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Research Article

Synchronization of General Complex Networks with Hybrid Couplings and Unknown Perturbations

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The issue of synchronization for a class of hybrid coupled complex networks with mixed delays (discrete delays and distributed delays) and unknown nonstochastic external perturbations is studied. The perturbations do not disappear even after all the dynamical nodes have reached synchronization. To overcome the bad effects of such perturbations, a simple but all-powerful robust adaptive controller is designed to synchronize the complex networks even without knowing a priori the functions and bounds of the perturbations. Based on Lyapunov stability theory, integral inequality Barbalat lemma, and Schur Complement lemma, rigorous proofs are given for synchronization of the complex networks. Numerical simulations verify the effectiveness of the new robust adaptive controller.

1. Introduction

Over the past decade, complex networks have attracted much attention from authors of many disciplines since the pioneer works of Watts and Strogatz [1, 2]. In fact, many phenomena in nature and our daily life can be explained by using complex networks, such as the Internet, World Wide Web, social networks, and neural networks. A complex network can be considered as a graph which consists of a set of nodes and edges connecting these nodes [3].

In recent years, chaos synchronization [3–7] has been intensively studied due to its important applications in many different areas, such as secure communication, biological systems, and information science [8–11]. Particularly, the synchronization of all the dynamical nodes in complex networks has become a hot research topic [3], and several results have been appeared in the literature. The authors of [12] studied the synchronization in complex networks with switching topology. In [13], Wu and Jiao investigated the synchronization in complex dynamical networks with nonsymmetric coupling. They showed that the synchronizability of a dynamical network with nonsymmetric coupling is not always characterized by its second-largest eigenvalue, even though all the eigenvalues of the nonsymmetric coupling matrix are real. Liu and Chen [14] gave some criteria for the global synchronization of complex networks in virtual of the left eigenvector corresponding to the zero eigenvalue of the coupling matrix. For a given network with identical node dynamics, the authors of [15] showed that two key factors influencing the network synchronizability are the network inner linking matrix and the eigenvalues of the network topological matrix. Some synchronization criteria were given in [16–19] for coupled neural networks with or without delayed couplings. In [20], the robust impulsive synchronization of coupled delayed neural networks with uncertainties is considered; several new criteria are obtained to guarantee the robust synchronization via impulses.

Complex networks have the properties of robustness and fragility. A complex network can synchronize itself when parameter mismatch is within some limit. If parameter mismatch exceeds this limit, networks cannot realize synchronization themselves. Thus the controlled synchronization of coupled networks is believed to be a rather significant topic in both theoretical research and practical applications [21–29]. Some effective control scheme has been proposed, for instance, state feedback control with constant control gains, impulsive control, intermittent control, and adaptive control. Adaptive control method receives particular attention of researchers in recently rears. In [3], the authors studied synchronization in complex networks by using distributed adaptive control scheme. By designing a simple adaptive controller, authors of [23] investigated the locally and globally adaptive synchronization of an uncertain complex dynamical network. Authors in [24] investigated synchronization of neural networks with time-varying delays and distributed delays via adaptive control method. By using the adaptive feedback control scheme, Chen and Zhou [25] studied synchronization of complex nondelayed networks and Cao et al. [26] investigated the complete synchronization in an array of linearly stochastically coupled identical networks with delays. By using adaptive pinning control method, Zhou et al. [27] studied local and global synchronization of complex networks without delays, authors of [28, 29] considered the global synchronization of the complex networks with nondelayed and delayed couplings and the authors of [30] investigated lag synchronization of complex networks via state feedback pinning strategy. Outer synchronization of complex delayed networks with uncertain parameters was considered by using adaptive coupling in [31]. However, models in the previous references are special; that is, each of them does not consider general complex networks in which every dynamical node has mixed delays (discrete delay and distributed delay), and the complex networks have nondelayed, discrete-delayed, and distributed-delayed couplings.

Complex networks are always affected by some unknown external perturbations due to environmental causes and human causes. White noises brought by some random fluctuations in the course of transmission and other probabilities causes have received extensive attention in the literatures [21, 24, 32–35]. However, not all the external perturbations are white noise, and some of them may be nonlinear and nonstochastic perturbations. When complex networks are disturbed by nonlinear and nonstochastic perturbations, the states of the nodes will be changed dramatically, which will affect the stability and synchronization of the complex networks. Due to the fragility of complex networks, if some important nodes are perturbed by such external perturbations, whole states of the network will be affected or even the network cannot operate normally. Hence, how to realize synchronization of all nodes for complex networks with uncertain nonlinear nonstochastic external perturbations is an urgent practical problem to be solved. Obviously, the controllers for stability and synchronization of stochastic perturbations are not applicable to the case of nonlinear nonstochastic perturbations, especially when the functions and bounds of the perturbations are unknown. Therefore, to enhance antiperturbations capability and to realize synchronization of complex networks, more effective controller should be designed.

Motivated by the previous analysis, in this paper, a class of more general complex networks is proposed. The new model has nondelayed, discrete-delayed, and distributed-delayed couplings, and every dynamical node has mixed delays. Unknown nonstochastic external perturbations to the complex networks are also considered. Then we study the global complete synchronization of the proposed model. A new simple but robust adaptive controller is designed to overcome the effects of such perturbations and synchronize the complex networks even without knowing the exact functions and bounds of the perturbations. Moreover, the adaptive controller can also synchronize coupled systems with stochastic perturbations since it includes existing adaptive controller as special case. Two cases are considered: all nodes or partial nodes are perturbed. All nodes should be controlled for the former case. Pinning control scheme can also be used for the latter case. Based on Lyapunov stability theory, integral inequality, Barbalat lemma, and Schur Complement lemma, rigorous proofs are given for synchronization of the complex networks with unknown perturbations of the previous two cases. It should be noted that our new adaptive controllers can also prevent external perturbations. Therefore, the new adaptive controllers are better than those in [23–29]. Numerical simulations verify the effectiveness of our theoretical results.

Notations. In the sequel, if not explicitly stated, matrices are assumed to have compatible dimensions. I_N denotes the identity matrix of N dimension. The Euclidean norm in $\mathbb{R}^n$ is denoted as $\|\cdot\|$; accordingly, for vector $x \in \mathbb{R}^n$, $\|x\| = \sqrt{x^T x}$, where T denotes transposition. $A = (a_{ij})_{m\times m}$ denotes a matrix of m dimension, $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$, and $A^s = (1/2)(A + A^T)$. $A > 0$ or $A < 0$ denotes that the matrix A is symmetric and positive or negative definite matrix. $\lambda_{\min}(A^s)$ is the minimum eigenvalues of the symmetric matrices A^s, and A_l denotes the matrix of the first l row-column pairs of A. A_l^c denotes the minor matrix of matrix A by removing all the first l row-column elements of A.

The rest of this paper is organized as follows. In Section 2, a class of general complex networks with mixed delays and external perturbations is proposed. Some necessary assumptions and lemmas are also given in this section. In Section 3, synchronization of the complex networks with all nodes perturbed is studied. Synchronization with only partial nodes perturbed is considered in Section 4. Then, in Section 5, numerical simulations are given to show the effectiveness of our results. Finally, in Section 6, conclusions are given.

2. Preliminaries

The general complex networks consisting of N identical nodes with external perturbations and mixed-delay couplings are described as

$$\begin{aligned}\dot{x}_i(t) &= Cx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau(t))) \\ &\quad + D\int_{t-\theta(t)}^{t} f(x_i(s))\,ds + I(t) + \alpha\sum_{j=1}^{N} u_{ij}\Phi x_j(t) \\ &\quad + \beta\sum_{j=1}^{N} v_{ij}\Upsilon x_j(t-\tau(t)) + \gamma\sum_{j=1}^{N} w_{ij}\Lambda\int_{t-\theta(t)}^{t} x_j(s)\,ds\end{aligned}$$

$$+\sigma_i\left(t, x_i(t), x_i(t-\tau(t)), \int_{t-\theta(t)}^{t} x_i(s)\,ds\right)$$
$$+R_i, \quad i=0,1,\ldots,N, \tag{1}$$

where $x_i(t) = [x_{i1}(t), \ldots, x_{in}(t)]^T \in \mathbb{R}^n$ represents the state vector of the ith node of the network at time t, and C, A, B, D are matrices with proper dimension. $f(\cdot)$ is a continuous vector function. $I(t)$ is the external input vector. $R_i \in \mathbb{R}^n$ is the control input. $\tau(t) > 0, \theta(t) > 0$ are time-varying discrete delay and distributed delay, respectively. Constants $\alpha > 0$, $\beta > 0$, $\gamma > 0$ are coupling strengths of the whole network corresponding to nondelay, discrete delay, and distributed delay, respectively. $\Phi, \Upsilon, \Lambda \in \mathbb{R}^{n\times n}$ are inner coupling matrices of the networks, which describe the individual coupling between two subsystems. Matrices $U = (u_{ij})_{N\times N}$, $V = (v_{ij})_{N\times N}$, $W = (w_{ij})_{N\times N}$ are outer couplings of the whole networks satisfying the following diffusive conditions:

$$u_{ij} \geq 0 \quad (i \neq j), \qquad u_{ii} = -\sum_{j=1, j\neq i}^{N} u_{ij},$$
$$v_{ij} \geq 0 \quad (i \neq j), \qquad v_{ii} = -\sum_{j=1, j\neq i}^{N} u_{ij}, \tag{2}$$
$$w_{ij} \geq 0 \quad (i \neq j), \qquad w_{ii} = -\sum_{j=1, j\neq i}^{N} w_{ij},$$

where $i, j = 1, 2, \ldots, N$. Vector $\sigma_i(t, x_i(t), x_i(t-\tau(t)), \int_{t-\theta(t)}^{t} x_i(s)ds) \in \mathbb{R}^n$ describes the unknown perturbation to ith node of the complex networks. In this paper, we always assume that $\dot{\tau} \leq h_\tau < 1$ and $\dot{\theta} \leq h_\theta < 1$. $\theta(t)$ is bounded and we denote $\theta_{\min} > 0$ the minimum of $\theta(t)$ and $\theta_{\max}$ the maximum of $\theta(t)$.

We assume that (1) has a unique continuous solution for any initial condition in the following form:

$$x_i(s) = \varphi_i(s), \quad -\varrho \leq s \leq 0, \quad i = 0, 1, 2, \ldots, N, \tag{3}$$

where $\varrho = \max\{\tau_{\max}, \sigma_{\max}\}$ and $\tau_{\max}$ is the maximum of $\tau(t)$.

For convenience of writing, in the sequel, we denote $\sigma_i(t, x_i(t), x_i(t-\tau(t)), \int_{t-\theta(t)}^{t} x_j(s)ds)$ with $\sigma_i(t)$.

The system of an isolate node without external perturbation is described as

$$\dot{z}(t) = Cz(t) + Af(z(t)) + Bf(z(t-\tau(t)))$$
$$+D\int_{t-\theta(t)}^{t} f(z(s))\,ds + I(t), \tag{4}$$

and $z(t)$ can be any desired state: equilibrium point, a nontrivial periodic orbit, or even a chaotic orbit.

Remark 1. The nonstochastic perturbations $\sigma_i(t, x_i(t), x_i(t-\tau(t)), \int_{t-\theta(t)}^{t} x_i(s)ds)$ are different from stochastic ones in the literature [21, 24, 32–35]. The distinct feature of the such stochastic perturbations is that the stochastic perturbations disappear when the synchronization goal is realized. However, perturbations of this paper still exist even when complete synchronization has been achieved. Therefore, the controllers in most of existing papers including those in [21, 24, 32–35] are invalid for perturbations of this paper.

When system (4) is perturbed, then (4) turns to the following system:

$$\dot{\overline{z}}(t) = C\overline{z}(t) + Af(\overline{z}(t)) + Bf(\overline{z}(t-\tau(t)))$$
$$+D\int_{t-\theta(t)}^{t} f(\overline{z}(s))\,ds + I(t) \tag{5}$$
$$+\sigma_i\left(t, \overline{z}(t), \overline{z}(t-\tau(t)), \int_{t-\theta(t)}^{t} \overline{z}(s)\,ds\right).$$

Generally, the state of a system will be changed when the system is perturbed. We assume that the state of system (5) remains to be any one of the previous three states but not necessarily the original one.

The following assumptions are needed in this paper:

(H_1) $f(0) \equiv 0$, and there exists positive constant h such that

$$\|f(u) - f(v)\| \leq h\|u - v\|, \quad \text{for any } u, v \in \mathbb{R}^n, \tag{6}$$

(H_2) $\sigma_{ik}(t, 0, 0, 0) \equiv 0$, and there exist positive constants M_{ik} such that $|\sigma_{ik}(t, u, v, w)| \leq M_{ik}$ for any bounded $u, v, w \in \mathbb{R}^n$, $i = 1, 2, \ldots, N$, $k = 1, 2, \ldots, n$.

Remark 2. Note that (4) unifies many well-known chaotic systems with or without delays, such as Chua system, Lorenz system, Rössler system, Chen system, and chaotic neural networks with mixed delays [12–29]. Hence, results of this paper are general.

Remark 3. Condition (H_2) is very mild. We do not impose the usual conditions such as Lipschitz condition, differentiability on the external perturbation functions. It can be discontinuous or even impulsive functions. If the state of (5) is a equilibrium point or a nontrivial periodic orbit, the condition (H_2) can be easily satisfied. If the state of (5) is a chaotic orbit, the condition (H_2) can also be satisfied. Since chaotic system has strange attractors, there exists a bounded region containing all attractors of it such that every orbit of the system never leaves them. Anyway, condition (H_2) can be satisfied for equilibrium point, a nontrivial periodic orbit, and a chaotic orbit. Moreover, we will subsequently prove that the complex networks (1) can be synchronized even without knowing the exact values of h and M_{ik}, $i = 1, 2, \ldots, N$, $k = 1, 2, \ldots, n$.

The aim of this paper is to synchronize all the states of complex networks (1) to the following manifold:

$$x_1(t) = x_2(t) = \cdots = x_N(t) = z(t), \tag{7}$$

where $z(t)$ is immune to external perturbations.

Lemma 4 ((Schur Complement) see [36]). *The linear matrix inequality (LMI)*

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} < 0 \quad (8)$$

is equivalent to any one of the following two conditions:

(L_1) $S_{11} < 0$, $S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$,

(L_2) $S_{22} < 0$, $S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$,

where $S_{11} = S_{11}^T$, $S_{22} = S_{22}^T$.

Lemma 5 (see [37]). *For any constant matrix* $D \in \mathbb{R}^{n\times n}$, $D^T = D > 0$, *scalar* $\sigma > 0$, *and vector function* $\omega : [0, \sigma] \to \mathbb{R}^n$, *one has*

$$\sigma \int_0^{\sigma} \omega^T(s) D\omega(s)\, ds \geq \left(\int_0^{\sigma} \omega(s)\, ds\right)^T D \int_0^{\sigma} \omega(s)\, ds \quad (9)$$

provided that the integrals are all well defined.

Lemma 6 ((Barbalat lemma) see [38]). *If* $f(t) : \mathbb{R} \to \mathbb{R}^+$ *is a uniformly continuous function for* $t \geq 0$ *and if the limit of the integral*

$$\lim_{t\to\infty} \int_0^t f(s)\, ds \quad (10)$$

exists and is finite, then $\lim_{t\to\infty} f(t) = 0$.

3. Synchronization with All the Nodes Perturbed

In this section, we consider the case when all the nodes are perturbed. To realize synchronization goal (7), we have to introduce an isolate node (4).

Let $e_i(t) = x_i(t) - z(t)$. Subtracting (4) from (1), we get the following error dynamical system:

$$\begin{aligned} \dot{e}_i(t) = {} & Ce_i(t) + Ag(e_i(t)) + Bg(e_i(t-\tau(t))) \\ & + D\int_{t-\theta(t)}^{t} g(e_i(s))\, ds + \alpha\sum_{j=1}^{N} u_{ij}\Phi e_j(t) \\ & + \beta\sum_{j=1}^{N} v_{ij}\Upsilon e_j(t-\tau(t)) + \sigma_i(t) + R_i \\ & + \gamma\sum_{j=1}^{N} w_{ij}\Lambda\int_{t-\theta(t)}^{t} e_j(s)\, ds, \end{aligned} \quad (11)$$

where $g(e_i) = f(x_i(t)) - f(z(t))$, $i = 1, 2, \ldots, N$.

From (H_1) and (H_2) we know that (11) admits a trivial solution $e_i(0) \equiv 0$, $i = 1, 2, \ldots, N$. Obviously, to reach the goal (7), we have only to prove that system (11) is asymptotically stable at the origin.

Theorem 7. *Under the assumption conditions* (H_1) *and* (H_2), *the networks* (1) *are synchronized with the following adaptive controllers:*

$$\begin{aligned} R_i &= -\alpha\varepsilon_i e_i(t) - \omega\beta_i \operatorname{sign}(e_i(t)), \\ \dot{\varepsilon}_i &= p_i e_i(t)^T e_i(t), \\ \dot{\beta}_i &= \xi_i \sum_{k=1}^{n} |e_{ik}(t)|, \end{aligned} \quad (12)$$

where $\omega > 1$, $p_i > 0$, *and* $\xi_i > 0$ *are arbitrary constants, respectively,* $i = 1, 2, \ldots, N$.

Proof. Define the Lyapunov function as

$$V(t) = V_1(t) + V_2(t) + V_3(t), \quad (13)$$

where

$$\begin{aligned} V_1(t) &= \frac{1}{2}\sum_{i=1}^{N} e_i^T(t) e_i(t) + \frac{1}{2}\sum_{i=1}^{N}\frac{\alpha(\varepsilon_i - k_i)^2}{p_i} + \frac{1}{2}\sum_{i=1}^{N}\frac{(M_i-\beta_i)^2}{\xi_i}, \\ V_2(t) &= \int_{t-\tau(t)}^{t} \eta^T(s) Q\eta(s)\, ds, \\ V_3(t) &= \int_{t-\theta(t)}^{t}\int_{\mu}^{t} \eta^T(s) G\eta(s)\, ds\, d\mu, \end{aligned} \quad (14)$$

$\eta(t) = (\|e_1(t)\|, \|e_2(t)\|, \ldots, \|e_N(t)\|)^T$, $M_i = \max_{1\leq k\leq n}\{M_{ik}\}$, k_i, $i = 1, 2, \ldots, N$, are constants, Q and G are symmetric positive definite matrices, and k_i, Q, and G are to be determined.

Differentiating $V_1(t)$ along the solution of (11) and from (H_1) and (H_2), we obtain

$$\begin{aligned} \dot{V}_1(t) = {} & \sum_{i=1}^{N} e_i^T(t)\dot{e}_i(t) + \alpha\sum_{i=1}^{N}(\varepsilon_i - k_i) e_i^T(t) e_i(t) \\ & - \sum_{i=1}^{N}(M_i - \beta_i)\sum_{k=1}^{n}|e_{ik}(t)| \\ \leq {} & \sum_{i=1}^{N}\Bigg[(\|C\| + \|A\| h - \alpha k_i)\|e_i(t)\|^2 \\ & + \|B\| h \|e_i(t)\| \|e_i(t-\tau(t))\| \\ & + \|D\| h \|e_i(t)\| \int_{t-\theta(t)}^{t} \|e_i(s)\|\, ds \\ & + \alpha\sum_{j=1, j\neq i}^{N} u_{ij}\|\Phi\| \|e_i(t)\| \|e_j(t)\| \\ & + \alpha\lambda_{\min}(\Phi^s) u_{ii}\|e_i(t)\|^2 \end{aligned}$$

$$
\begin{aligned}
&+\beta\sum_{j=1}^{N}\left|v_{ij}\right|\|\Upsilon\|\left\|e_i(t)\right\|\left\|e_j(t-\tau(t))\right\| \\
&\left.+\gamma\sum_{j=1}^{N}\left|w_{ij}\right|\|\Lambda\|\left\|e_i(t)\right\|\int_{t-\theta(t)}^{t}\left\|e_j(s)\right\|ds\right] \\
&=\eta^T(t)\left((\|C\|+\|A\|h)I_N+\alpha\left(\|\Phi\|\widehat{U}^s-K\right)\right)\eta(t) \\
&\quad+\eta^T(t)\left(\|B\|hI_N+\beta\|\Upsilon\||V|\right)\eta(t-\tau(t)) \\
&\quad+\eta^T(t)\left(\|D\|hI_N+\gamma\|\Lambda\||W|\right)\int_{t-\theta(t)}^{t}\eta(s)\,ds \\
&\leq\eta^T(t)\Big((\|C\|+\|A\|h+1)I_N \\
&\qquad+\alpha\left(\|\Phi\|\widehat{U}^s-K\right)\Big)\eta(t) \\
&\quad+\eta^T(t-\tau(t))\overline{B}^T\overline{B}\eta(t-\tau(t)) \\
&\quad+\left(\int_{t-\theta(t)}^{t}\eta(s)\,ds\right)^T\overline{D}^T\overline{D}\int_{t-\theta(t)}^{t}\eta(s)\,ds,
\end{aligned}
\tag{15}
$$

where $K=\operatorname{diag}(k_1,k_2,\ldots,k_N)$, $\widehat{U}=(\widehat{u}_{ij})_{N\times N}$, $\widehat{u}_{ij}=u_{ij}$, $i\neq j$, $\widehat{u}_{ii}=(\lambda_{\min}(\Phi^s)/\|\Phi\|)u_{ii}$, $\overline{B}=\|B\|hI_N+\beta\|\Upsilon\||V|$, $\overline{D}=\|D\|hI_N+\gamma\|\Lambda\||W|$, $|V|=(|v_{ij}|)_{N\times N}$, $|W|=(|w_{ij}|)_{N\times N}$, and we have used the following deduction:

$$
\begin{aligned}
&\sum_{i=1}^{N}\left[e_i^T(t)\sigma_i(t)-\omega\sum_{k=1}^{n}\beta_i\left|e_{ik}(t)\right|-\sum_{k=1}^{n}(M_i-\beta_i)\left|e_{ik}(t)\right|\right] \\
&\qquad\leq\sum_{i=1}^{N}\sum_{k=1}^{n}\left[\left|e_{ik}(t)\right|M_{ik}-M_i\left|e_{ik}\right|-(\omega-1)\beta_i\left|e_{ik}\right|\right] \\
&\qquad\leq-\sum_{i=1}^{N}\sum_{k=1}^{n}(\omega-1)\beta_i\left|e_{ik}\right|\leq 0.
\end{aligned}
\tag{16}
$$

Differentiating $V_2(t)$, we get

$$
\begin{aligned}
\dot{V}_2(t)&=\eta^T(t)Q\eta(t)-(1-\dot{\tau}(t))\eta^T(t-\tau(t))Q\eta(t-\tau(t)) \\
&\leq\eta^T(t)Q\eta(t)-\left(1-h_\tau\right)\eta^T(t-\tau(t))Q\eta(t-\tau(t)).
\end{aligned}
\tag{17}
$$

Differentiating $V_3(t)$ from Lemma 5 we have

$$
\begin{aligned}
\dot{V}_3(t)&=\theta(t)\eta^T(t)G\eta(t)-\left(1-\dot{\theta}(t)\right)\int_{t-\theta(t)}^{t}\eta^T(s)G\eta(s)\,ds \\
&\leq\theta_{\max}\eta^T(t)G\eta(t) \\
&\quad-\frac{1-h_\theta}{\theta_{\min}}\left(\int_{t-\theta(t)}^{t}\eta(s)\,ds\right)^T G\int_{t-\theta(t)}^{t}\eta(s)\,ds.
\end{aligned}
\tag{18}
$$

Take $Q=(1/(1-h_\tau))\overline{B}^T\overline{B}$, $G=(\theta_{\min}/(1-h_\theta))\overline{D}^T\overline{D}$. From the definition of $V(t)$ we reach the following inequality:

$$
\begin{aligned}
\dot{V}(t)\leq\alpha\eta^T(t)\Bigg[&\frac{1}{\alpha}(\|C\|+\|A\|h+1)I_N \\
&+\|\Phi\|\widehat{U}^s+\frac{1}{\alpha(1-h_\tau)}\overline{B}^T\overline{B} \\
&+\frac{\theta_{\min}\theta_{\max}}{\alpha(1-h_\theta)}\overline{D}^T\overline{D}-K\Bigg]\eta(t).
\end{aligned}
\tag{19}
$$

Let $k_i=\lambda_{\max}((1/\alpha)(\|C\|+\|A\|h+1)I_N+\|\Phi\|\widehat{U}^s+(1/\alpha(1-h_\tau))\overline{B}^T\overline{B}+(\theta_{\min}\theta_{\max}/(\alpha(1-h_\theta)))\overline{D}^T\overline{D})+1$, where $\lambda_{\max}((1/\alpha)(\|C\|+\|A\|h+1)I_N+\|\Phi\|\widehat{U}^s+(1/\alpha(1-h_\tau))\overline{B}^T\overline{B}+(\theta_{\min}\theta_{\max}/\alpha(1-h_\theta))\overline{D}^T\overline{D})$ denotes the maximum eigenvalue of $(1/\alpha)(\|C\|+\|A\|h+1)I_N+\|\Phi\|\widehat{U}^s+(1/\alpha(1-h_\tau))\overline{B}^T\overline{B}+(\theta_{\min}\theta_{\max}/\alpha(1-h_\theta))\overline{D}^T\overline{D}$. Then, from the previous inequality, we get

$$
\dot{V}(t)\leq-\alpha\eta^T(t)\eta(t).
\tag{20}
$$

Integrating both sides of the previous equation from 0 to t yields

$$
V(0)\geq V(t)+\alpha\sum_{i=1}^{N}\int_0^t\left\|e_i(s)\right\|^2ds\geq\alpha\sum_{i=1}^{N}\int_0^t\left\|e_i(s)\right\|^2ds.
\tag{21}
$$

Therefore,

$$
\alpha\lim_{t\to\infty}\sum_{i=1}^{N}\int_0^t\left\|e_i(s)\right\|^2ds\leq V(0).
\tag{22}
$$

In view of Lemma 6 and the previous inequality, one can easily get

$$
\lim_{t\to\infty}\sum_{i=1}^{N}\left\|e_i(t)\right\|^2=0,
\tag{23}
$$

which in turn means

$$
\lim_{t\to\infty}\left\|e_i(t)\right\|=0,\quad i=1,2,\ldots,N.
\tag{24}
$$

This completes the proof. □

4. Synchronization with Partial Nodes Perturbed

Usually, only partial nodes of complex networks are perturbed. If some important nodes are perturbed, then the entire network will not work correctly. Theoretically speaking, nodes with larger degree (undirected networks) or larger outdegree (directed networks) are more vulnerable to perturbation [39], since the states of these nodes have more effect on networks than those with smaller degree (undirected networks) or outdegree (directed networks). On the other hand, the real-world complex networks normally have a large number of nodes; it is usually impractical and impossible to control a complex networks by adding the controllers to all nodes. Therefore, from both practical point of view and the view of reducing control cost, we can use the scheme of pinning control [27–29, 40–42] to prevent external perturbations and synchronize complex networks.

In this section, we assume that matrix U is irreducible in the sense that there is no isolate cluster in the network and there are l_1 nodes affected by external perturbations.

Without loss of generality, rearrange the order of the nodes in the network, and take the first l ($l \geq l_1$) nodes to be controlled. Thus, the pinning controlled network can be described as

$$\begin{aligned}
\dot{x}_i(t) &= Cx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau(t))) \\
&\quad + D\int_{t-\theta(t)}^{t} f(x_i(s))\,ds + \alpha\sum_{j=1}^{N} u_{ij}\Phi x_j(t) \\
&\quad + I(t) + \beta\sum_{j=1}^{N} v_{ij}\Upsilon x_j(t-\tau(t)) \\
&\quad + \gamma\sum_{j=1}^{N} w_{ij}\Lambda\int_{t-\theta(t)}^{t} x_j(s)\,ds + \sigma_i(t) \\
&\quad + R_i, \quad i = 1,2,\ldots,l_1, \\
\dot{x}_i(t) &= Cx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau(t))) \\
&\quad + D\int_{t-\theta(t)}^{t} f(x_i(s))\,ds + \alpha\sum_{j=1}^{N} u_{ij}\Phi x_j(t) \\
&\quad + I(t) + \beta\sum_{j=1}^{N} v_{ij}\Upsilon x_j(t-\tau(t)) \\
&\quad + \gamma\sum_{j=1}^{N} w_{ij}\Lambda\int_{t-\theta(t)}^{t} x_j(s)\,ds + R_i, \\
&\qquad i = l_1+1, l_1+2, \ldots, l, \\
\dot{x}_i(t) &= Cx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau(t))) \\
&\quad + D\int_{t-\theta(t)}^{t} f(x_i(s))\,ds + \alpha\sum_{j=1}^{N} u_{ij}\Phi x_j(t) \\
&\quad + I(t) + \beta\sum_{j=1}^{N} v_{ij}\Upsilon x_j(t-\tau(t)) \\
&\quad + \gamma\sum_{j=1}^{N} w_{ij}\Lambda\int_{t-\theta(t)}^{t} x_j(s)\,ds, \\
&\qquad i = l+1, l+2, \ldots, N,
\end{aligned} \tag{25}$$

where R_i, $i = 1,2,\ldots,l$, are control inputs.

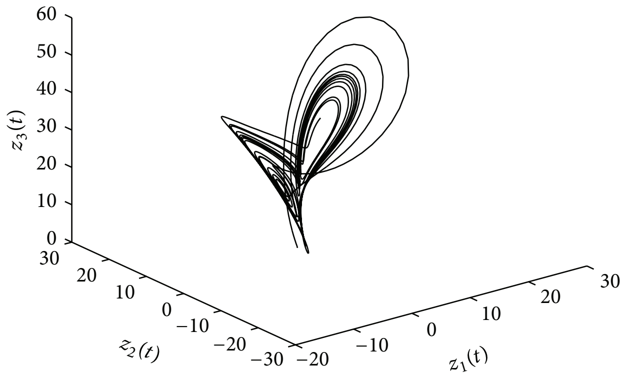

Figure 1: Chaotic trajectory of (44).

Let $e_i(t) = x_i(t) - z(t)$. Subtracting (4) from (25) we obtain the following error dynamical system:

$$\begin{aligned}
\dot{e}_i(t) &= Ce_i(t) + Ag(e_i(t)) + Bg(e_i(t-\tau(t))) \\
&\quad + D\int_{t-\theta(t)}^{t} g(e_i(s))\,ds + \alpha\sum_{j=1}^{N} u_{ij}\Phi e_j(t) \\
&\quad + \beta\sum_{j=1}^{N} v_{ij}\Upsilon e_j(t-\tau(t)) + \gamma\sum_{j=1}^{N} w_{ij}\Lambda\int_{t-\theta(t)}^{t} e_j(s)\,ds \\
&\quad + \sigma_i(t) + R_i, \quad i = 1,2,\ldots,l_1, \\
\dot{e}_i(t) &= Ce_i(t) + Ag(e_i(t)) + Bg(e_i(t-\tau(t))) \\
&\quad + D\int_{t-\theta(t)}^{t} g(e_i(s))\,ds + \alpha\sum_{j=1}^{N} u_{ij}\Phi e_j(t) \\
&\quad + \beta\sum_{j=1}^{N} v_{ij}\Upsilon e_j(t-\tau(t)) + \gamma\sum_{j=1}^{N} w_{ij}\Lambda\int_{t-\theta(t)}^{t} e_j(s)\,ds \\
&\quad + R_i, \quad i = l_1+1, l_1+2, \ldots, l, \\
\dot{e}_i(t) &= Ce_i(t) + Ag(e_i(t)) + Bg(e_i(t-\tau(t))) \\
&\quad + D\int_{t-\theta(t)}^{t} g(e_i(s))\,ds + \alpha\sum_{j=1}^{N} u_{ij}\Phi e_j(t) \\
&\quad + \beta\sum_{j=1}^{N} v_{ij}\Upsilon e_j(t-\tau(t)) + \gamma\sum_{j=1}^{N} w_{ij}\Lambda\int_{t-\theta(t)}^{t} e_j(s)\,ds, \\
&\qquad i = l+1, l+2, \ldots, N.
\end{aligned} \tag{26}$$

Similar to Theorem 7, to reach the goal (7), we have only to prove that system (26) is asymptotically stable at the origin.

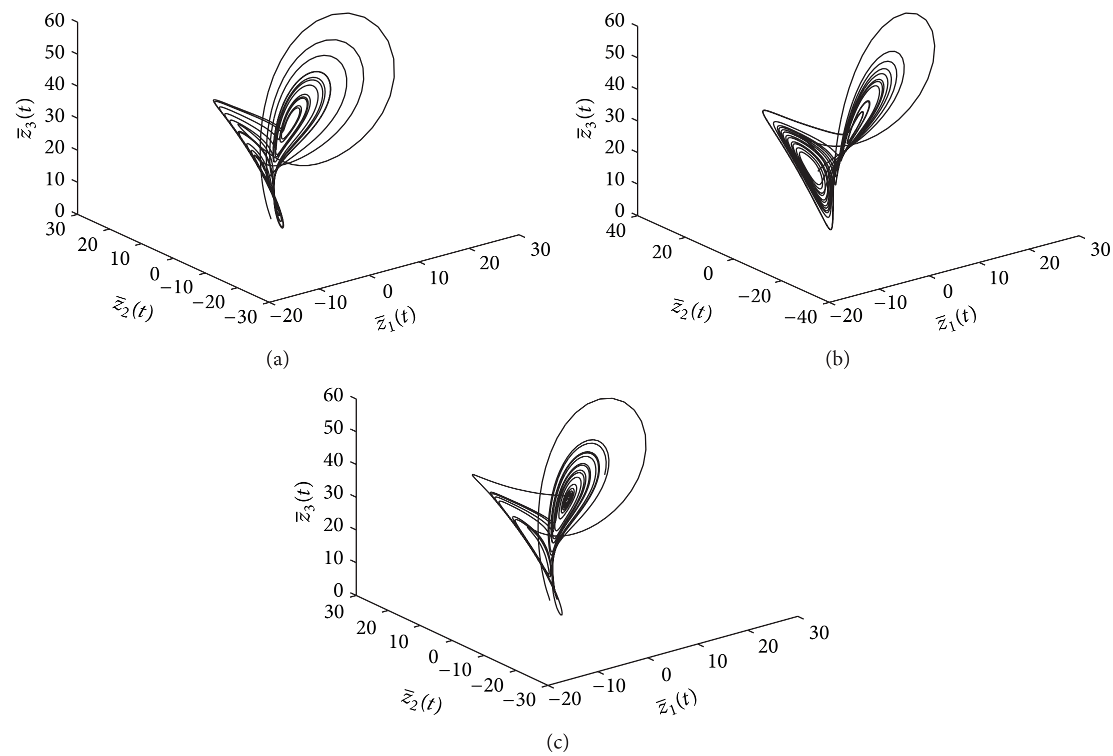

Figure 2: Chaotic trajectories of (46) with $\sigma_1(t)$ (a), $\sigma_2(t)$ (b), $\sigma_3(t)$ (c).

Theorem 8. *Suppose that matrix U is irreducible and the assumptions (H_1) and (H_2) hold. If*

$$2\alpha \|\Phi\| \left(\widehat{U}^s\right)_l^c + \Sigma I_{N-l} < 0, \tag{27}$$

then the complex networks (25) *are synchronized with the adaptive pinning controllers*

$$\begin{aligned} R_i &= -\alpha\varepsilon_i e_i(t) - \omega\beta_i \operatorname{sign}(e_i(t)), \\ \dot{\varepsilon}_i &= p_i e_i(t)^T e_i(t), \\ \dot{\beta}_i &= \xi_i \sum_{k=1}^{n} |e_{ik}(t)|, \end{aligned} \tag{28}$$

where $i = 1, 2, \ldots, l$, $\Sigma = 2(\|C\| + \|A\|h) + ((\theta_{\max}\theta_{\min} + 1 - h_\theta)/(1 - h_\theta))\|\|D\|hI_N + \gamma\|\Lambda\|\|W\| + ((2 - h_\tau)/(1 - h_\tau))\|\|B\|hI_N + \beta\|\Upsilon\|\|V\|$, and the other parameters are the same as those of Theorem 7.

Proof. We define another Lyapunov function as

$$\overline{V}(t) = \overline{V}_1(t) + V_2(t) + V_3(t), \tag{29}$$

where

$$\begin{aligned} \overline{V}_1(t) &= \frac{1}{2}\sum_{i=1}^{N} e_i^T(t)\, e_i(t) \\ &\quad + \frac{1}{2}\sum_{i=1}^{l} \frac{\alpha(\varepsilon_i - k_i)^2}{p_i} + \frac{1}{2}\sum_{i=1}^{l_1} \frac{(M_i - \beta_i)^2}{\xi_i}, \end{aligned} \tag{30}$$

k_i, $i = 1, 2, \ldots, l$, are constants to be determined, and $V_2(t)$ and $V_3(t)$ are defined as those in the proof of Theorem 7.

In view of (H_1) and (H_2), differentiating $\overline{V}_1(t)$ along the solution of (26) yields

$$\begin{aligned} \dot{\overline{V}}_1(t) &= \sum_{i=1}^{N} e_i^T(t)\,\dot{e}_i(t) + \alpha\sum_{i=1}^{l}(\varepsilon_i - k_i)\, e_i^T(t)\, e_i(t) \\ &\quad - \sum_{i=1}^{l_1}(M_i - \beta_i)\sum_{k=1}^{n}|e_{ik}(t)| \\ &\le \sum_{i=1}^{N}\Big[(\|C\| + \|A\|h)\|e_i(t)\|^2 \\ &\quad + \|B\|h\|e_i(t)\|\|e_i(t-\tau(t))\| \\ &\quad + \|D\|h\|e_i(t)\|\int_{t-\theta(t)}^{t}\|e_i(s)\|\,ds\Big] \\ &\quad - \alpha\sum_{i=1}^{l}k_i\|e_i(t)\|^2 + \alpha\sum_{i=1}^{N}\lambda_{\min}^{\Phi^s}u_{ii}\|e_i(t)\|^2 \\ &\quad + \alpha\sum_{i=1}^{N}\sum_{j=1, j\ne i}^{N} u_{ij}\|\Phi\|\|e_i(t)\|\|e_j(t)\| \\ &\quad + \beta\sum_{i=1}^{N}\sum_{j=1}^{N}|v_{ij}|\|\Upsilon\|\|e_i(t)\|\|e_j(t-\tau(t))\| \end{aligned}$$

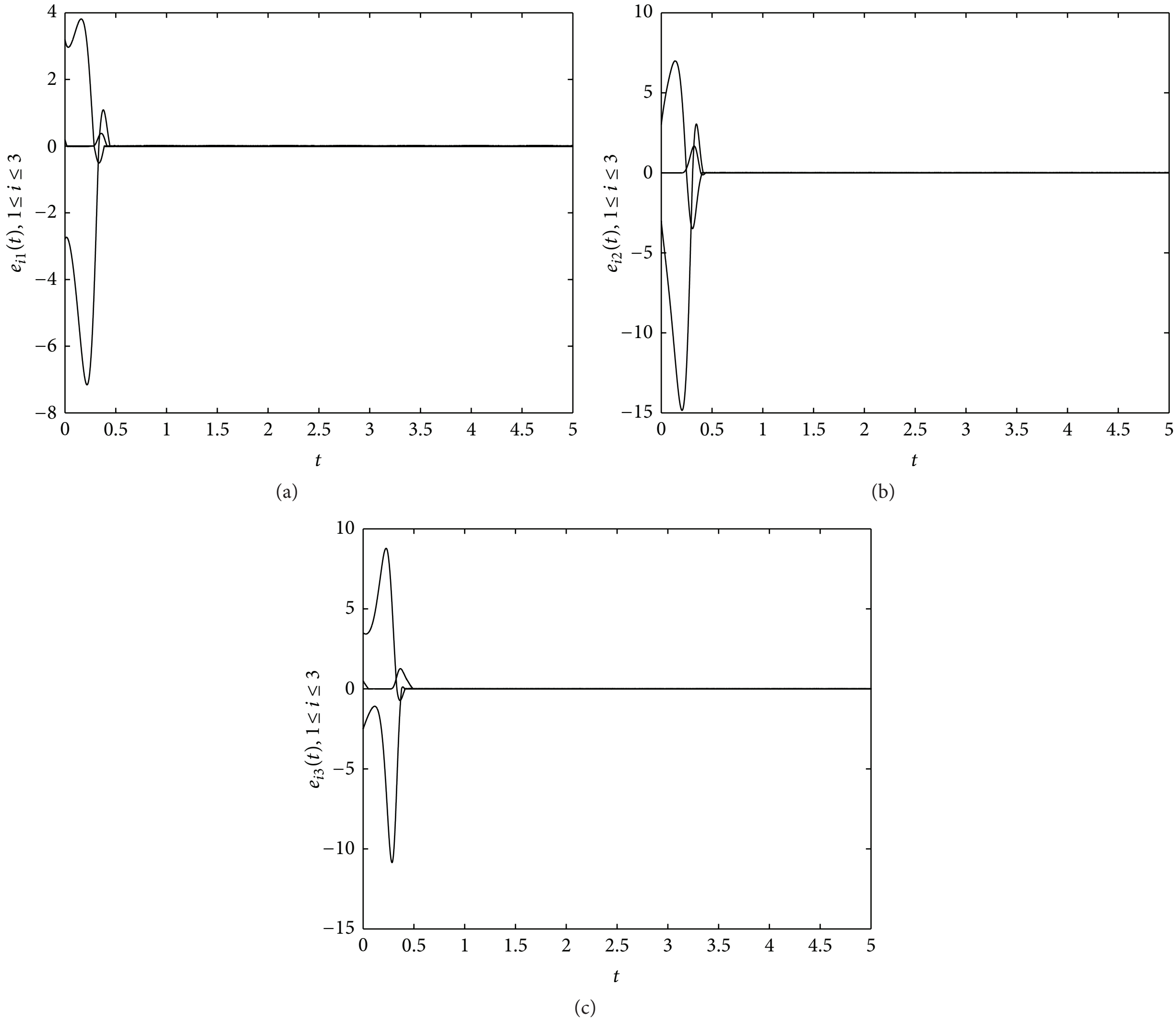

Figure 3: Synchronization errors of (47): (a) $e_{i1}(t)$, (b) $e_{i2}(t)$, (c) $e_{i3}(t)$, $i = 1, 2, 3$.

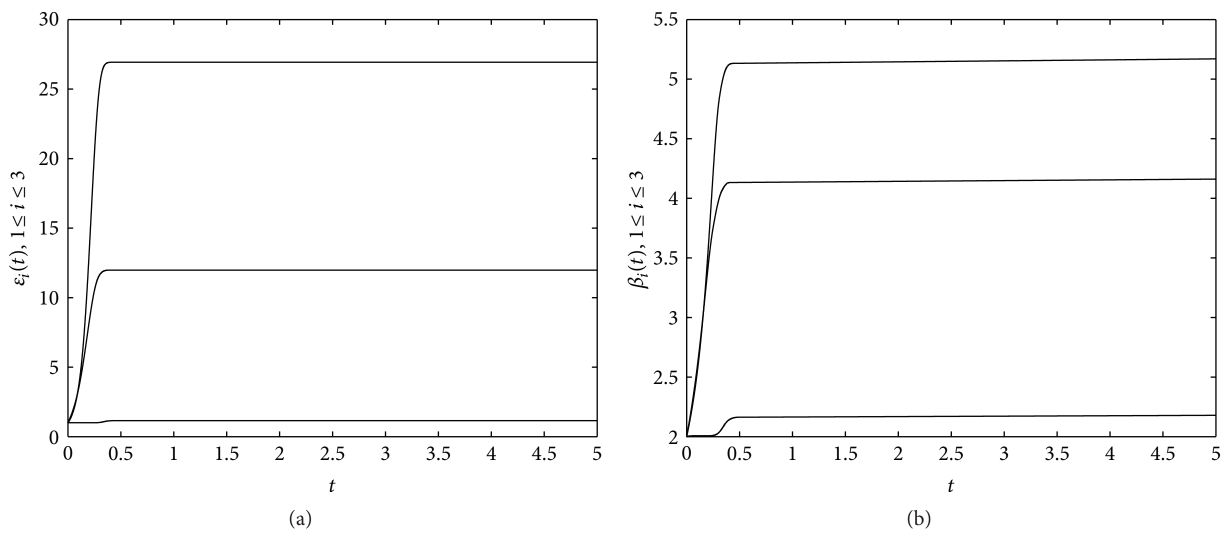

Figure 4: The adaptive control gains of (47): (a) the adaptive gains ε_i, $1 \le i \le 3$, (b) the adaptive gains β_i, $1 \le i \le 3$.

$$+\gamma\sum_{i=1}^{N}\sum_{j=1}^{N}\left|w_{ij}\right|\|\Lambda\|\left\|e_i(t)\right\|\int_{t-\theta(t)}^{t}\left\|e_j(s)\right\|ds$$
$$=\eta^T(t)\left((\|C\|+\|A\|h)I_N+\alpha\left(\|\Phi\|\widehat{U}^s-\overline{K}\right)\right)\eta(t)$$
$$+\eta^T(t)\left(\|B\|hI_N+\beta\|\Upsilon\||V|\right)\eta(t-\tau(t))$$
$$+\eta^T(t)\left(\|D\|hI_N+\gamma\|\Lambda\||W|\right)\int_{t-\theta(t)}^{t}\eta(s)\,ds, \tag{31}$$

where $\overline{K}=\mathrm{diag}(k_1,\ldots,k_l,\underbrace{0,\ldots,0}_{N-l})$, and the following deduction is used:

$$\sum_{i=1}^{l_1}e_i^T(t)\sigma_i(t)-\omega\sum_{i=1}^{l_1}\sum_{k=1}^{n}\beta_i\left|e_{ik}(t)\right|$$
$$-\sum_{i=1}^{l_1}\sum_{k=1}^{n}\left(M_i-\beta_i\right)\left|e_{ik}(t)\right|-\omega\sum_{i=l_1+1}^{l}\sum_{k=1}^{n}\beta_i\left|e_{ik}(t)\right|$$
$$\le\sum_{i=1}^{l_1}\sum_{k=1}^{n}\left[\left|e_{ik}(t)\right|M_{ik}-M_i\left|e_{ik}\right|-(\omega-1)\beta_i\left|e_{ik}\right|\right] \tag{32}$$
$$-\omega\sum_{i=l_1+1}^{l}\sum_{k=1}^{n}\beta_i\left|e_{ik}(t)\right|$$
$$\le-\sum_{i=1}^{l_1}\sum_{k=1}^{n}(\omega-1)\beta_i\left|e_{ik}\right|-\omega\sum_{i=l_1+1}^{l}\sum_{k=1}^{n}\beta_i\left|e_{ik}(t)\right|\le 0.$$

Combining (31) with (17) and (25), we have

$$\dot{\overline{V}}(t)\le\frac{1}{2}\zeta^T(t)\mathbf{\Pi}\zeta(t), \tag{33}$$

where $\zeta(t)=(\eta^T(t),\eta^T(t-\tau(t)),(\int_{t-\theta(t)}^{t}\eta(s)ds)^T)^T$ and

$$\mathbf{\Pi}=\begin{bmatrix}\Pi_{11} & \Pi_{12} & \Pi_{13}\\ \Pi_{12}^T & -2\left(1-h_\tau\right)Q & 0\\ \Pi_{13}^T & 0 & -\dfrac{2\left(1-h_\theta\right)}{\theta_{\min}}G\end{bmatrix} \tag{34}$$

with $\Pi_{11}=2((\|C\|+\|A\|h)I_N+\alpha(\|\Phi\|\widehat{U}^s-K)+Q+\theta_{\max}G)$, $\Pi_{12}=\|B\|hI_N+\beta\|\Upsilon\||V|$, $\Pi_{13}=\|D\|hI_N+\gamma\|\Lambda\||W|$.

According to Lemma 4, $\mathbf{\Pi}<0$ is equivalent to

$$\Delta=2\left((\|C\|+\|A\|h)I_N+\alpha\left(\|\Phi\|\widehat{U}^s-K\right)+Q+\theta_{\max}G\right)$$
$$+\frac{1}{2\left(1-h_\tau\right)}\left(\|B\|hI_N+\beta\|\Upsilon\||V|\right)Q^{-1}$$
$$\times\left(\|B\|hI_N+\beta\|\Upsilon\||V|^T\right)$$
$$+\frac{\theta_{\min}}{2\left(1-h_\theta\right)}\left(\|D\|hI_N+\gamma\|\Lambda\||W|\right)G^{-1}$$
$$\times\left(\|D\|hI_N+\gamma\|\Lambda\||W|^T\right)<0. \tag{35}$$

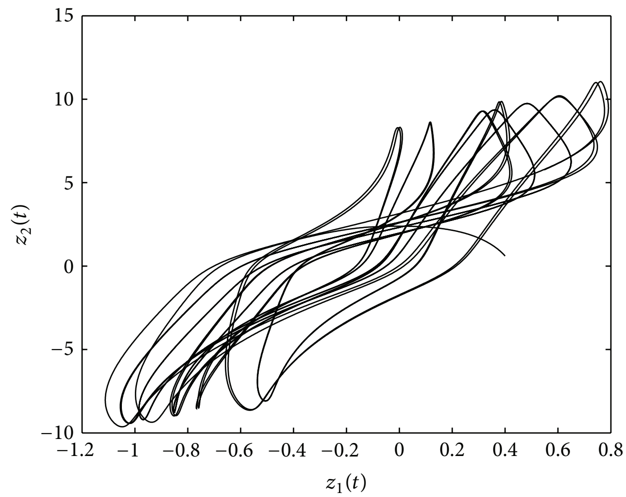

Figure 5: Chaotic trajectory of (49).

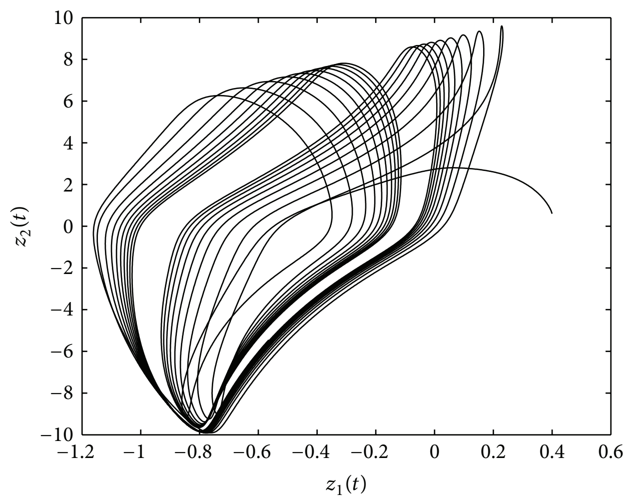

Figure 6: Chaotic trajectory of (51).

Let $Q=(1/2(1-h_\tau))\|\|B\|hI_N+\beta\|\Upsilon\||V|\|I_N$, $G=(\theta_{\min}/2(1-h_\theta))\|\|D\|hI_N+\gamma\|\Lambda\||W|\|I_N$. We have

$$\Delta\le 2\left(\|C\|+\|A\|h\right)I_N+2\alpha\left(\|\Phi\|\widehat{U}^s-K\right)$$
$$+\frac{\theta_{\max}\theta_{\min}+1-h_\theta}{1-h_\theta}\left\|\|D\|hI_N+\gamma\|\Lambda\||W|\right\|I_N$$
$$+\frac{2-h_\tau}{1-h_\tau}\left\|\|B\|hI_N+\beta\|\Upsilon\||V|\right\|I_{N-1} \tag{36}$$
$$=\begin{bmatrix}\Delta_{11} & 2\alpha\|\Phi\|\left(\widehat{U}^s\right)_*\\ 2\alpha\|\Phi\|\left(\widehat{U}^s\right)_*^T & 2\alpha\|\Phi\|\left(\widehat{U}^s\right)_l^c+\Sigma I_{N-l}\end{bmatrix},$$

where $\Delta_{11}=2\alpha\|\Phi\|(\widehat{U}^s)_l-2\alpha K_l+\Sigma I_l$, $2\alpha\|\Phi\|(\widehat{U}^s)_*$ is matrix with appropriate dimension.

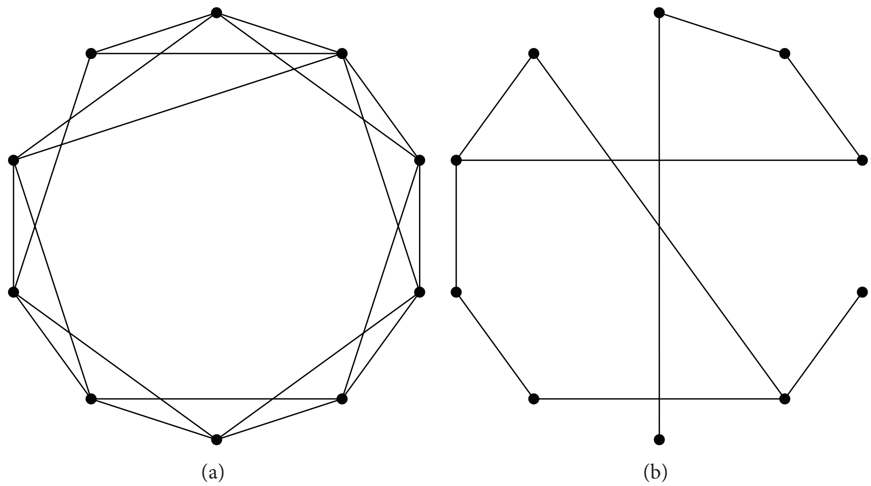

Figure 7: WS Small-Worlds with 10 nodes. In (a) each node connects 4 nodes, and the rewire probability is 0.2; in (b) each node connects 2 nodes, and the rewire probability is 0.4.

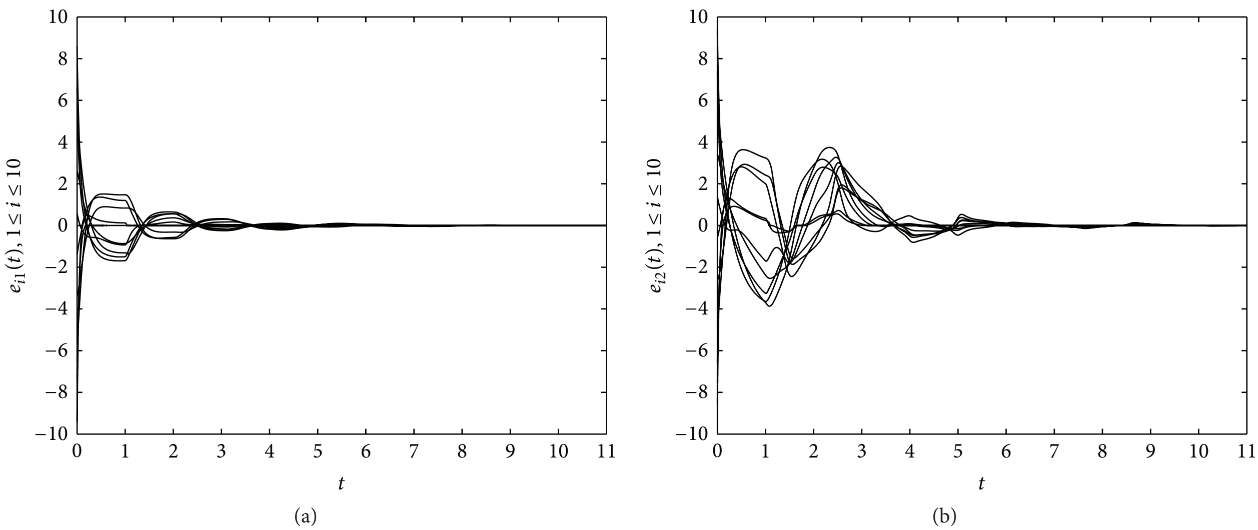

Figure 8: Synchronization errors of (52): (a) e_{i1}, (b) e_{i2}, $1 \le i \le 10$.

Since $2\alpha\|\Phi\|(\widehat{U}^s)_l^c + \Sigma I_{N-l} < 0$ and there exist positive constants $k_1, k_2, \ldots, k_l$ such that

$$2\alpha\|\Phi\|\left(\widehat{U}^s\right)_l - 2\alpha K_l + \Sigma I_l - (2\alpha\|\Phi\|)^2\left(\widehat{U}^s\right)_* \times \left(2\alpha\|\Phi\|\left(\widehat{U}^s\right)_l^c + \Sigma I_{N-l}\right)^{-1}\left(\widehat{U}^s\right)_*^T < 0, \quad (37)$$

again, from Lemma 4 we obtain $\Delta < 0$. Hence, $\mathbf{\Pi} < 0$. Denote $\lambda_{\min}$ to be the minimum eigenvalue of $-\mathbf{\Pi}$; then

$$\dot{\overline{V}}(t) \le -\lambda_{\min}\sum_{i=1}^{N}\|e_i(t)\|^2 \le 0. \quad (38)$$

Integrating both sides of the previous equation from 0 to t yields

$$\begin{aligned}\overline{V}(0) &\ge \overline{V}(t) + \lambda_{\min}\sum_{i=1}^{N}\int_0^t \|e_i(s)\|^2 ds \\ &\ge \lambda_{\min}\sum_{i=1}^{N}\int_0^t \|e_i(s)\|^2 ds.\end{aligned} \quad (39)$$

Therefore,

$$\lim_{t\to\infty}\lambda_{\min}\sum_{i=1}^{N}\int_0^t \|e_i(s)\|^2 ds \le \overline{V}(0). \quad (40)$$

By Lemma 6 we obtain

$$\lim_{t\to\infty}\lambda_{\min}\sum_{i=1}^{N}\|e_i(t)\|^2 = 0, \quad (41)$$

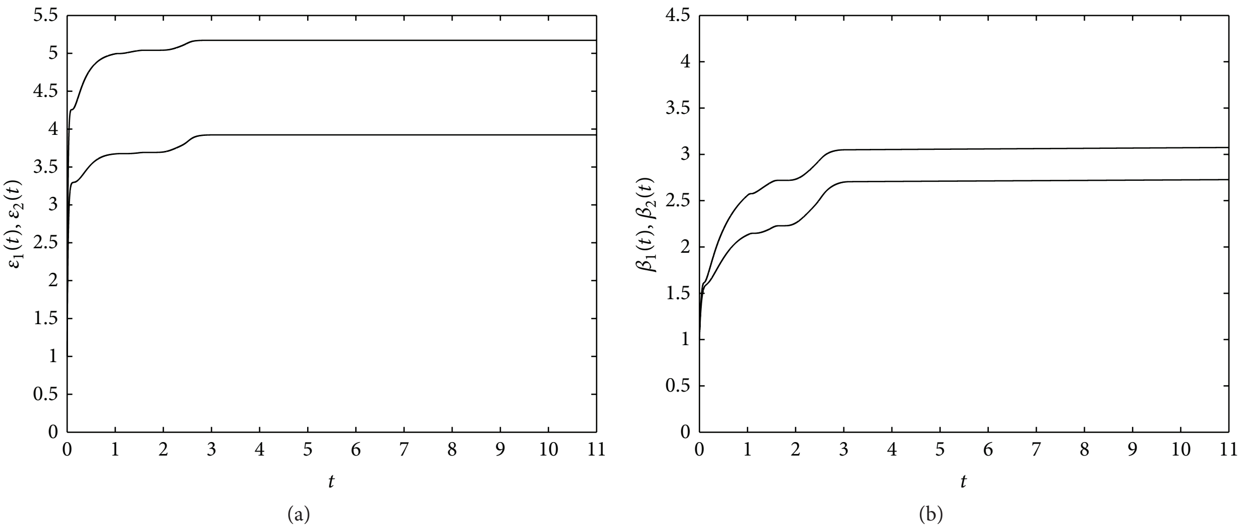

FIGURE 9: The adaptive pinning control gains of (52): (a) the adaptive pinning control gains ε_i, $i = 1, 2$. (b) The adaptive pinning control gains β_i, $i = 1, 2$.

which in turn means

$$\lim_{t\to\infty} \|e_i(t)\| = 0, \quad i = 1, 2, \ldots, N. \tag{42}$$

This completes the proof. □

When there is no external perturbation, that is, $\sigma_i(t) = 0$, $i = 1, 2, \ldots, N$, one can easily get the following corollaries from Theorems 7 and 8, respectively. We omit their proofs here.

Corollary 9. *Suppose that $\sigma_i(t) = 0$, $i = 1, 2, \ldots, N$, and the assumption condition (H_1) holds. Then complex networks* (1) *are synchronized with the adaptive controllers* (12)*. Moreover, the scalar ω can be relaxed to any positive constant.*

Corollary 10. *Suppose that matrix U is irreducible and the assumption (H_1) holds. The complex networks* (25) *are synchronized with the adaptive pinning controllers* (28)*, if* (27) *holds. Moreover, the scalar ω can be relaxed to any positive constant.*

Remark 11. From the inequalities (16) and (32) one can see that the designed adaptive controllers (12) and (28) are very useful. They can overcome the bad effects of the uncertain nonlinear perturbations without knowing the exact functions and bounds of the perturbations as long as the perturbed systems are chaotic. Especially, when there are only partial nodes perturbed (the first l_1 nodes in the system (25)), the designed controllers still are effective to stabilize the error system by adding them to nodes with and without such perturbations, (see the inequality (32)). Obviously, in the case of no perturbation, the parameter ω can also be taken as 0. When $\omega = 0$, the controllers (12) and (28) turn out to be the usual adaptive controller, which is extensively utilized to synchronize coupled systems with or without stochastic perturbations [8, 23–34, 40–42]. However, the controllers in [8, 23–34, 40–42] cannot synchronize coupled systems with nonstochastic perturbations. Therefore, the designed controllers can deal with both stochastic and nonstochastic perturbations to the systems, and hence they have better robustness than usual adaptive controllers.

Remark 12. Model (1) can be extended to the following more general complex networks:

$$\begin{aligned}\dot{x}_i(t) &= Cx_i(t) + Af(x_i(t)) + Bf_\tau(x_i(t-\tau(t))) \\ &\quad + D\int_{t-\theta(t)}^{t} f_\theta(x_i(s))\,ds + I(t) \\ &\quad + \alpha\sum_{j=1}^{N} u_{ij}\Phi x_j(t) + \beta\sum_{j=1}^{N} v_{ij}\Upsilon x_j(t-\tau(t)) \\ &\quad + \gamma\sum_{j=1}^{N} w_{ij}\Lambda\int_{t-\theta(t)}^{t} x_j(s)\,ds + \sigma_i(t) + R_i, \\ &\qquad i = 0, 1, \ldots, N.\end{aligned} \tag{43}$$

Moreover, we can also consider stochastic perturbations [21] and Markovian jump [43, 44] in (43) to get more general results. For simplicity, we omit the corresponding results and only consider model (1).

5. Numerical Examples

In this section, we provide two examples to illustrate the general model and the advantage of the new adaptive controller.

Example 13. The Lorenz system is described as

$$\dot{z}(t) = Cz(t) + Af(z(t)), \tag{44}$$

where

$$C=\begin{bmatrix}-10 & 10 & 0\\ 28 & -1 & 0\\ 0 & 0 & 8/3\end{bmatrix},\qquad A=\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \tag{45}$$

$f(z(t)) = (0, -z_1(t)z_3(t), z_1(t)z_2(t))^T$. When initial values are taken as $z_1(0) = 0.8$, $z_2(0) = 2$, $z_3(0) = 2.5$, chaotic trajectory of (44) can be seen in Figure 1.

The following three perturbed Lorenz systems are chaotic:

$$\dot{\overline{z}}(t) = C\overline{z}(t) + Af(\overline{z}(t)) + \sigma_i(t), \quad i = 1, 2, 3, \tag{46}$$

where $\sigma_1(t) = (0.1\overline{z}_1^2(t), 0.2\overline{z}_2(t), 0.2\overline{z}_3(t))^T$, $\sigma_2(t) = (0.1\overline{z}_1(t), 0.05\overline{z}_2^2(t), \sin\overline{z}_3(t))^T$, $\sigma_3(t) = (0.1\overline{z}_1(t), \cos\overline{z}_2(t), \sin\overline{z}_3(t))^T$. Chaotic trajectories of the three perturbed Lorenz systems are showed in Figure 2 with the same initial values $\overline{z}_1(0) = 0.8$, $\overline{z}_2(0) = 2$, $\overline{z}_3(0) = 2.5$.

Now consider the following complex networks with each node as the previous perturbed Lorenz system:

$$\begin{aligned}\dot{x}_i(t) &= Cx_i(t) + Af(x_i(t))\\ &\quad + \alpha\sum_{j=1}^{N}u_{ij}\Phi x_j(t) + \sigma_i(t) + R_i, \quad i = 0, 1, 3,\end{aligned} \tag{47}$$

where $\alpha = 0.5$, $\Phi = I_3$ and

$$U=\begin{bmatrix}-1 & 1 & 0\\ 1 & -1 & 0\\ 1 & 1 & -2\end{bmatrix}. \tag{48}$$

Obviously, conditions (H_1) and (H_2) are satisfied. According to Theorem 7, the complex networks (47) can be synchronized with adaptive controllers (12).

The initial conditions of the numerical simulations are as follows: $\omega = 4$, step $= 0.0005$, $x_1(0) = (-2, -1, 0)^T$, $x_2(0) = (1, 2, 3)^T$, $x_3(0) = (4, 5, 6)^T$, $\varepsilon_i(0) = 1$, $\beta_i(0) = 2$, $p_i = \xi_i = 0.5$, $i = 1, 2, 3$. Figure 3 describes the synchronization errors $e_{ij}(t) = x_{ij}(t) - z_j(t)$, $i, j = 1, 2, 3$. Figure 4 shows the adaptive feedback gains. Numerical simulations verify the effectiveness of Theorem 7.

Example 14. Consider the following chaotic neural networks with mixed delays:

$$\begin{aligned}\dot{z}(t) &= Cz(t) + Af(z(t)) + Bf(z(t-\tau(t)))\\ &\quad + D\int_{t-\sigma(t)}^{t} f(z(s))\,ds + I(t),\end{aligned} \tag{49}$$

where $z(t) = (z_1(t), z_2(t))^T$, $\tau(t) = 1$, $\sigma(t) = 0.3$, $f(z(t)) = (\tanh(z_1(t), \tanh(z_2(t))^T$,

$$C=\begin{bmatrix}-1.2 & 0\\ 0 & -1\end{bmatrix},\qquad A=\begin{bmatrix}3 & -0.3\\ 8 & 5\end{bmatrix},$$
$$I=\begin{bmatrix}0\\ 2\end{bmatrix},\qquad B=\begin{bmatrix}-1.4 & 0.1\\ 0.3 & -8\end{bmatrix},\qquad D=\begin{bmatrix}-1.2 & 0.1\\ -2.8 & -1\end{bmatrix}. \tag{50}$$

In the case that the initial condition is chosen as $z_1(t) = 0.4$, $z_2(t) = 0.6$, for all $t \in [-1, 0]$, the chaotic attractor can be seen in Figure 5.

The perturbed system of (49) is

$$\begin{aligned}\dot{\overline{z}}(t) &= C\overline{z}(t) + Af(\overline{z}(t)) + Bf(\overline{z}(t-\tau(t)))\\ &\quad + D\int_{t-\sigma(t)}^{t} f(\overline{z}(s))\,ds + I(t) + \sigma(t),\end{aligned} \tag{51}$$

where $\sigma(t) = (0.2\overline{z}_1(t-1), 0.2\int_{t-0.3}^{t}\overline{z}_2(s)ds)^T$. The chaotic attractor of (51) can be seen in Figure 6 with $\overline{z}_1(t) = 0.4$, $\overline{z}_2(t) = 0.6$, for all $t \in [-1, 0]$.

Now consider the following complex networks with each node as the previous neural networks with mixed delays (49), while the second node is disturbed with the previous $\sigma(t)$.

$$\begin{aligned}\dot{x}_i(t) &= Cx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau(t)))\\ &\quad + D\int_{t-\theta(t)}^{t} f(x_i(s))\,ds + I(t) + \alpha\sum_{j=1}^{10}u_{ij}\Phi x_j(t)\\ &\quad + \beta\sum_{j=1}^{10}v_{ij}\Upsilon x_j(t-\tau(t)) + \gamma\sum_{j=1}^{10}w_{ij}\Lambda\int_{t-\theta(t)}^{t} x_j(s)\,ds\\ &\quad + \sigma_i(t) + R_i, \quad i = 0, 1, \ldots, 10,\end{aligned} \tag{52}$$

where $\alpha = 3$, $\beta = \gamma = 1$, $\sigma_2(t) = \sigma(t)$, else $\sigma_i(t) = 0$.

Figure 7 depicts the WS Small-World networks [2] corresponding to nondelay (a), discrete delay, and distributed delay (b). The corresponding Laplacian matrices are shown as following:

$$U=\begin{bmatrix}-4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1\\ 1 & -5 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1\\ 1 & 1 & -4 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 1 & -4 & 1 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & -4 & 1 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 1 & -4 & 1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 1 & -4 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -4 & 1 & 1\\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & -4 & 0\\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -3\end{bmatrix}, \tag{53}$$

$$V=W=\begin{bmatrix}-2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & -3 & 0 & 1 & 0 & 0 & 1\\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & -2 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -3 & 1\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -2\end{bmatrix}. \tag{54}$$

Take the first two nodes (corresponding to matrix U) to be controlled. According Theorem 8, the complex networks (52) can be synchronized with adaptive controllers (27).

The initial conditions of the numerical simulations are as follows: $\omega = 2$, step $= 0.0005$, $x_i(0) = (-11 + 2i, -10 + 2i)^T$, $i = 1, 2, \ldots, 10$. $\varepsilon_i(0) = \beta_i(0) = p_i = \xi_i = 1$, $i = 1, 2$. Figure 8 describes the synchronization errors $e_{ij}(t) = x_{ij}(t) - z_j(t)$, $i = 1, 2, \ldots, 10$, $j = 1, 2$. Figure 9 depicts the adaptive feedback gains. Numerical simulations verify the effectiveness of Theorem 8.

6. Conclusions

External perturbations to networks are unavoidable in practice. On the other hand, many chaotic models have discrete delay and distributed delay. Therefore, in this paper, we introduced a class of hybrid coupled complex networks with mixed delays and unknown nonstochastic external perturbations. A simple robust adaptive controller is designed to synchronize the complex networks even without knowing a priori the bounds and the exact functions of the perturbations. It should be emphasized that we do not assume that the coupling matrix is symmetric or diagonal. The controller can enhance robustness and reduce fragility of complex networks; hence, it has great practical significance. Moreover, we also verify the effectiveness of the theoretical results by numerical simulations.

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