Research Article

Approximation for the Hierarchical Constrained Variational Inequalities over the Fixed Points of Nonexpansive Semigroups

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The purpose of the present paper is to study the hierarchical constrained variational inequalities of finding a point x^* such that $x^* \in \Omega, \langle (A - \gamma f)x^* - (I - B)Sx^*, x - x^* \rangle \geq 0, \ \forall x \in \Omega$, where Ω is the set of the solutions of the following variational inequality: $x^* \in {_F}, \langle (A - S)x^*, x - x^* \rangle \geq 0, \ \forall x \in {_F}$, where A, B are two strongly positive bounded linear operators, f is a ρ-contraction, S is a nonexpansive mapping, and ${_F}$ is the fixed points set of a nonexpansive semigroup $\{T(s)\}_{s\geq 0}$. We present a double-net convergence hierarchical to some elements in ${_F}$ which solves the above hierarchical constrained variational inequalities.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. Recall that a self-mapping f of C is said to be contractive if there exists a constant $\rho \in [0, 1)$ such that $\|f(x) - f(y)\| \leq \rho\|x - y\|$ for all $x, y \in C$. A mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1}$$

We denote by Fix(T) the set of fixed points of T; that is, Fix$(T) = \{x \in C : Tx = x\}$. A bounded linear operator B is called strongly positive on H if there exists a constant $\tilde{\gamma} > 0$ such that

$$\langle Bx, x \rangle \geq \tilde{\gamma}\|x\|^2, \quad \forall x \in H. \tag{2}$$

It is well known that the variational inequality for an operator, $\varphi : H \to H$, over a nonempty, closed, and convex set, $C \subset H$, is to find a point $x^* \in C$ with the property

$$\langle \varphi(x^*), y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{3}$$

The set of the solutions of the variational inequality (3) is denoted by VI(C, φ). If the mapping φ is a monotone operator, then we say that VI(3) is monotone. It is well known

that if φ is Lipschitzian and strongly monotone, then for small enough $\delta > 0$, the mapping $P_C(I - \delta\varphi)$ is a contraction on C and so the sequence $\{x_n\}$ of Picard iterates, given by $x_n = P_C(I - \delta\varphi)x_{n-1}$ $(n \geq 1)$, converges strongly to the unique solution of the VI(3). This sort of VI(3) where φ is strongly monotone and Lipschitzian is originated from Yamada [1]. However, if φ is only monotone (not strongly monotone), then their iterative methods do not apply to VI.

Many practical problems such as signal processing and network resource allocation are formulated as the variational inequality over the set of the solutions of some nonlinear mappings (e.g., the fixed point set of nonexpansive mappings), and algorithms to solve these problems have been proposed. Iterative algorithms have been presented for the convex optimization problem with a fixed point constraint along with proof that these algorithms strongly converge to the unique solution of problems with a strongly monotone operator. The strong monotonicity condition guarantees the uniqueness of the solution. For some related works on the variational inequalities, please see [2–23] and the references therein. Particulary, the variational inequality problems over the fixed points of nonexpansive mappings have been considered. The reader can consult [16, 24]. On the other hand, we note that in the literature, nonlinear ergodic theorems for nonexpansive semigroups have been considered by many authors; see, for example, [25–32]. In this paper, we will

consider a general variational inequality problem with the variational inequality constraint is the fixed points of nonexpansive semigroups.

The purpose of the present paper is to study the hierarchical constrained variational inequalities of finding a point x^* such that

$$x^* \in \Omega, \quad \langle (A - \gamma f) x^* - (I - B) Sx^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega, \tag{4}$$

where Ω is the set of the solutions of the following variational inequality:

$$x^* \in {}_F, \quad \langle (A - S) x^*, x - x^* \rangle \geq 0, \quad \forall x \in {}_F, \tag{5}$$

where A and B are two strongly positive bounded linear operators, f is a ρ-contraction, S is a nonexpansive mapping, and ${}_F$ is the fixed points set of a nonexpansive semigroup $\{T(s)\}_{s \geq 0}$. We present a double-net convergence hierarchical to some elements in ${}_F$ which solves the above hierarchical constrained variational inequalities.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. The metric (or the nearest point) projection from H onto C is the mapping $P_C : H \rightarrow C$ which assigns to each point $x \in C$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C). \tag{6}$$

It is well known that P_C is a nonexpansive mapping and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{7}$$

Moreover, P_C is characterized by the following property:

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \quad \forall x \in H, \ y \in C. \tag{8}$$

Recall that a family $\{T(s)\}_{s \geq 0}$ of mappings of H into itself is called a nonexpansive semigroup if it satisfies the following conditions:

(S1) $T(0)x = x$ for all $x \in H$;

(S2) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$;

(S3) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in H$ and $s \geq 0$;

(S4) for all $x \in H$, $s \rightarrow T(s)x$ is continuous.

We denote by $\text{Fix}(T(s))$ the set of fixed points of $T(s)$ and by ${}_F$ the set of all common fixed points of $\{T(s)\}_{s \geq 0}$; that is, ${}_F = \bigcap_{s \geq 0} \text{Fix}(T(s))$. It is known that ${}_F$ is closed and convex.

We need the following lemmas for proving our main results.

Lemma 1 (see [33]). *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $\{T(s)\}_{s \geq 0}$ be a nonexpansive semigroup on C. Then, for every $h \geq 0$,*

$$\lim_{t \to \infty} \sup_{x \in C} \left\| \frac{1}{t} \int_0^t T(s) x \, ds - T(h) \frac{1}{t} \int_0^t T(s) x \, ds \right\| = 0. \tag{9}$$

Lemma 2 (see [34]). *Let C be a closed convex subset of a real Hilbert space H and let $S : C \rightarrow C$ be a nonexpansive mapping. Then, the mapping $I - S$ is demiclosed. That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightarrow x^*$ weakly and $(I - S)x_n \rightarrow y$ strongly, then $(I - S)x^* = y$.*

Lemma 3 (see [35]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that a mapping $F : C \rightarrow H$ is monotone and weakly continuous along segments (i.e., $F(x + ty) \rightarrow F(x)$ weakly, as $t \rightarrow 0$, whenever $x + ty \in C$ for $x, y \in C$). Then the variational inequality*

$$x^* \in C, \quad \langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{10}$$

is equivalent to the dual variational inequality

$$x^* \in C, \quad \langle Fx, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{11}$$

3. Main Results

Now we consider the following hierarchical variational inequality with the variational inequality constraint over the fixed points set of nonexpansive semigroups $\{T(s)\}_{s \geq 0}$.

Problem 1. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \rightarrow H$ be a ρ-contraction with coefficient $\rho \in [0, 1)$ and let $S : C \rightarrow C$ be a nonexpansive mapping. Let $\{T(s)\}_{s \geq 0}$ be a nonexpansive semigroup on C and let $A, B : H \rightarrow H$ be two strongly positive bounded linear operators with coefficients $\widetilde{\lambda}$ ($1 \leq \widetilde{\lambda} < 2$) and $\widetilde{\gamma}$ ($0 < \widetilde{\gamma} < 1$), respectively. Let γ be a constant satisfying $0 < \gamma\rho < \widetilde{\gamma}$.

Now, our objective is to find x^* such that

$$x^* \in \Omega, \quad \langle (A - \gamma f) x^* - (I - B) Sx^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega, \tag{12}$$

where $\Omega := \text{VI}({}_F, A - S)$ is the set of the solutions of the following variational inequality:

$$x^* \in {}_F, \quad \langle (A - S) x^*, x - x^* \rangle \geq 0, \quad \forall x \in {}_F. \tag{13}$$

We observe that $(A - \gamma f) - (I - B)S$ is strongly monotone and Lipschitz continuous. In fact, we have

$$\langle (A - \gamma f) x - (I - B) Sx - [(A - \gamma f) y - (I - B) Sy], x - y \rangle$$

$$= \langle Ax - Ay, x - y \rangle - \gamma \langle f(x) - f(y), x - y \rangle$$

$$- (I - B) \langle Sx - Sy, x - y \rangle \geq \widetilde{\lambda} \|x - y\|^2 - \gamma\rho \|x - y\|^2$$

$$- (1 - \widetilde{\gamma}) \|x - y\|^2 = (\widetilde{\lambda} - 1 + \widetilde{\gamma} - \gamma\rho) \|x - y\|^2,$$

$$\|(A - \gamma f) x - (I - B) Sx - [(A - \gamma f) y - (I - B) Sy]\|$$

$$\leq \|Ax - Ay\| + \gamma \|f(x) - f(y)\| + \|I - B\| \|Sx - Sy\|$$

$$\leq \|A\| \|x - y\| + \gamma\rho \|x - y\| + (1 - \widetilde{\gamma}) \|x - y\|$$

$$= (1 + \|A\| + \gamma\rho - \widetilde{\gamma}) \|x - y\|. \tag{14}$$

Hence, the existence and the uniqueness of the solution to Problem 1 are guaranteed.

In order to solve the above hierarchical constrained variational inequality, we present the following double net.

Algorithm 4. Set $\kappa = 1/(\widetilde{\lambda} + \widetilde{\gamma} - \gamma\rho)$. Then, $0 < \kappa \leq 1$. For each $(s, t) \in (0, \kappa) \times (0, 1)$, we define a double net $\{x_{s,t}\}$ implicitly by

$$
x_{s,t} = P_C \Bigg[s \left(t\gamma f \left(x_{s,t} \right) + \left(I - tB \right) S x_{s,t} \right) \\
+ \left(I - sA \right) \frac{1}{\lambda_s} \int_0^{\lambda_s} T\left(\nu \right) x_{s,t} d\nu \Bigg]. \tag{15}
$$

Note that this implicit manner algorithm is well defined. In fact, we define the mapping

$$
x \longmapsto W_{s,t}\left(x \right) := P_C \Bigg[s \left(t\gamma f \left(x \right) + \left(I - tB \right) S x \right) \\
+ \left(I - sA \right) \frac{1}{\lambda_s} \int_0^{\lambda_s} T\left(\nu \right) x d\nu \Bigg], \tag{16}
$$

$$
(s, t) \in (0, \kappa) \times (0, 1).
$$

Note that this self-mapping is a contraction. As a matter of fact, we have

$$
\left\| W_{s,t}\left(x \right) - W_{s,t}\left(y \right) \right\| \\
= \Bigg\| P_C \Bigg[s \left(t\gamma f \left(x \right) + \left(I - tB \right) S x \right) \\
+ \left(I - sA \right) \frac{1}{\lambda_s} \int_0^{\lambda_s} T\left(\nu \right) x d\nu \Bigg] \\
- P_C \Bigg[s \left(t\gamma f \left(y \right) + \left(I - tB \right) S y \right) \\
+ \left(I - sA \right) \frac{1}{\lambda_s} \int_0^{\lambda_s} T\left(\nu \right) y d\nu \Bigg] \Bigg\| \tag{17}
$$

$$
\leq st\gamma \left\| f\left(x \right) - f\left(y \right) \right\| \\
+ s \left\| I - tB \right\| \left\| S x - S y \right\| \\
+ \left\| I - sA \right\| \Bigg\| \frac{1}{\lambda_s} \int_0^{\lambda_s} T\left(\nu \right) x d\nu \\
- \frac{1}{\lambda_s} \int_0^{\lambda_s} T\left(\nu \right) y d\nu \Bigg\| \\
\leq \left[1 - \left(\widetilde{\lambda} - 1 \right) s - \left(\widetilde{\gamma} - \gamma\rho \right) st \right] \left\| x - y \right\|.
$$

Since $(s, t) \in (0, \kappa) \times (0, 1)$, $0 < 1 - (\widetilde{\lambda} - 1)s - (\widetilde{\gamma} - \gamma\rho)st < 1$. Hence, $W_{s,t}$ is a contraction. Therefore, by Banach's Contraction Principle, $W_{s,t}$ has a unique fixed point which is denoted by $x_{s,t} \in C$.

Next we show the behavior of the net $\{x_{s,t}\}$ as $s \to 0$ and $t \to 0$ successively.

Theorem 5. *Assume that $VI(F, A - S) \neq \emptyset$. Then, for each fixed $t \in (0, 1)$, the net $\{x_{s,t}\}$ defined by (15) converges in norm, as $s \to 0+$, to a solution $x_t \in F$. Moreover, as $t \to 0+$, the net $\{x_t\}$ converges in norm to the unique solution x^* of Problem 1.*

Proof. We first show that the sequence $\{x_{s,t}\}$ is bounded. Take $y^* \in F$. From (15), we have

$$
\left\| x_{s,t} - y^* \right\| = \Bigg\| P_C \Bigg[s \left(t\gamma f \left(x_{s,t} \right) + \left(I - tB \right) S x_{s,t} \right) \\
+ \left(I - sA \right) \frac{1}{\lambda_s} \int_0^{\lambda_s} T\left(\nu \right) x d\nu \Bigg] - y^* \Bigg\| \\
\leq \Bigg\| s \left(t\gamma f \left(x_{s,t} \right) + \left(I - tB \right) S x_{s,t} \right) \\
+ \left(I - sA \right) \frac{1}{\lambda_s} \int_0^{\lambda_s} T\left(\nu \right) x d\nu - y^* \Bigg\| \\
= \Bigg\| st\gamma \left(f\left(x_{s,t} \right) - f\left(y^* \right) \right) \\
+ s \left(I - tB \right) \left(S x_{s,t} - S y^* \right) \\
+ st\gamma f\left(y^* \right) + s \left(I - tB \right) S y^* - sA y^* \\
+ \left(I - sA \right) \left(\frac{1}{\lambda_s} \int_0^{\lambda_s} T\left(\nu \right) x d\nu - y^* \right) \Bigg\| \tag{18}
$$

$$
\leq st\gamma \left\| f\left(x_{s,t} \right) - f\left(y^* \right) \right\| \\
+ s \left\| I - tB \right\| \left\| S x_{s,t} - S y^* \right\| \\
+ \left\| I - sA \right\| \times \Bigg\| \frac{1}{\lambda_s} \int_0^{\lambda_s} T\left(\nu \right) x d\nu - y^* \Bigg\| \\
+ s \left\| t\gamma f\left(y^* \right) + \left(I - tB \right) S y^* - A y^* \right\| \\
\leq st\gamma\rho \left\| x_{s,t} - y^* \right\| + s \left(1 - t\widetilde{\gamma} \right) \left\| x_{s,t} - y^* \right\| \\
+ \left(1 - s\widetilde{\lambda} \right) \left\| x_{s,t} - y^* \right\| \\
+ s \left\| t\gamma f\left(y^* \right) + \left(I - tB \right) S y^* - A y^* \right\| \\
= \left[1 - \left(\widetilde{\lambda} - 1 \right) s - \left(\widetilde{\gamma} - \gamma\rho \right) st \right] \left\| x_{s,t} - y^* \right\| \\
+ s \left\| t\gamma f\left(y^* \right) + \left(I - tB \right) S y^* - A y^* \right\|.
$$

Hence

$$
\left\| x_{s,t} - y^* \right\| \leq \frac{1}{\left(\widetilde{\gamma} - \gamma\rho \right) t + \widetilde{\lambda} - 1} \\
\times \left\| t\gamma f\left(y^* \right) + \left(I - tB \right) S y^* - A y^* \right\|. \tag{19}
$$

It follows that for each fixed $t \in (0, 1)$, $\{x_{s,t}\}$ is bounded.

Next, we show that $\lim_{s \to 0} \| T(\tau) x_{s,t} - x_{s,t} \| = 0$ for all $0 \leq \tau < \infty$ and consequently, as $s \to 0+$, the entire net $\{x_{s,t}\}$ converges in norm to $x_t \in F$.

For each fixed $t \in (0,1)$, we set $R_t := (1/((\widetilde{\gamma} - \gamma\rho)t + \widetilde{\lambda} - 1))\|t\gamma f(y^*) + (I - tB)Sy^* - Ay^*\|$. It is clear that for each fixed $t \in (0,1)$, $\{x_{s,t}\} \subset B(y^*, R_t)$. Notice that

$$\left\| \frac{1}{\lambda_s} \int_0^{\lambda_s} T(\nu) x_{s,t} d\nu - y^* \right\| \le \|x_{s,t} - y^*\| \le R_t. \tag{20}$$

Moreover, we observe that if $x \in B(y^*, R_t)$, then

$$\|T(s)x - y^*\| \le \|T(s)x - T(s)y^*\| \le \|x - y^*\| \le R_t, \tag{21}$$

that is, $B(y^*, R_t)$ is $T(s)$-invariant for all s.

From (15), we deduce

$$\|T(\tau) x_{s,t} - x_{s,t}\|$$
$$\le \left\| T(\tau) x_{s,t} - T(\tau) \frac{1}{\lambda_s} \int_0^{\lambda_s} T(\nu) x_{s,t} d\nu \right\|$$
$$+ \left\| T(\tau) \frac{1}{\lambda_s} \int_0^{\lambda_s} T(\nu) x_{s,t} d\nu - \frac{1}{\lambda_s} \int_0^{\lambda_s} T(\nu) x_{s,t} d\nu \right\|$$
$$+ \left\| \frac{1}{\lambda_s} \int_0^{\lambda_s} T(\nu) x_{s,t} d\nu - x_{s,t} \right\|$$
$$\le \left\| T(\tau) \frac{1}{\lambda_s} \int_0^{\lambda_s} T(\nu) x_{s,t} d\nu - \frac{1}{\lambda_s} \int_0^{\lambda_s} T(\nu) x_{s,t} d\nu \right\|$$
$$+ 2 \left\| x_{s,t} - \frac{1}{\lambda_s} \int_0^{\lambda_s} T(\nu) x_{s,t} d\nu \right\|$$
$$\le 2s \left\| t\gamma f(x_{s,t}) + (I - tB) S x_{s,t} \right.$$
$$\left. - \frac{A}{\lambda_s} \int_0^{\lambda_s} T(\nu) x_{s,t} d\nu \right\|$$
$$+ \left\| T(\tau) \frac{1}{\lambda_s} \int_0^{\lambda_s} T(\nu) x_{s,t} d\nu - \frac{1}{\lambda_s} \int_0^{\lambda_s} T(\nu) x_{s,t} d\nu \right\|. \tag{22}$$

Since $\{x_{s,t}\}$ is bounded, $\{f(x_{s,t})\}$ and $\{Sx_{s,t}\}$ are also bounded. Then, from Lemma 1, we deduce for all $0 \le \tau < \infty$ and fixed $t \in (0,1)$ that

$$\lim_{s \to 0} \|T(\tau) x_{s,t} - x_{s,t}\| = 0. \tag{23}$$

Set $y_{s,t} = s(t\gamma f(x_{s,t}) + (I - tB)Sx_{s,t}) + (I - sA)(1/\lambda_s) \int_0^{\lambda_s} T(\nu) x_{s,t} d\nu$ for all $(s,t) \in (0,\kappa) \times (0,1)$. We then have $x_{s,t} = P_C y_{s,t}$, and for any $y^* \in F$,

$$x_{s,t} - y^* = x_{s,t} - y_{s,t} + y_{s,t} - y^*$$
$$= x_{s,t} - y_{s,t} + s(t\gamma f(x_{s,t}) + (I - tB)Sx_{s,t})$$
$$+ (I - sA) \frac{1}{\lambda_s} \int_0^{\lambda_s} T(\nu) x_{s,t} d\nu - y^* = x_{s,t} - y_{s,t}$$
$$+ st\gamma(f(x_{s,t}) - f(y^*)) + s(I - tB)(Sx_{s,t} - Sy^*)$$
$$+ s(t\gamma f(y^*) + (I - tB)Sy^* - Ay^*)$$
$$+ (I - sA)\left(\frac{1}{\lambda_s} \int_0^{\lambda_s} T(\nu) x_{s,t} d\nu - y^* \right). \tag{24}$$

Notice that

$$\langle x_{s,t} - y_{s,t}, x_{s,t} - y^* \rangle \le 0. \tag{25}$$

Thus, we have

$$\|x_{s,t} - y^*\|^2 = \langle x_{s,t} - y_{s,t}, x_{s,t} - y^* \rangle$$
$$+ st\gamma \langle f(x_{s,t}) - f(y^*), x_{s,t} - y^* \rangle$$
$$+ s(I - tB) \langle Sx_{s,t} - Sy^*, x_{s,t} - y^* \rangle$$
$$+ (I - sA)$$
$$\times \left\langle \frac{1}{\lambda_s} \int_0^{\lambda_s} T(\nu) x_{s,t} d\nu - y^*, x_{s,t} - y^* \right\rangle$$
$$+ s \langle t\gamma f(y^*) + (I - tB)Sy^* - Ay^*, x_{s,t} - y^* \rangle$$
$$\le st\gamma \|f(x_{s,t}) - f(y^*)\| \|x_{s,t} - y^*\|$$
$$+ s\|I - tB\| \|Sx_{s,t} - Sy^*\| \|x_{s,t} - y^*\|$$
$$+ s \langle t\gamma f(y^*) + (I - tB)Sy^* - Ay^*, x_{s,t} - y^* \rangle$$
$$+ \|I - sA\| \left\| \frac{1}{\lambda_s} \int_0^{\lambda_s} T(\nu) x_{s,t} d\nu - y^* \right\| \|x_{s,t} - y^*\|$$
$$\le st\gamma\rho \|x_{s,t} - y^*\|^2 + s(1 - t\widetilde{\gamma}) \|x_{s,t} - y^*\|^2$$
$$+ s \langle t\gamma f(y^*) + (I - tB)Sy^* - Ay^*, x_{s,t} - y^* \rangle$$
$$+ (1 - s\widetilde{\lambda}) \|x_{s,t} - y^*\|^2$$
$$= [1 - (\widetilde{\lambda} - 1)s - (\widetilde{\gamma} - \gamma\rho)st] \|x_{s,t} - y^*\|^2$$
$$+ s \langle t\gamma f(y^*) + (I - tB)Sy^* - Ay^*, x_{s,t} - y^* \rangle. \tag{26}$$

So

$$\|x_{s,t} - y^*\|^2 \leq \frac{1}{(\tilde{\gamma} - \gamma\rho)t + \tilde{\lambda} - 1}$$
$$\times \langle t\gamma f(y^*) + (I - tB)Sy^* - Ay^*, x_{s,t} - y^* \rangle,$$
$$y^* \in F. \tag{27}$$

Assume $\{s_n\} \subset (0, 1)$ such that $s_n \to 0$ as $n \to \infty$. By (27), we obtain immediately that

$$\|x_{s_n,t} - y^*\|^2 \leq \frac{1}{(\tilde{\gamma} - \gamma\rho)t + \tilde{\lambda} - 1}$$
$$\times \langle t\gamma f(y^*) + (I - tB)Sy^* - Ay^*, x_{s_n,t} - y^* \rangle,$$
$$y^* \in F. \tag{28}$$

Since $\{x_{s_n,t}\}$ is bounded, there exists a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ such that $\{x_{s_{n_i},t}\}$ converges weakly to a point x_t. From (23) and Lemma 2, we get $x_t \in F$. We can substitute x_t for y^* in (28) to get

$$\|x_{s_n,t} - x_t\|^2 \leq \frac{1}{(\tilde{\gamma} - \gamma\rho)t + \tilde{\lambda} - 1}$$
$$\times \langle t\gamma f(x_t) + (I - tB)Sx_t - Ax_t, x_{s_n,t} - x_t \rangle. \tag{29}$$

The weak convergence of $\{x_{s_n,t}\}$ to x_t actually implies that $x_{s_n,t} \to x_t$ strongly. This has proved the relative norm-compactness of the net $\{x_{s,t}\}$ as $s \to 0+$ for each fixed $t \in (0, 1)$.

In (28), we take the limit as $n \to \infty$ to get

$$\|x_t - y^*\|^2 \leq \frac{1}{(\tilde{\gamma} - \gamma\rho)t + \tilde{\lambda} - 1}$$
$$\times \langle t\gamma f(y^*) + (I - tB)Sy^* - Ay^*, x_t - y^* \rangle,$$
$$\forall y^* \in F. \tag{30}$$

In particular, x_t solves the following variational inequality:

$$x_t \in F, \quad \langle Ay^* - t\gamma f(y^*)$$
$$- (I - tB)Sy^*, y^* - x_t \rangle \geq 0, \quad \forall y^* \in F. \tag{31}$$

Note that the mapping $A - t\gamma f - (I - tB)S$ is monotone for all $t \in (0, 1)$, since

$$\langle Ax - t\gamma f(x) - (I - tB)Sx$$
$$- (Ay - t\gamma f(y) - (I - tB)Sy), x - y \rangle$$
$$= \langle Ax - Ay, x - y \rangle - t\gamma \langle f(x) - f(y), x - y \rangle$$
$$- (I - tB)\langle Sx - Sy, x - y \rangle \geq \tilde{\lambda}\|x - y\|^2$$
$$- t\gamma\rho\|x - y\|^2 - (1 - t\tilde{\gamma})\|x - y\|^2$$
$$= [\tilde{\lambda} - 1 + (\tilde{\gamma} - \gamma\rho)t]\|x - y\|^2 \geq 0. \tag{32}$$

By Lemma 3, (31) is equivalent to its dual VI:

$$x_t \in F, \quad \langle Ax_t - t\gamma f(x_t) - (I - tB)Sx_t, y^* - x_t \rangle \geq 0,$$
$$\forall y^* \in F. \tag{33}$$

Next we show that as $s \to 0+$, the entire net $\{x_{s,t}\}$ converges in norm to $x_t \in F$. We assume $x_{s'_n,t} \to x'_t$, where $s'_n \to 0$. Similarly, by the above proof, we deduce $x'_t \in F$ which solves the following variational inequality:

$$x'_t \in F, \quad \langle Ax'_t - t\gamma f(x'_t) - (I - tB)Sx'_t, y^* - x'_t \rangle \geq 0,$$
$$\forall y^* \in F. \tag{34}$$

In (33), we take $y^* = x'_t$ to get

$$\langle Ax_t - t\gamma f(x_t) - (I - tB)Sx_t, x'_t - x_t \rangle \geq 0. \tag{35}$$

In (34), we take $y^* = x_t$ to get

$$\langle Ax'_t - t\gamma f(x'_t) - (I - tB)Sx'_t, x_t - x'_t \rangle \geq 0. \tag{36}$$

Adding up (35) and (36) yields

$$\langle Ax_t - Ax'_t, x_t - x'_t \rangle - t\gamma \langle f(x_t) - f(x'_t), x_t - x'_t \rangle$$
$$- (I - tB)\langle Sx_t - Sx'_t, x_t - x'_t \rangle \leq 0. \tag{37}$$

At the same time we note that

$$\langle Ax_t - Ax'_t, x_t - x'_t \rangle - t\gamma \langle f(x_t) - f(x'_t), x_t - x'_t \rangle$$
$$- (I - tB)\langle Sx_t - Sx'_t, x_t - x'_t \rangle$$
$$\geq [\tilde{\lambda} - 1 + (\tilde{\gamma} - \gamma\rho)t]\|x_t - x'_t\|^2$$
$$\geq 0. \tag{38}$$

Therefore, by (37) and (38), we deduce

$$x'_t = x_t. \tag{39}$$

Hence the entire net $\{x_{s,t}\}$ converges in norm to $x_t \in F$ as $s \to 0+$.

As $t \to 0+$, the net $\{x_t\}$ converges to the unique solution x^* of Problem 1.

In (33), we take any $y^* \in \Omega$ to deduce

$$\langle Ax_t - t\gamma f(x_t) - (I - tB) Sx_t, y^* - x_t \rangle \geq 0. \tag{40}$$

By virtue of the monotonicity of $A-S$ and the fact that $y^* \in \Omega$, we have

$$\langle Ax_t - Sx_t, y^* - x_t \rangle \leq \langle Ay^* - Sy^*, y^* - x_t \rangle \leq 0. \tag{41}$$

We can rewrite (40) as

$$\langle t[Ax_t - \gamma f(x_t) - (I - B) Sx_t] \\ + (1 - t)(Ax_t - Sx_t), y^* - x_t \rangle \geq 0. \tag{42}$$

It follows from (41) and (42) that

$$\langle Ax_t - \gamma f(x_t) - (I - B) Sx_t, y^* - x_t \rangle \geq 0, \quad \forall y^* \in \Omega. \tag{43}$$

Hence

$$\begin{aligned}
\|x_t - y^*\|^2 &\leq \langle x_t - y^*, x_t - y^* \rangle \\
&\quad + \langle \gamma f(x_t) + (I - B) Sx_t - Ax_t, x_t - y^* \rangle \\
&= \gamma \langle f(x_t) - f(y^*), x_t - y^* \rangle \\
&\quad + (I - B) \langle Sx_t - Sy^*, x_t - y^* \rangle \\
&\quad + \langle Ay^* - Ax_t, x_t - y^* \rangle \\
&\quad + \langle \gamma f(y^*) + (I - B) Sy^* - Ay^*, x_t - y^* \rangle \\
&\leq \gamma\rho\|x_t - y^*\|^2 + (1 - \tilde\gamma)\|x_t - y^*\|^2 \\
&\quad - \tilde\lambda\|x_t - y^*\|^2 \\
&\quad + \langle \gamma f(y^*) + (I - B) Sy^* - Ay^*, x_t - y^* \rangle \\
&= [1 - (\tilde\lambda + \tilde\gamma - \gamma\rho)]\|x_t - y^*\|^2 \\
&\quad + \langle \gamma f(y^*) + (I - B) Sy^* - Ay^*, x_t - y^* \rangle.
\end{aligned} \tag{44}$$

Therefore,

$$\|x_t - y^*\|^2 \leq \frac{1}{\tilde\lambda + \tilde\gamma - \gamma\rho} \\ \times \langle \gamma f(y^*) + (I - B) Sy^* - Ay^*, x_t - y^* \rangle, \\ y^* \in \Omega. \tag{45}$$

In particular,

$$\|x_t - y^*\| \leq \frac{1}{\tilde\lambda + \tilde\gamma - \gamma\rho} \\ \times \|\gamma f(y^*) + (I - B) Sy^* - Ay^*\|, \\ \forall t \in (0, 1), \tag{46}$$

which implies that $\{x_t\}$ is bounded.

We next prove that $\omega_w(x_t) \subset \Omega$; namely, if (t_n) is a null sequence in $(0, 1)$ such that $x_{t_n} \to x'$ weakly as $n \to \infty$, then $x' \in \Omega$. To see this, we use (33) to get

$$\langle (A - S) x_t, y^* - x_t \rangle \\ \geq \frac{t}{1 - t} \langle \gamma f(x_t) + (I - B) Sx_t \\ - Ax_t, y^* - x_t \rangle, \quad y^* \in F. \tag{47}$$

However, since $A - S$ is monotone,

$$\langle (A - S) y^*, y^* - x_t \rangle \geq \langle (A - S) x_t, y^* - x_t \rangle. \tag{48}$$

Combining the last two relations yields

$$\langle (A - S) y^*, y^* - x_t \rangle \\ \geq \frac{t}{1 - t} \langle \gamma f(x_t) + (I - B) Sx_t \\ - Ax_t, y^* - x_t \rangle, \quad y^* \in F. \tag{49}$$

Letting $t = t_n \to 0+$ as $n \to \infty$ in (49), we get

$$\langle (A - S) y^*, y^* - x' \rangle \geq 0, \quad y^* \in F. \tag{50}$$

The equivalent dual VI of (50) is

$$\langle (A - S) x', y^* - x' \rangle \geq 0, \quad y^* \in F. \tag{51}$$

Namely, x' is a solution of VI(13); hence $x' \in \Omega$.

We further prove that $x' = x^*$, the unique solution of VI(12). As a matter of fact, we have by (45)

$$\|x_{t_n} - x'\|^2 \leq \frac{1}{\tilde\lambda + \tilde\gamma - \gamma\rho} \\ \times \langle \gamma f(x') + (I - B) Sx' - Ax', x_{t_n} - x' \rangle, \\ x' \in \Omega. \tag{52}$$

Therefore, the weak convergence to x' of $\{x_{t_n}\}$ right implies that $x_{t_n} \to x'$ in norm. Now we can let $t = t_n \to 0$ in (45) to get

$$\langle \gamma f(y^*) + (I - B) Sy^* - Ay^*, y^* - x' \rangle \leq 0, \quad \forall y^* \in \Omega, \tag{53}$$

which is equivalent to its dual VI

$$\langle \gamma f(x') + (I - B) Sx' - Ax', y^* - x' \rangle \leq 0, \quad \forall y^* \in \Omega. \tag{54}$$

It turns out that $x' \in \Omega$ solves VI(12). By uniqueness, we have $x' = x^*$. This is sufficient to guarantee that $x_t \to x^*$ in norm, as $t \to 0+$. This completes the proof. $\qquad\square$

Corollary 6. *For each* $(s,t) \in (0,\kappa) \times (0,1)$, *let* $\{x_{s,t}\}$ *be a double net defined by*

$$x_{s,t} = P_C \left[s\left(t\gamma f\left(x_{s,t}\right) + (1 - tB) Sx_{s,t}\right) \right.$$
$$\left. + (1 - s)\frac{1}{\lambda_s} \int_0^{\lambda_s} T(\nu) x_{s,t} d\nu \right], \tag{55}$$

for all $(s,t) \in (0,\kappa) \times (0,1)$. *Then, for each fixed* $t \in (0,1)$, *the net* $\{x_{s,t}\}$ *defined by (55) converges in norm, as* $s \to 0+$, *to a solution* $x_t \in F$. *Moreover, as* $t \to 0+$, *the net* $\{x_t\}$ *converges in norm to* x^* *which solves the following variational inequality:*

$$x^* \in \Omega, \quad \left\langle (I - \gamma f) x^* - (I - B) Sx^*, x - x^* \right\rangle \geq 0,$$
$$\forall x \in \Omega, \tag{56}$$

where Ω *is the set of the solutions of the following variational inequality:*

$$x^* \in F, \quad \left\langle (I - S) x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in F. \tag{57}$$

Corollary 7. *For each* $(s,t) \in (0,\kappa) \times (0,1)$, *let* $\{x_{s,t}\}$ *be a double net defined by*

$$x_{s,t} = P_C \left[s\left((1 - t) Sx_{s,t}\right) + (1 - s)\frac{1}{\lambda_s} \int_0^{\lambda_s} T(\nu) x_{s,t} d\nu \right], \tag{58}$$

for all $(s,t) \in (0,\kappa) \times (0,1)$. *Then, for each fixed* $t \in (0,1)$, *the net* $\{x_{s,t}\}$ *defined by (58) converges in norm, as* $s \to 0+$, *to a solution* $x_t \in F$. *Moreover, as* $t \to 0+$, *the net* $\{x_t\}$ *converges to the minimum norm solution* x^* *of the following variational inequality:*

$$x^* \in F, \quad \left\langle (I - S) x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in F. \tag{59}$$

Proof. In (55), we take $f = 0$ and $B = I$. Then (55) reduces to (58). Hence, the net $\{x_t\}$ defined by (58) converges in norm to $x^* \in \Omega$ which satisfies

$$x^* \in \Omega, \quad \left\langle x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in \Omega. \tag{60}$$

This indicates that

$$\|x^*\|^2 \leq \left\langle x^*, x \right\rangle \leq \|x^*\| \|x\|, \quad \forall x \in \Omega. \tag{61}$$

Therefore, x^* is the minimum norm solution of the VI(59). This completes the proof. $\qquad\square$

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