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Research Article

Korovkin Second Theorem via *B*-Statistical *A*-Summability

M. Mursaleen[1] and A. Kiliçman[2]

[1] *Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India*
[2] *Department of Mathematics and Institute for Mathematical Research, University Putra Malaysia, 43400 Serdang, Selangor, Malaysia*

Correspondence should be addressed to A. Kiliçman; akilicman@putra.upm.edu.my

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Korovkin type approximation theorems are useful tools to check whether a given sequence $(L_n)_{n\geq 1}$ of positive linear operators on $C[0,1]$ of all continuous functions on the real interval $[0,1]$ is an approximation process. That is, these theorems exhibit a variety of test functions which assure that the approximation property holds on the whole space if it holds for them. Such a property was discovered by Korovkin in 1953 for the functions 1, x, and x^2 in the space $C[0,1]$ as well as for the functions 1, cos, and sin in the space of all continuous 2π-periodic functions on the real line. In this paper, we use the notion of B-statistical A-summability to prove the Korovkin second approximation theorem. We also study the rate of B-statistical A-summability of a sequence of positive linear operators defined from $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$.

1. Introduction and Preliminaries

Let $\mathbb{N}$ be the set of all natural numbers, $K \subseteq \mathbb{N}$, and $K_n = \{k \leq n : k \in K\}$. Then the *natural density* of K is defined by

$$\delta(K) = \lim_n \frac{1}{n}\left|K_n\right| = \lim_n (C_1\chi_K)_n, \tag{1}$$

if the limit exists, where the vertical bars indicate the number of elements in the enclosed set, $C_1 = (C,1)$ is the Cesàro matrix of order 1, and χ_K denotes the characteristic sequence of K given by

$$(\chi_K)_i = \begin{cases} 0, & \text{if } i \notin K, \\ 1, & \text{if } i \in K. \end{cases} \tag{2}$$

A sequence $x = (x_k)$ is said to be *statistically convergent* to L if for every $\varepsilon > 0$, the set $K_\varepsilon := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has natural density zero (cf. Fast [1]); that is, for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n}\left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right| = 0. \tag{3}$$

In this case, we write $L = \text{st} - \lim x$. By the symbol st we denote the set of all statistically convergent sequences. Statistical convergence of double sequences is studied in [2, 3].

A matrix $A = (a_{nk})_{n,k=0}^{\infty}$ is called *regular* if it transforms a convergent sequence into a convergent sequence leaving the limit invariant. The well-known necessary and sufficient conditions (Silverman-Toeplitz) for A to be regular are

(i) $\|A\| = \sup_n \sum_k |a_{nk}| < \infty$;

(ii) $\lim_n a_{nk} = 0$, for each k;

(iii) $\lim_n \sum_k a_{nk} = 1$.

Freedmann and Sember [4] generalized the natural density by replacing C_1 with an arbitrary nonnegative regular matrix A. A subset K of $\mathbb{N}$ has A-density if

$$\delta_A(K) = \lim_n \sum_{k\in K} a_{nk} \tag{4}$$

exists. Connor [5] and Kolk [6] extended the idea of statistical convergence to A-statistical convergence by using the notion of A-density.

A sequence x is said to be *A-statistically convergent* to L if $\delta_A(K_\epsilon) = 0$ for every $\epsilon > 0$. In this case we write $\text{st}_A - \lim x_k = L$. By the symbol st_A we denote the set of all A-statistically convergent sequences.

In [7], Edely and Mursaleen generalized these statistical summability methods by defining the statistical A-summability and studied its relationship with A-statistical convergence.

Let $A=(a_{ij})$ be a nonnegative regular matrix. A sequence x is said to be statistically A-summable to L if for every $\epsilon > 0$, $\delta(\{i \le n : |y_i - L| \ge \epsilon\}) = 0$; that is,

$$\lim_n \frac{1}{n} \left|\{i \le n : |y_i - L| \ge \epsilon\}\right| = 0, \quad (5)$$

where $y_i = A_i(x)$. Thus x is statistically A-summable to L if and only if Ax is statistically convergent to L. In this case we write $L = (A)_{\mathrm{st}} - \lim x = \mathrm{st} - \lim Ax$. By $(A)_{\mathrm{st}}$ we denote the set of all statistically A-summable sequences. A more general case of statistically A-summability is discussed in [8].

Quite recently, Edely [9] defined the concept of B-statistical A-summability for nonnegative regular matrices A and B which generalizes all the variants and generalizations of statistical convergence, for example, lacunary statistical convergence [10], λ-statistical convergence [11], A-statistical convergence [6], statistical A-summability [7], statistical $(C,1)$-summability [12], statistical $(H,1)$-summability [13], statistical $(\overline{N},p)$-summability [14], and so forth.

Let $A = (a_{ij})$ and $B = (b_{nk})$ be two nonnegative regular matrices. A sequence $x = (x_k)$ of real numbers is said to be B-statistically$A$-summable to L if for every $\epsilon > 0$, the set $K(\epsilon) = \{i : |y_i - L| \ge \epsilon\}$ has B-density zero, thus

$$\begin{aligned}\delta_B(K(\epsilon)) &= \lim_n \sum_{k\in K(\epsilon)} b_{nk} = \lim_n \left(B\chi_{K(\epsilon)}\right)\\ &= \lim_n \sum_k b_{nk}\chi_{K(\epsilon)}(k) = 0,\end{aligned} \quad (6)$$

where $y_i = A_i(x) = \sum_j a_{ij}x_j$. In this case we denote by $L = (A)_{\mathrm{st}_B} - \lim x = \mathrm{st}_B - \lim Ax$. The set of all B-statistically A-summable sequences will be denoted by $(A)_{\mathrm{st}_B}$.

Remark 1. (1) If $A = I$ (unit matrix), then $(A)_{\mathrm{st}_B}$ is reduced to the set of B-statistically convergent sequences which can be further reduced to lacunary statistical convergence and λ-statistical convergence for particular choice of the matrix B.

(2) If $B = (C,1)$ matrix, then $(A)_{\mathrm{st}_B}$ is reduced to the set of statistically A-summable sequences.

(3) If $A = B = (C,1)$ matrix, then $(A)_{\mathrm{st}_B}$ is reduced to the set of statistically $(C,1)$-summable sequences.

(4) If $B = (C,1)$ matrix and $A = (a_{jk})$ are defined by

$$a_{jk} = \begin{cases} \dfrac{p_k}{P_j} & \text{if } 0 \le k \le j,\\ 0 & \text{otherwise,}\end{cases} \quad (7)$$

then $(A)_{\mathrm{st}_B}$ is reduced to the set of statistically $(\overline{N},p)$-summable sequences, where $p = (p_k)$ is a sequence of nonnegative numbers, such that $p_0 > 0$ and

$$P_j = \sum_{k=0}^{j} p_k \longrightarrow \infty \quad (j \longrightarrow \infty). \quad (8)$$

(5) If $B = (C,1)$ matrix and $A = (a_{jk})$ are defined by

$$a_{jk} = \begin{cases} \dfrac{1}{kl_j} & \text{if } 0 \le k \le j,\\ 0 & \text{otherwise,}\end{cases} \quad (9)$$

where $l_j = \sum_{k=0}^{j}(1/(k+1))$, then $(A)_{\mathrm{st}_B}$ is reduced to the set of statistically $(H,1)$-summable sequences.

(6) If a sequence is convergent, then it is B-statistically A-summable, since Ax converges and has B-density zero, but not conversely.

(7) The spaces st, st_B, $(A)_{\mathrm{st}}$, and $(A)_{\mathrm{st}_B}$ are not comparable, even if $A = B(\neq (C,1))$.

(8) If a sequence is A-summable, then it is B-statistically A-summable.

(9) If a sequence is bounded and A-statistically convergent, then it is A-summable and hence statistically A-summable ([7], see Theorem 2.1) and B-statistically A-summable but not conversely.

Example 2. (1) Let us define $A = (a_{ij})$, $B = (b_{nk})$, and $x = (x_k)$ by

$$a_{ij} = \begin{cases} 1; & \text{if } j = i^2,\\ 0; & \text{otherwise,}\end{cases}$$

$$b_{nk} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & \cdot & \cdot \\ \frac{1}{5} & 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad (10)$$

$$x_k = \begin{cases} 1; & \text{if } k \text{ is odd},\\ 0; & \text{if } k \text{ is even}.\end{cases}$$

Then

$$\sum_{j=1}^{\infty} a_{ij}x_j = \begin{cases} 1; & \text{if } i \text{ is odd},\\ 0; & \text{if } i \text{ is even}.\end{cases} \quad (11)$$

Here $x \notin \mathrm{st}$, $x \notin (A)_{\mathrm{st}}$, $x \notin \mathrm{st}_A$, and $x \notin (A)_{\mathrm{st}_A}$, but x is B-statistically A-summable to 1, since $\delta_B\{i : |y_i - 1| \ge \epsilon\} = 0$. On the other hand we can see that x is B-summable and hence x is B-statistically B-summable, A-statistically B-summable, B-statistically convergent, and statistically B-summable.

Let $F(\mathbb{R})$ denote the linear space of all real-valued functions defined on $\mathbb{R}$. Let $C(\mathbb{R})$ be the space of all functions

f continuous on $\mathbb{R}$. We know that $C(\mathbb{R})$ is a Banach space with norm

$$\|f\|_\infty := \sup_{x\in\mathbb{R}} |f(x)|, \quad f \in C(\mathbb{R}). \tag{12}$$

We denote by $C_{2\pi}(\mathbb{R})$ the space of all 2π-periodic functions $f \in C(\mathbb{R})$ which is a Banach space with

$$\|f\|_{2\pi} = \sup_{t\in\mathbb{R}} |f(t)|. \tag{13}$$

The classical Korovkin first and second theorems statewhatfollows [15, 16]:

Theorem I. *Let (T_n) be a sequence of positive linear operators from $C[0,1]$ into $F[0,1]$. Then $\lim_n\|T_n(f,x) - f(x)\|_\infty = 0$, for all $f \in C[0,1]$ if and only if $\lim_n\|T_n(f_i,x) - e_i(x)\|_\infty = 0$, for $i = 0,1,2$, where $e_0(x) = 1$, $e_1(x) = x$, and $e_2(x) = x^2$.*

Theorem II. *Let (T_n) be a sequence of positive linear operators from $C_{2\pi}(\mathbb{R})$ into $F(\mathbb{R})$. Then $\lim_n\|T_n(f,x) - f(x)\|_\infty = 0$, for all $f \in C_{2\pi}(\mathbb{R})$ if and only if $\lim_n\|T_n(f_i,x) - f_i(x)\|_\infty = 0$, for $i = 0,1,2$, where $f_0(x) = 1$, $f_1(x) = \cos x$, and $f_2(x) = \sin x$.*

We write $L_n(f;x)$ for $L_n(f(s);x)$, and we say that L is a positive operator if $L(f;x) \geq 0$ for all $f(x) \geq 0$.

The following result was studied by Duman [17] which is A-statistical analogue of Theorem II.

Theorem A. *Let $A = (a_{nk})$ be a nonnegative regular matrix, and let (T_k) be a sequence of positive linear operators from $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$. Then for all $f \in C_{2\pi}(\mathbb{R})$*

$$\mathrm{st}_A - \lim_{k\to\infty} \|T_k(f;x) - f(x)\|_{2\pi} = 0 \tag{14}$$

if and only if

$$\begin{gathered} \mathrm{st}_A - \lim_{k\to\infty} \|T_k(1;x) - 1\|_{2\pi} = 0, \\ \mathrm{st}_A - \lim_{k\to\infty} \|T_k(\cos t;x) - \cos x\|_{2\pi} = 0, \\ \mathrm{st}_A - \lim_{k\to\infty} \|T_k(\sin t;x) - \sin x\|_{2\pi} = 0. \end{gathered} \tag{15}$$

Recently, Karakuş and Demirci [18] proved Theorem II for statistical A-summability.

Theorem B. *Let $A = (a_{nk})$ be a nonnegative regular matrix, and let (T_k) be a sequence of positive linear operators from $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$. Then for all $f \in C_{2\pi}(\mathbb{R})$*

$$\mathrm{st} - \lim_{n\to\infty} \left\| \sum_k a_{nk} T_k(f;x) - f(x) \right\|_{2\pi} = 0 \tag{16}$$

if and only if

$$\begin{gathered} \mathrm{st} - \lim_{n\to\infty} \left\| \sum_k a_{nk} T_k(1;x) - 1 \right\|_{2\pi} = 0, \\ \mathrm{st} - \lim_{n\to\infty} \left\| \sum_k a_{nk} T_k(\cos t;x) - \cos x \right\|_{2\pi} = 0, \\ \mathrm{st} - \lim_{n\to\infty} \left\| \sum_k a_{nk} T_k(\sin t;x) - \sin x \right\|_{2\pi} = 0. \end{gathered} \tag{17}$$

Several mathematicians have worked on extending or generalizing the Korovkin's theorems in many ways and to several settings, including function spaces, abstract Banach lattices, Banach algebras, and Banach spaces. This theory is very useful in real analysis, functional analysis, harmonic analysis, measure theory, probability theory, summability theory, and partial differential equations. But the foremost applications are concerned with constructive approximation theory which uses it as a valuable tool. Even today, the development of Korovkin-type approximation theory is far frombeingcomplete. Note that the first and the second theorems of Korovkin are actually equivalent to the algebraic and the trigonometric version, respectively, of the classical Weierstrass approximation theorem [19]. Recently, such type of approximation theorems has been proved by many authors by using the concept of statistical convergence and its variants, for example, [20–28]. Further Korovkin type approximation theorems for functions of two variables are proved in [29–32]. In [29, 33] authors have used the concept of almost convergence. In this paper, we prove Korovkin second theorem by applying the notion of B-statistical A-summability. We give here an example to justify that our result is stronger than Theorems II, A, and B. We also study the rate of B-statistical A-summability of a sequence of positive linear operators defined from $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$.

2. Main Result

Now, we prove Theorem II for B-statistically A-summability.

Theorem 3. *Let $A = (a_{nk})$ and $B = (b_{nk})$ be nonnegative regular matrices, and let (T_k) be a sequence of positive linear operators from $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$. Then for all $f \in C_{2\pi}(\mathbb{R})$*

$$\mathrm{st}_B - \lim_{n\to\infty} \left\| \sum_k a_{nk} T_k(f;x) - f(x) \right\|_{2\pi} = 0 \tag{18}$$

if and only if

$$\begin{gathered} \mathrm{st}_B - \lim_{n\to\infty} \left\| \sum_k a_{nk} T_k(1;x) - 1 \right\|_{2\pi} = 0, \\ \mathrm{st}_B - \lim_{n\to\infty} \left\| \sum_k a_{nk} T_k(\cos t;x) - \cos x \right\|_{2\pi} = 0, \\ \mathrm{st}_B - \lim_{n\to\infty} \left\| \sum_k a_{nk} T_k(\sin t;x) - \sin x \right\|_{2\pi} = 0. \end{gathered} \tag{19}$$

Proof. Since each of 1, $\cos x$, and $\sin x$ belongs to $C_{2\pi}(\mathbb{R})$, conditions (19) follow immediately from (18). Let the conditions (19) hold and $f \in C_{2\pi}(\mathbb{R})$. Let I be a closed subinterval of length 2π of $\mathbb{R}$. Fix $x \in I$. By the continuity of f at x, it follows that for given $\varepsilon > 0$ there is a number $\delta > 0$, such that for all t

$$|f(t) - f(x)| < \varepsilon, \tag{20}$$

whenever $|t - x| < \delta$. Since f is bounded, it follows that

$$|f(t) - f(x)| \le 2\|f\|_{2\pi}, \tag{21}$$

for all $t \in \mathbb{R}$. For all $t \in (x - \delta, 2\pi + x - \delta]$, it is well known that

$$|f(t) - f(x)| < \varepsilon + \frac{2\|f\|_{2\pi}}{\sin^2(\delta/2)}\psi(t), \tag{22}$$

where $\psi(t) = \sin^2((t - x)/2)$. Since the function $f \in C_{2\pi}(\mathbb{R})$ is 2π-periodic, the inequality (22) holds for $t \in \mathbb{R}$.

Now, operating $T_k(1; x)$ to this inequality, we obtain

$$\begin{aligned} &|T_k(f; x) - f(x)| \\ &\quad \le (\varepsilon + |f(x)|)|T_k(1; x) - 1| + \varepsilon \\ &\quad + \frac{\|f\|_{2\pi}}{\sin^2(\delta/2)}\{|T_k(1; x) - 1| + |\cos x| \\ &\qquad \times |T_k(\cos t; x) - \cos x| \\ &\qquad + |\sin x|\,|T_k(\sin t; x) - \sin x|\} \\ &\quad \le \varepsilon + \left(\varepsilon + |f(x)| + \frac{\|f\|_{2\pi}}{\sin^2(\delta/2)}\right) \\ &\quad \times \{|T_k(1; x) - 1| + |T_k(\cos t; x) - \cos x| \\ &\qquad + |T_k(\sin t; x) - \sin x|\}. \end{aligned} \tag{23}$$

Now, taking $\sup_{x\in I}$, we get

$$\begin{aligned} &\|T_k(f; x) - f(x)\|_\infty \\ &\quad \le \varepsilon + K(\|T_k(1; x) - 1\|_{2\pi} + \|T_k(\cos t; x) - \cos x\|_{2\pi} \\ &\qquad + \|T_k(\sin t; x) - \sin x\|_{2\pi}), \end{aligned} \tag{24}$$

where $K := \varepsilon + \|f\|_{2\pi} + (\|f\|_{2\pi}/\sin^2(\delta/2))$. Now replace $T_k(\cdot, x)$ by $\sum_k a_{mk}T_k(\cdot, x)$ and then by $B_m(\cdot, x)$ in (24) on both sides. For a given $r > 0$ choose $\varepsilon' > 0$, such that $\varepsilon' < r$. Define the following sets

$$\begin{aligned} D &= \{m \le n : \|B_m(f, x) - f(x)\|_{2\pi} \ge r\}, \\ D_1 &= \left\{m \le n : \|B_m(f_1, x) - f_0\|_{2\pi} \ge \frac{r - \varepsilon'}{3K}\right\}, \\ D_2 &= \left\{m \le n : \|B_m(f_2, x) - f_1\|_{2\pi} \ge \frac{r - \varepsilon'}{3K}\right\}, \\ D_3 &= \left\{m \le n : \|B_m(f_3, x) - f_2\|_{2\pi} \ge \frac{r - \varepsilon'}{3K}\right\}. \end{aligned} \tag{25}$$

Then $D \subset D_1 \cup D_2 \cup D_3$, and so $\delta_B(D) \le \delta_B(D_1) + \delta_B(D_2) + \delta_B(D_3)$. Therefore, using conditions (19) we get (18).

This completes the proof of the theorem. □

3. Rate of B-Statistical A-Summability

In this section, we study the rate of B-statistical A-summability of a sequence of positive linear operators defined from $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$.

Definition 4. Let $A = (a_{ij})$ and $B = (b_{nk})$ be two nonnegative regular matrices. Let (α_n) be a positive nonincreasing sequence. We say that the sequence $x = (x_k)$ is B-statistically A-summable to the number L with the rate $o(\alpha_n)$ if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{\alpha_n} \sum_{k\in K(\epsilon)} b_{nk} = 0, \tag{26}$$

where $K(\epsilon) = \{i : |y_i - L| \ge \epsilon\}$ and $y_i = A_i(x) = \sum_j a_{ij}x_j$ as described above. In this case, we write $x_k - L = (A)_{\mathrm{st}_B} - o(\alpha_n)$.

As usual we have the following auxiliary result whose proof is standard.

Lemma 5. *Let (α_n) and (β_n) be two positive nonincreasing sequences. Let $x = (x_k)$ and $y = (y_k)$ be two sequences, such that $x_k - L_1 = (A)_{\mathrm{st}_B} - o(\alpha_n)$ and $y_k - L_2 = (A)_{\mathrm{st}_B} - o(\beta_n)$. Then*

(i) $c(x_k - L_1) = (A)_{\mathrm{st}_B} - o(\alpha_n)$, *for any scalar c,*

(ii) $(x_k - L_1) \pm (y_k - L_2) = (A)_{\mathrm{st}_B} - o(\gamma_n)$,

(iii) $(x_k - L_1)(y_k - L_2) = (A)_{\mathrm{st}_B} - o(\alpha_n\beta_n)$,

where $\gamma_n = \max\{\alpha_n, b_n\}$.

Now, we recall the notion of modulus of continuity. The modulus of continuity of $f \in C_{2\pi}(\mathbb{R})$, denoted by $\omega(f, \delta)$, is defined by

$$\omega(f, \delta) = \sup_{|x-y|<\delta} |f(x) - f(y)|. \tag{27}$$

It is well known that

$$|f(x) - f(y)| \le \omega(f, \delta)\left(\frac{|x - y|}{\delta} + 1\right). \tag{28}$$

Then prove the following result.

Theorem 6. *Let (T_k) be a sequence of positive linear operators from $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$. Suppose that*

(i) $\|T_k(1; x) - 1\|_{2\pi} = (A)_{\mathrm{st}_B} - o(\alpha_n)$,

(ii) $\omega(f, \lambda_k) = (A)_{\mathrm{st}_B} - o(\beta_n)$, *where* $\lambda_k = \sqrt{T_k(\varphi_x; x)}$ *and* $\varphi_x(y) = \sin^2((y - x)/2)$.

Then for all $f \in C_{2\pi}(\mathbb{R})$, we have

$$\|T_k(f; x) - f(x)\|_{2\pi} = (A)_{\mathrm{st}_B} - o(\gamma_n), \tag{29}$$

where $\gamma_n = \max\{\alpha_n, \beta_n\}$.

Proof. Let $f \in C_{2\pi}(\mathbb{R})$ and $x \in [-\pi, \pi]$. Using (28), we have

$$\begin{aligned}
|T_k(f;x) - f(x)| &\le T_k(|f(y) - f(x)|; x) \\
&\quad + |f(x)| \, |T_k(1;x) - 1| \\
&\le T_k\left(\frac{|x-y|}{\delta} + 1; x\right)\omega(f,\delta) \\
&\quad + |f(x)| \, |T_k(1;x) - 1| \\
&\le T_k\left(1 + \frac{\pi^2}{\delta^2}\sin^2\left(\frac{y-x}{2}\right); x\right)\omega(f,\delta) \\
&\quad + |f(x)| \, |T_k(1;x) - 1| \\
&\le \left(T_k(1;x) + \frac{\pi^2}{\delta^2}T_k(\varphi_x;x)\right)\omega(f,\delta) \\
&\quad + |f(x)| \, |T_k(1;x) - 1|.
\end{aligned} \tag{30}$$

Put $\delta = \lambda_k = \sqrt{T_k(\varphi_x;x)}$. Hence we get

$$\begin{aligned}
&\|T_k(f;x) - f(x)\|_{2\pi} \\
&\quad \le \|f\|_{2\pi}\|T_k(1;x) - 1\|_{2\pi} + (1+\pi^2)\,\omega(f,\lambda_k) \\
&\qquad + \omega(f,\lambda_k)\,\|T_k(1;x) - 1\|_{2\pi} \\
&\quad \le K\{\|T_k(1;x) - 1\|_{2\pi} + \omega(f,\lambda_k) \\
&\qquad + \omega(f,\lambda_k)\,\|T_k(1;x) - 1\|_{2\pi}\},
\end{aligned} \tag{31}$$

where $K = \max\{\|f\|_{2\pi}, 1+\pi^2\}$. Hence

$$\begin{aligned}
&\|T_k(f;x) - f(x)\|_{2\pi} \\
&\quad \le K\{\|T_k(1;x) - 1\|_{2\pi} + \omega(f,\lambda_k) + \omega(f,\lambda_k) \\
&\qquad \times \|T_k(1;x)\,p_k - 1\|_{2\pi}\}.
\end{aligned} \tag{32}$$

Now, using Definition 4 and Conditions (i) and (ii), we get the desired result.

This completes the proof of the theorem. □

4. Example and Concluding Remark

In the following we construct an example of a sequence of positive linear operators satisfying the conditions of Theorem 3 but does not satisfy the conditions of Theorems II, A, and B.

For any $n \in \mathbb{N}$, denote by $S_n(f)$ the nth partial sum of the Fourier series of f; that is,

$$S_n(f)(x) = \frac{1}{2}a_0(f) + \sum_{k=1}^{n} a_k(f)\cos kx + b_k(f)\sin kx. \tag{33}$$

For any $n \in \mathbb{N}$, write

$$F_n(f) := \frac{1}{n+1}\sum_{k=0}^{n} S_k(f). \tag{34}$$

A standard calculation gives that for every $t \in \mathbb{R}$

$$\begin{aligned}
F_n(f;x) &:= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\frac{1}{n+1}\sum_{k=0}^{n}\frac{\sin((2k+1)(x-t)/2)}{\sin((x-t)/2)}\,dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\frac{1}{n+1}\sum_{k=0}^{n}\frac{\sin^2((n+1)(x-t)/2)}{\sin^2((x-t)/2)}\,dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\varphi_n(x-t)\,dt,
\end{aligned} \tag{35}$$

where

$$\varphi_n(x) := \begin{cases} \dfrac{\sin^2((n+1)(x-t)/2)}{(n+1)\sin^2((x-t)/2)} & \text{if } x \text{ is not a multiple of } 2\pi, \\ n+1 & \text{if } x \text{ is a multiple of } 2\pi. \end{cases} \tag{36}$$

The sequence $(\varphi_n)_{n\in\mathbb{N}}$ is a positive kernel which is called the *Fejér kernel*, and the corresponding operators F_n, $n \ge 1$, are called the *Fejér convolution operators*. We have

$$\begin{aligned}
F_n(1;x) &= 1, \\
F_n(\cos t;x) &= \left(\frac{n}{n+1}\right)\cos x, \\
F_n(\sin t;x) &= \left(\frac{n}{n+1}\right)\sin x.
\end{aligned} \tag{37}$$

Note that the Theorems II, A, and B hold for the sequence (F_n). In fact, we have for every $f \in C_{2\pi}(\mathbb{R})$,

$$\lim_{n\to\infty} F_n(f) = f. \tag{38}$$

Let $A = (a_{ij})$, $B = (b_{nk})$, and $x = (x_k)$ be defined as in Example 2. Let $L_n : C_{2\pi}(\mathbb{R}) \to C_{2\pi}(\mathbb{R})$ be defined by

$$L_n(f;x) = x_n F_n(f;x). \tag{39}$$

Then x is not statistically convergent, not A-statistically convergent, and not statistically A-summable, but it is B-statistically A–summable to 1. Since x is B-statistically A-summable to 1, it is easy to see that the operator L_n satisfies the conditions (19), and hence Theorem 3 holds. But on the other hand, Theorems II, A, and B do not hold for our operator defined by (39), since x (and so L_n) is not statistically convergent, not A-statistically convergent, and not statistically A-summable.

Hence our Theorem 3 is stronger than all the above three theorems.

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