of memory type. Cavalcanti et al. [19] proved the uniform decay rates of solutions to a degenerate system with a memory condition at the boundary. Santos and Junior [20] investigated the stability of solutions for Kirchhoff plate equations with a boundary memory condition. Rivera et al. [21] showed the asymptotic behavior to a von Karman plate with boundary memory conditions. Park and Kang [22] studied the exponential decay for the Kirchhoff plate equations with nonlinear dissipation and boundary memory condition. They proved that the energy decays uniformly exponentially or algebraically with the same rate of decay as the relaxation functions. In the present work, we generalize the earlier decay results of the solution of (1)–(5). More precisely, we show that the energy decays at the rate similar to the relaxation functions, which are not necessarily decaying like polynomial or exponential functions. In fact, our result allows a larger class of relaxation functions. Recently, Messaoudi and Soufyane [23], Mustafa and Messaoudi [24], and Santos and Soufyane [25] proved the general decay for the wave equation, Timoshenko system, and von Karman plate system with viscoelastic boundary conditions, respectively.

The organization of this paper is as follows. In Section 2, we present some notations and material needed for our work and state the existence result to system (1)–(5). In Section 3, we prove the general decay of the solutions to the degenerate equation with a memory condition at the boundary.

2. Preliminaries

In this section, we introduce some notations and establish the existence of solutions of the problem (1)–(5).

Note that, because of condition (2), the solution of system (1)–(5) must belong to the following space:

$$W = \left\{ v \in H^2(\Omega) : v = \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma_0 \right\}. \tag{7}$$

Let us define the bilinear form $a(\cdot,\cdot)$ as follows:

$$a(u,v) = \int_\Omega \left\{ u_{xx}v_{xx} + u_{yy}v_{yy} + \mu\left(u_{xx}v_{yy} + u_{yy}v_{xx}\right) + 2(1-\mu)u_{xy}v_{xy} \right\} dx\,dy. \tag{8}$$

Since $\Gamma_0 \neq \emptyset$, we know that $a(u,u)$ is equivalent to the $H^2(\Omega)$ norm on W; that is,

$$c_0\|u\|^2_{H^2(\Omega)} \le a(u,u) \le C_0\|u\|^2_{H^2(\Omega)}, \tag{9}$$

and here and in the sequel, we denote by c_0 and C_0 generic positive constants. Simple calculation, based on the integration by parts formula, yields

$$\left(\Delta^2 u, v\right) = a(u,v) + \left(\mathscr{B}_2 u, v\right)_\Gamma - \left(\mathscr{B}_1 u, \frac{\partial v}{\partial \nu}\right)_\Gamma. \tag{10}$$

We assume that there exists $x_0 \in \mathbb{R}^n$ such that

$$\Gamma_0 = \left\{x \in \Gamma : \nu(x)\cdot(x-x_0) \le 0\right\}, \tag{11}$$

$$\Gamma_1 = \left\{x \in \Gamma : \nu(x)\cdot(x-x_0) > 0\right\}. \tag{12}$$

If we denote the compactness of Γ_1 by $m(x) = x - x_0$, condition (12) implies that there exists a small positive constant δ_0 such that $0 < \delta_0 \le m(x)\cdot\nu(x)$, for all $x \in \Gamma_1$.

Next, we will use (3) and (4) to estimate the values $\mathscr{B}_1$ and $\mathscr{B}_2$ on Γ_1. Denoting by

$$(g * \varphi)(t) = \int_0^t g(t-s)\varphi(s)\,ds \tag{13}$$

the convolution product operator and differentiating (3) and (4), we arrive at the following Volterra equations:

$$\begin{aligned} \mathscr{B}_2 u + \frac{1}{g_1(0)} g_1' * \mathscr{B}_2 u &= \frac{1}{g_1(0)} u', \\ \mathscr{B}_1 u + \frac{1}{g_2(0)} g_2' * \mathscr{B}_1 u &= -\frac{1}{g_2(0)} \frac{\partial u'}{\partial \nu}. \end{aligned} \tag{14}$$

Applying the Volterra inverse operator, we get

$$\begin{aligned} \mathscr{B}_2 u &= \frac{1}{g_1(0)}\left\{u' + k_1 * u'\right\}, \\ \mathscr{B}_1 u &= -\frac{1}{g_2(0)}\left\{\frac{\partial u'}{\partial \nu} + k_2 * \frac{\partial u'}{\partial \nu}\right\}, \end{aligned} \tag{15}$$

where the resolvent kernels satisfy

$$k_i + \frac{1}{g_i(0)} g_i' * k_i = -\frac{1}{g_i(0)} g_i', \quad \forall i = 1,2. \tag{16}$$

Denoting that $\tau_1 = 1/g_1(0)$ and $\tau_2 = 1/g_2(0)$, we have

$$\begin{aligned} \mathscr{B}_2 u &= \tau_1\left\{u' + k_1(0)u - k_1(t)u_0 + k_1' * u\right\}, \\ \mathscr{B}_1 u &= -\tau_2\left\{\frac{\partial u'}{\partial \nu} + k_2(0)\frac{\partial u}{\partial \nu} - k_2(t)\frac{\partial u_0}{\partial \nu} + k_2' * \frac{\partial u}{\partial \nu}\right\}. \end{aligned} \tag{17}$$

Therefore, we use (17) instead of the boundary conditions (3) and (4).

Let us denote that

$$(g \diamond v)(t) := \int_0^t g(t-s)\left|v(t) - v(s)\right|^2 ds. \tag{18}$$

The following lemma states an important property of the convolution operator.

Lemma 1. *For $g, v \in C^1([0,\infty) : \mathbb{R})$, one has*

$$(g * v)v' = -\frac{1}{2}g(t)|v(t)|^2 + \frac{1}{2}g' \diamond v - \frac{1}{2}\frac{d}{dt} \times \left[g \diamond v - \left(\int_0^t g(s)\,ds\right)|v|^2\right]. \tag{19}$$

The proof of this lemma follows by differentiating the term $g \diamond v$.

Lemma 2 (see [26]). *Suppose that $f \in L^2(\Omega)$, $g \in H^{1/2}(\Gamma_1)$ and $h \in H^{3/2}(\Gamma_1)$; then, any solution of*

$$a(v,w) = \int_\Omega fw\,dx + \int_{\Gamma_1} gw\,d\Gamma + \int_{\Gamma_1} h\frac{\partial w}{\partial \nu}d\Gamma, \quad \forall w \in W, \tag{20}$$

satisfies $v \in H^4(\Omega)$ and also

$$\begin{aligned} \Delta^2 v = f, \qquad v = \frac{\partial v}{\partial \nu} = 0 \quad on\ \Gamma_0, \\ \mathscr{B}_1 v = h, \qquad \mathscr{B}_2 v = g \quad on\ \Gamma_1. \end{aligned} \tag{21}$$

We formulate the following assumptions.

(A1) Let $f \in C^1(\mathbb{R})$ satisfy

$$f(s)s \geq 0, \quad \forall s \in \mathbb{R}. \tag{22}$$

Additionally, we suppose that f is superlinear; that is,

$$f(s)s \geq (2+\eta)F(s), \quad F(z) = \int_0^z f(s)\,ds, \quad \forall s \in \mathbb{R}, \tag{23}$$

for some $\eta > 0$ with the following growth condition:

$$|f(x) - f(y)| \leq c\left(1 + |x|^{\rho-1} + |y|^{\rho-1}\right)|x-y|, \quad \forall x, y \in \mathbb{R} \tag{24}$$

for some $c > 0$ and $\rho \geq 1$ such that $(n-2)\rho \leq n$.

(A2) $K \in C^1(\overline{\Omega})$; $H_0^2(\Omega) \cap L^\infty(\Omega)$ with $K(x) \geq 0$, for all $x \in \Omega$, and satisfy the following condition

$$\nabla K \cdot m \geq 0 \quad \text{in } \Omega. \tag{25}$$

The well-posedness of system (1)–(5) is given by the following theorem.

Theorem 3 (see [27]). *Consider assumptions (A1)-(A2) and let $k_i \in C^2(\mathbb{R}^+)$ be such that*

$$k_i, -k_i', k_i'' \geq 0 \quad (i = 1, 2). \tag{26}$$

If $u_0 \in W \cap H^4(\Omega)$, $u_1 \in W$, satisfying the compatibility condition

$$\mathscr{B}_1 u_0 = -\tau_2 \frac{\partial u_1}{\partial \nu}, \qquad \mathscr{B}_2 u_0 = \tau_1 u_1 \quad on\ \Gamma_1, \tag{27}$$

then there is only one solution u of the system (1)–(5) *satisfying*

$$\begin{gathered} u \in L^\infty\left(0, T : W \cap H^4(\Omega)\right), \qquad u' \in L^\infty(0, T : W), \\ u'' \in L^\infty\left(0, T : L^2(\Omega)\right). \end{gathered} \tag{28}$$

3. General Decay

In this section, we show that the solution of system (1)–(5) may have a general decay not necessarily of exponential or polynomial type. For this we consider that the resolvent kernels satisfy the following hypothesis.

(H) $k_i : \mathbb{R}_+ \to \mathbb{R}_+$ is twice differentiable function such that

$$k_i(0) > 0, \qquad \lim_{t\to\infty} k_i(t) = 0, \qquad k_i'(t) \leq 0, \tag{29}$$

and there exists a nonincreasing continuous function $\xi_i : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$k_i''(t) \geq -\xi_i(t)k_i'(t), \quad i = 1, 2, \ \forall t \geq 0. \tag{30}$$

The following identity will be used later.

Lemma 4 (see [26]). *For every $v \in H^4(\Omega)$ and for every $\mu \in \mathbb{R}$, one has*

$$\begin{aligned} \int_\Omega (m \cdot \nabla v)\Delta^2 v\,dx &= a(v,v) \\ &+ \frac{1}{2}\int_\Gamma m \cdot \nu \left[v_{xx}^2 + v_{yy}^2 + 2\mu v_{xx}v_{yy} + 2(1-\mu)v_{xy}^2\right]d\Gamma \\ &+ \int_\Gamma \left[(\mathscr{B}_2 v)\,m \cdot \nabla v - (\mathscr{B}_1 v)\frac{\partial}{\partial \nu}(m \cdot \nabla v)\right]d\Gamma. \end{aligned} \tag{31}$$

Let us introduce the energy function

$$\begin{aligned} E(t) = {} & \frac{1}{2}\int_\Omega K(x)|u'|^2 dx + \frac{1}{2}a(u,u) \\ & + \int_\Omega F(u)\,dx + \frac{\tau_1}{2}\int_{\Gamma_1}\left(k_1(t)|u|^2 - k_1' \diamond u\right)d\Gamma \\ & + \frac{\tau_2}{2}\int_{\Gamma_1}\left(k_2(t)\left|\frac{\partial u}{\partial \nu}\right|^2 - k_2' \diamond \frac{\partial u}{\partial \nu}\right)d\Gamma. \end{aligned} \tag{32}$$

Now, we establish some inequalities for the strong solution of system (1)–(5).

Lemma 5. *The energy functional E satisfies, along the solution of* (1)–(5), *the estimate*

$$\begin{aligned} E'(t) \leq {} & -\frac{\tau_1}{2}\int_{\Gamma_1}\left(|u'|^2 - k_1^2(t)|u_0|^2 - k_1'(t)|u|^2 + k_1'' \diamond u\right)d\Gamma \\ & - \frac{\tau_2}{2}\int_{\Gamma_1}\left(\left|\frac{\partial u'}{\partial \nu}\right|^2 - k_2^2(t)\left|\frac{\partial u_0}{\partial \nu}\right|^2\right. \\ & \left. - k_2'(t)\left|\frac{\partial u}{\partial \nu}\right|^2 + k_2'' \diamond \frac{\partial u}{\partial \nu}\right)d\Gamma. \end{aligned} \tag{33}$$

Proof. Multiplying (1) by u', integrating over Ω, and using (10), we get

$$\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega K\left|u'\right|^2 dx + a(u,u) + 2\int_\Omega F(u)\,dx\right\} = -\int_{\Gamma_1}(\mathscr{B}_2 u)\,u'\,d\Gamma + \int_{\Gamma_1}(\mathscr{B}_1 u)\frac{\partial u'}{\partial\nu}\,d\Gamma. \tag{34}$$

Substituting the boundary terms by (17) and using Lemma 1 and the Young inequality, our conclusion follows. □

Let us consider the following binary operator:

$$(k\circ u)(t) := \int_0^t k(t-s)(u(t)-u(s))\,ds. \tag{35}$$

Then applying the Holder inequality for $0\le\alpha\le 1$ we have

$$|(k\circ u)(t)|^2 \le \left[\int_0^t |k(s)|^{2(1-\alpha)}ds\right]\left(|k|^{2\alpha}\diamond u\right)(t). \tag{36}$$

Let us define the functional

$$\psi(t) = \int_\Omega\left[m\cdot\nabla u + \left(\frac{n}{2}-\theta\right)u\right]Ku'\,dx. \tag{37}$$

The following lemma plays an important role in the construction of the desired functional.

Lemma 6. *Suppose that the initial data* $(u_0,u_1)\in(H^4(\Omega)\cap W)\times W$, *satisfying the compatibility condition* (27). *Then, the solution of system* (1)–(5) *satisfies*

$$\begin{aligned}\psi'(t) \le{}& \frac{1}{2}\int_{\Gamma_1} m\cdot\nu K\left|u'\right|^2 d\Gamma\\ &-\theta\int_\Omega K\left|u'\right|^2dx - \left(1+\frac{n}{2}-\theta-\epsilon\lambda_0\right)a(u,u)\\ &-\left(\frac{n\eta}{2}-2\theta-\eta\theta\right)\int_\Omega F(u)\,dx + \frac{2{\tau_1}^2}{\epsilon}\\ &\times\int_{\Gamma_1}\left\{\left|u'\right|^2 + k_1^2(t)\,|u|^2 + k_1^2(t)\,\left|u_0\right|^2 + \left|k_1'\circ u\right|^2\right\}d\Gamma\\ &+\frac{2{\tau_2}^2}{\epsilon}\int_{\Gamma_1}\left\{\left|\frac{\partial u'}{\partial\nu}\right|^2 + k_2^2(t)\left|\frac{\partial u}{\partial\nu}\right|^2 + k_2^2(t)\left|\frac{\partial u_0}{\partial\nu}\right|^2\right.\\ &\qquad\left.+\left|k_2'\circ\frac{\partial u}{\partial\nu}\right|^2\right\}d\Gamma\\ &-\left(\frac{1}{2}-\frac{\epsilon\lambda_0}{\delta_0}\right)\int_{\Gamma_1} m\cdot\nu\left[u_{xx}^2+u_{yy}^2+2\mu u_{xx}u_{yy}\right.\\ &\qquad\left.+2(1-\mu)\,u_{xy}^2\right]d\Gamma.\end{aligned} \tag{38}$$

Proof. Differentiating ψ using (1) and Lemma 4, we get

$$\begin{aligned}\psi'(t) ={}& \int_\Omega\left[m\cdot\nabla u' + \left(\frac{n}{2}-\theta\right)u'\right]Ku'\,dx\\ &+\int_\Omega\left[m\cdot\nabla u + \left(\frac{n}{2}-\theta\right)u\right]Ku''\,dx\\ ={}& \frac{1}{2}\int_{\Gamma_1} m\cdot\nu K\left|u'\right|^2 d\Gamma - \theta\int_\Omega K\left|u'\right|^2dx\\ &-\frac{1}{2}\int_\Omega\nabla K\cdot m\left|u'\right|^2dx - \left(1+\frac{n}{2}-\theta\right)a(u,u)\\ &+n\int_\Omega F(u)\,dx - \left(\frac{n}{2}-\theta\right)\int_\Omega f(u)\,u\,dx\\ &-\frac{1}{2}\int_\Gamma m\cdot\nu\left[u_{xx}^2+u_{yy}^2+2\mu u_{xx}u_{yy}+2(1-\mu)\,u_{xy}^2\right]d\Gamma\\ &-\int_\Gamma(\mathscr{B}_2 u)\left[(m\cdot\nabla u) + \left(\frac{n}{2}-\theta\right)u\right]d\Gamma\\ &+\int_\Gamma(\mathscr{B}_1 u)\left[\frac{\partial}{\partial\nu}(m\cdot\nabla u) + \left(\frac{n}{2}-\theta\right)\frac{\partial u}{\partial\nu}\right]d\Gamma.\end{aligned} \tag{39}$$

Let us next examine the integrals over Γ_0 in (39). Since $u = \partial u/\partial\nu = 0$ on Γ_0, we have $B_1u = B_2u = 0$ on Γ_0 and

$$\begin{gathered}\frac{\partial}{\partial\nu}(m\cdot\nabla u) = (m\cdot\nu)\,\Delta u,\\ u_{xx}^2 + u_{yy}^2 + 2\mu u_{xx}u_{yy} + 2(1-\mu)\,u_{xy}^2 = (\Delta u)^2 \quad \text{on } \Gamma_0,\end{gathered} \tag{40}$$

since

$$u_{xx}u_{yy} - u_{xy}^2 = 0 \quad \text{on } \Gamma_0. \tag{41}$$

Therefore, from (39) and (40), we have

$$\begin{aligned}\psi'(t) \le{}& \frac{1}{2}\int_{\Gamma_1} m\cdot\nu K\left|u'\right|^2 d\Gamma - \theta\int_\Omega K\left|u'\right|^2dx\\ &-\frac{1}{2}\int_\Omega\nabla K\cdot m\left|u'\right|^2dx - \left(1+\frac{n}{2}-\theta\right)a(u,u)\\ &+n\int_\Omega F(u)\,dx - \left(\frac{n}{2}-\theta\right)\int_\Omega f(u)\,u\,dx\\ &+\frac{1}{2}\int_{\Gamma_0} m\cdot\nu(\Delta u)^2 d\Gamma\\ &-\frac{1}{2}\int_{\Gamma_1} m\cdot\nu\left[u_{xx}^2+u_{yy}^2+2\mu u_{xx}u_{yy}+2(1-\mu)\,u_{xy}^2\right]d\Gamma\\ &-\int_{\Gamma_1}(\mathscr{B}_2 u)\left[(m\cdot\nabla u) + \left(\frac{n}{2}-\theta\right)u\right]d\Gamma\\ &+\int_{\Gamma_1}(\mathscr{B}_1 u)\left[\frac{\partial}{\partial\nu}(m\cdot\nabla u) + \left(\frac{n}{2}-\theta\right)\frac{\partial u}{\partial\nu}\right]d\Gamma.\end{aligned} \tag{42}$$

Using the Young inequality, we get

$$\left|\int_{\Gamma_1}(\mathscr{B}_2u)\left[(m\cdot\nabla u)+\left(\frac{n}{2}-\theta\right)u\right]d\Gamma\right| \leq \frac{1}{2\epsilon}\int_{\Gamma_1}|\mathscr{B}_2u|^2d\Gamma+\epsilon\int_{\Gamma_1}\left(|m\cdot\nabla u|^2+\left(\frac{n}{2}-\theta\right)^2|u|^2\right)d\Gamma, \tag{43}$$

$$\left|\int_{\Gamma_1}(\mathscr{B}_1u)\left[\frac{\partial}{\partial\nu}(m\cdot\nabla u)+\left(\frac{n}{2}-\theta\right)\frac{\partial u}{\partial\nu}\right]d\Gamma\right| \leq \frac{1}{2\epsilon}\int_{\Gamma_1}|\mathscr{B}_1u|^2d\Gamma + \epsilon\int_{\Gamma_1}\left(\left|\frac{\partial}{\partial\nu}(m\cdot\nabla u)\right|^2+\left(\frac{n}{2}-\theta\right)^2\left|\frac{\partial u}{\partial\nu}\right|^2\right)d\Gamma, \tag{44}$$

where ϵ is a positive constant. Since the bilinear form $a(u,u)$ is strictly coercive on W, using the trace theory, we obtain

$$\int_{\Gamma_1}\left(|m\cdot\nabla u|^2+\left(\frac{n}{2}-\theta\right)^2|u|^2\right)d\Gamma + \int_{\Gamma_1}\left(\left|\frac{\partial}{\partial\nu}(m\cdot\nabla u)\right|^2+\left(\frac{n}{2}-\theta\right)^2\left|\frac{\partial u}{\partial\nu}\right|^2\right)d\Gamma \leq \lambda_0 a(u,u)+\frac{\lambda_0}{\delta_0} \times\int_{\Gamma_1}m\cdot\nu\left[u_{xx}^2+u_{yy}^2+2\mu u_{xx}u_{yy}+2(1-\mu)u_{xy}^2\right]d\Gamma, \tag{45}$$

where λ_0 is a constant depending on Ω, μ, θ and n. Substituting inequalities (43)–(45) into (42) and taking into account that $m\cdot\nu\leq 0$ on Γ_0, as well as (23) and (25), we have

$$\begin{aligned}\psi'(t)\leq{}&\frac{1}{2}\int_{\Gamma_1}m\cdot\nu K|u'|^2d\Gamma-\theta\int_\Omega K|u'|^2dx\\&-\left(1+\frac{n}{2}-\theta-\epsilon\lambda_0\right)a(u,u)\\&-\left(\frac{n\eta}{2}-2\theta-\eta\theta\right)\int_\Omega F(u)\,dx\\&+\frac{1}{2\epsilon}\int_{\Gamma_1}\left(|\mathscr{B}_1u|^2+|\mathscr{B}_2u|^2\right)d\Gamma\\&-\left(\frac{1}{2}-\frac{\epsilon\lambda_0}{\delta_0}\right)\int_{\Gamma_1}m\cdot\nu\left[u_{xx}^2+u_{yy}^2+2\mu u_{xx}u_{yy}+2(1-\mu)u_{xy}^2\right]d\Gamma.\end{aligned} \tag{46}$$

Since the boundary conditions (17) can be written as

$$\mathscr{B}_2u=\tau_1\left\{u'+k_1(t)u-k_1(t)u_0-k_1'\circ u\right\},$$
$$\mathscr{B}_1u=-\tau_2\left\{\frac{\partial u'}{\partial\nu}+k_2(t)\frac{\partial u}{\partial\nu}-k_2(t)\frac{\partial u_0}{\partial\nu}-k_2'\circ\frac{\partial u}{\partial\nu}\right\}, \tag{47}$$

our conclusion follows. □

Let us introduce the Lyapunov functional

$$\mathscr{L}(t)=NE(t)+\psi(t), \tag{48}$$

with $N>0$. Now, we are in a position to show the main result of this paper.

Theorem 7. *Let $(u_0,u_1)\in W\times L^2(\Omega)$. Suppose that the resolvent kernels k_1, k_2 satisfy the condition (H). Then, there exist constants $\omega, C>0$ such that, for some t_0 large enough, the solution of (1)–(5) satisfies*

$$E(t)\leq CE(0)e^{-\omega\int_0^t\xi(s)ds},\quad \forall t\geq t_0,\ \text{if } u_0=\frac{\partial u_0}{\partial\nu}=0\quad\text{on }\Gamma_1. \tag{49}$$

Otherwise,

$$E(t)\leq C\left(E(0)+\int_0^t k_0(s)e^{\omega\int_{t_0}^s\xi(\tau)d\tau}ds\right)e^{-\omega\int_0^t\xi(s)ds}, \tag{50}$$

for all $t\geq t_0$, where

$$\xi(t)=\min\{\xi_1(t),\xi_2(t)\},\qquad k_0(t)=\int_{\Gamma_1}k_1^2(t)|u_0|^2d\Gamma+\int_{\Gamma_1}k_2^2(t)\left|\frac{\partial u_0}{\partial\nu}\right|^2d\Gamma. \tag{51}$$

Proof. Applying inequality (36) with $\alpha=1/2$ in Lemma 6 and from Lemma 5, we obtain

$$\begin{aligned}\mathscr{L}'(t)\leq{}&-\int_\Omega\theta K|u'|^2dx\\&-\frac{\tau_1N}{2}\int_{\Gamma_1}\left\{|u'|^2-k_1^2(t)|u_0|^2-k_1'(t)|u|^2+k_1''\diamond u\right\}d\Gamma\\&-\frac{\tau_2N}{2}\int_{\Gamma_1}\left\{\left|\frac{\partial u'}{\partial\nu}\right|^2-k_2^2(t)\left|\frac{\partial u_0}{\partial\nu}\right|^2-k_2'(t)\left|\frac{\partial u}{\partial\nu}\right|^2+k_2''\diamond\frac{\partial u}{\partial\nu}\right\}d\Gamma\\&-\left(1+\frac{n}{2}-\theta-\epsilon\lambda_0\right)a(u,u)-\left(\frac{n\eta}{2}-2\theta-\eta\theta\right)\\&\times\int_\Omega F(u)\,dx+\frac{2\tau_1{}^2}{\epsilon}\int_{\Gamma_1}\left\{|u'|^2+k_1^2(t)|u|^2+k_1^2(t)|u_0|^2-k_1(0)k_1'\diamond u\right\}d\Gamma\end{aligned}$$

$$+\frac{2\tau_2{}^2}{\epsilon}\int_{\Gamma_1}\left\{\left|\frac{\partial u'}{\partial \nu}\right|^2+k_2^2(t)\left|\frac{\partial u}{\partial \nu}\right|^2+k_2^2(t)\left|\frac{\partial u_0}{\partial \nu}\right|^2 -k_2(0)\,k_2'\diamond\frac{\partial u}{\partial \nu}\right\}d\Gamma$$
$$+\frac{1}{2}\int_{\Gamma_1}m\cdot\nu K\left|u'\right|^2 d\Gamma-\left(\frac{1}{2}-\frac{\epsilon\lambda_0}{\delta_0}\right)\times\int_{\Gamma_1}m\cdot\nu\left[u_{xx}^2+u_{yy}^2+2\mu u_{xx}u_{yy}+2(1-\mu)u_{xy}^2\right]d\Gamma. \quad (52)$$

We take θ and ϵ so small such that

$$\frac{n\eta}{2}-2\theta-\eta\theta>0,\qquad 1+\frac{n}{2}-\theta-\epsilon\lambda_0>0,\qquad \frac{1}{2}-\frac{\epsilon\lambda_0}{\delta_0}>0. \quad (53)$$

Since $K\in L^\infty(\Omega)$ and then choosing N large enough, we obtain

$$\mathscr{L}'(t)\le -c_0E(t)+c\int_{\Gamma_1}k_1^2(t)\left|u_0\right|^2d\Gamma+c\int_{\Gamma_1}k_2^2(t)\left|\frac{\partial u_0}{\partial \nu}\right|^2d\Gamma -c\int_{\Gamma_1}k_1'\diamond u\,d\Gamma-c\int_{\Gamma_1}k_2'\diamond\frac{\partial u}{\partial \nu}d\Gamma,\quad \forall t\ge t_0. \quad (54)$$

On the other hand, we can choose N even larger so that

$$\mathscr{L}(t)\sim E(t). \quad (55)$$

If $\xi(t)=\min\{\xi_1(t),\xi_2(t)\}$, $t\ge t_0$, then, using (30) and (33), we have

$$\begin{aligned}\xi(t)\,\mathscr{L}'(t)&\le -c_0\xi(t)E(t)+c\xi(t)\int_{\Gamma_1}k_1^2(t)\left|u_0\right|^2d\Gamma+c\xi(t)\int_{\Gamma_1}k_2^2(t)\left|\frac{\partial u_0}{\partial \nu}\right|^2d\Gamma\\&\quad-c\xi_1(t)\int_{\Gamma_1}k_1'\diamond u\,d\Gamma-c\xi_2(t)\int_{\Gamma_1}k_2'\diamond\frac{\partial u}{\partial \nu}d\Gamma\\&\le -c_0\xi(t)E(t)+c\xi(t)\int_{\Gamma_1}k_1^2(t)\left|u_0\right|^2d\Gamma+c\xi(t)\int_{\Gamma_1}k_2^2(t)\left|\frac{\partial u_0}{\partial \nu}\right|^2d\Gamma\\&\quad+c\int_{\Gamma_1}k_1''\diamond u\,d\Gamma+c\int_{\Gamma_1}k_2''\diamond\frac{\partial u}{\partial \nu}d\Gamma\\&\le -c_0\xi(t)E(t)+c\xi(t)\int_{\Gamma_1}k_1^2(t)\left|u_0\right|^2d\Gamma+c\xi(t)\int_{\Gamma_1}k_2^2(t)\left|\frac{\partial u_0}{\partial \nu}\right|^2d\Gamma-cE'(t),\quad\forall t\ge t_0,\end{aligned} \quad (56)$$

which gives

$$\xi(t)\,\mathscr{L}'(t)+cE'(t)\le -c_0\xi(t)E(t)+c\xi(t)\int_{\Gamma_1}k_1^2(t)\left|u_0\right|^2d\Gamma+c\xi(t)\int_{\Gamma_1}k_2^2(t)\left|\frac{\partial u_0}{\partial \nu}\right|^2d\Gamma,\quad\forall t\ge t_0. \quad (57)$$

Using the fact that ξ is a nonincreasing continuous function as ξ_1 and ξ_2 are nonincreasing, and so ξ is differentiable, with $\xi'(t)\le 0$, for a.e. t, then we infer that

$$\begin{aligned}(\xi\mathscr{L}+cE)'(t)&\le\xi(t)\,\mathscr{L}'(t)+cE'(t)\\&\le -c_0\xi(t)E(t)+c\xi(t)\int_{\Gamma_1}k_1^2(t)\left|u_0\right|^2d\Gamma+c\xi(t)\int_{\Gamma_1}k_2^2(t)\left|\frac{\partial u_0}{\partial \nu}\right|^2d\Gamma,\quad\forall t\ge t_0.\end{aligned} \quad (58)$$

Since using (55),

$$F=\xi\mathscr{L}+cE\sim E, \quad (59)$$

we obtain, for some positive constant ω,

$$F'(t)\le-\omega\xi(t)F(t)+c\int_{\Gamma_1}k_1^2(t)\left|u_0\right|^2d\Gamma+c\int_{\Gamma_1}k_2^2(t)\left|\frac{\partial u_0}{\partial \nu}\right|^2d\Gamma,\quad\forall t\ge t_0. \quad (60)$$

Case 1. If $u_0=\partial u_0/\partial\nu=0$ on Γ_1, then (60) reduces to

$$F'(t)\le-\omega\xi(t)F(t),\quad\forall t\ge t_0. \quad (61)$$

A simple integration over (t_0,t) yields

$$F(t)\le F(t_0)\,e^{-\omega\int_{t_0}^{t}\xi(s)ds},\quad\forall t\ge t_0. \quad (62)$$

By using (33) and (59), we then obtain for some positive constant C

$$E(t)\le CE(0)\,e^{-\omega\int_{0}^{t}\xi(s)ds},\quad\forall t\ge t_0. \quad (63)$$

Thus, estimate (49) is proved.

Case 2. If $(u_0,(\partial u_0/\partial\nu))\ne(0,0)$ on Γ_1, then (60) gives

$$F'(t)\le-\omega\xi(t)F(t)+ck_0(t),\quad\forall t\ge t_0, \quad (64)$$

where

$$k_0(t) = \int_{\Gamma_1} k_1^2(t)\left|u_0\right|^2 d\Gamma + \int_{\Gamma_1} k_2^2(t)\left|\frac{\partial u_0}{\partial \nu}\right|^2 d\Gamma. \quad (65)$$

In this case, we introduce

$$G(t) := F(t) - ce^{-\omega\int_{t_0}^{t}\xi(s)ds}\int_{t_0}^{t} k_0(s)\, e^{\omega\int_{t_0}^{s}\xi(\tau)d\tau} ds. \quad (66)$$

A simple differentiation of G, using (64), leads to

$$\begin{aligned} G'(t) &= F'(t) + \omega\xi(t)\, ce^{-\omega\int_{t_0}^{t}\xi(s)ds} \\ &\quad \times \int_{t_0}^{t} k_0(s)\, e^{\omega\int_{t_0}^{s}\xi(\tau)d\tau} ds - ck_0(t) \\ &\le -\omega\xi(t)\, G(t), \quad \forall t \ge t_0. \end{aligned} \quad (67)$$

Again, a simple integration over (t_0, t) yields

$$G(t) \le G(t_0)\, e^{-\omega\int_{t_0}^{t}\xi(s)ds}, \quad \forall t \ge t_0, \quad (68)$$

which implies, for all $t \ge t_0$,

$$F(t) \le \left(F(t_0) + c\int_{t_0}^{t} k_0(s)\, e^{\omega\int_{t_0}^{s}\xi(\tau)d\tau} ds\right) e^{-\omega\int_{t_0}^{t}\xi(s)ds}. \quad (69)$$

By using (59), we deduce that

$$E(t) \le C\left(E(0) + \int_{0}^{t} k_0(s)\, e^{\omega\int_{t_0}^{s}\xi(\tau)d\tau} ds\right) e^{-\omega\int_{t_0}^{t}\xi(s)ds}, \quad \forall t \ge t_0. \quad (70)$$

Consequently, by the boundedness of ξ, (50) is established. □

Remark 8. Note that the exponential and polynomial decay estimates are only particular cases of (49) and (50). More precisely, we have exponential decay for $\xi_1(t) \equiv c_1$ and $\xi_2(t) \equiv c_2$ and polynomial decay for $\xi_1(t) = c_1(1+t)^{-1}$ and $\xi_2(t) \equiv c_2$, where c_1 and c_2 are positive constants.

Example 9. As in [24], we give some examples to illustrate the energy decay rates given by (49).

(1) If $k_1(t) = k_2(t) = ae^{-b(1+t)^p}$, $0 < p \le 1$, then, for $i = 1, 2$, $k_i''(t) \ge -\xi(t)k_i'(t)$, where $\xi(t) = bp(1+t)^{p-1}$. For suitably chosen positive constants a and b, k_i satisfies (H) and (49) gives

$$E(t) \le ce^{-\omega b(1+t)^p}. \quad (71)$$

(2) If $k_1(t) = a_1/(1+t)^q$, $q > 0$, and $k_2(t) = a_2e^{-b(1+t)^p}$, $0 < p \le 1$, then, for $i = 1, 2$, $k_i''(t) \ge -\xi(t)k_i'(t)$, where $\xi(t) = q(1+t)^{-1}$. Then

$$E(t) \le \frac{c}{(1+t)^{\omega q}}. \quad (72)$$

The aforementioned two examples are included in the following more general one.

(3) For any nonincreasing functions $k_i(t)$, $i = 1, 2$, which satisfy (H), $\xi_i = -k'/k$ are also nonincreasing differentiable functions, and $c\xi_1(t) \le \xi_2(t)$, for some $0 < c \le 1$, and (49) gives

$$E(t) \le c\left[k_1(t)\right]^{\omega}. \quad (73)$$

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