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Research Article

Conservation Laws of Two (2 + 1)-Dimensional Nonlinear Evolution Equations with Higher-Order Mixed Derivatives

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In this paper, conservation laws for the (2 + 1)-dimensional ANNV equation and KP-BBM equation with higher-order mixed derivatives are studied. Due to the existence of higher-order mixed derivatives, Ibragimov's "new conservation theorem" cannot be applied to the two equations directly. We propose two modification rules which ensure that the theorem can be applied to nonlinear evolution equations with any mixed derivatives. Formulas of conservation laws for the ANNV equation and KP-BBM equation are given. Using these formulas, many nontrivial and time-dependent conservation laws for these equations are derived.

1. Introduction

The construction of explicit forms of conservation laws plays an important role in the study of nonlinear science, as they are used for the development of appropriate numerical methods and for mathematical analysis, in particular, existence, uniqueness, and stability analysis [1–3]. In addition, the existence of a large number of conservation laws of a partial differential equation (system) is a strong indication of its integrability. The famous Noether's theorem [4] has provided a systematic way of determining conservation laws for Euler-Lagrange equations, once their Noether symmetries are known, but this theorem relies on the availability of classical Lagrangians. To find conservation laws of differential equations without classical Lagrangians, researchers have made various generalizations of Noether's theorem [5–16]. Among those, the new conservation theorem given by Ibragimov [5] is one of the most frequently used methods.

For any linear or nonlinear differential equations, Ibragimov's new conservation theorem offers a procedure for constructing explicit conservation laws associated with the known Lie, Lie-Backlund, or nonlocal symmetries. Furthermore, it does not require the existence of classical Lagrangians. Using the conservation laws formulas given by the theorem, conservation laws for lots of equations have been studied [6–16]. When applying Ibragimov's theorem to a given nonlinear evolution equation with mixed derivatives, we must be careful with the mixed derivatives. If we apply the conservation laws formulas to equations with mixed derivatives directly, it will result in errors. In [9], we have proposed two modification rules to apply Ibragimov's theorem to study conservation laws of two evolution equations with mixed derivatives, but the mixed derivatives are all second order and not the highest derivative term. In this paper, we will give two new modification rules and then use Ibragimov's theorem to study conservation laws of the following ANNV equation [17–20]:

$$u_{yt} + u_{xxxy} - 3u_{xx}u_y - 3u_x u_{xy} = 0, \tag{1}$$

and KP-BBM equation [21–24]

$$u_{xt} + u_{xx} - 2\alpha u_x^2 - 2\alpha u u_{xx} - \beta u_{xxxt} + \gamma u_{yy} = 0, \tag{2}$$

where α, β, and γ are constants. Both in (1) and (2), the highest derivative terms u_{xxxy} and u_{xxxt} are mixed. Furthermore, there are other lower-order mixed derivatives in addition to the higher-order mixed derivatives.

The rest of the paper is organized as follows. In Section 2, we recall the main concepts and theorems used in this paper. In Section 3, taking the ANNV equation as an example, we first give two new modification rules which ensure the theorem can be applied to nonlinear evolution equations with any mixed derivatives. Then formulas of conservation laws and explicit conservation laws for the ANNV equation are obtained. In Section 4, conservation laws for the KP-BBM equation with higher-order mixed derivative term are derived by means of Ibragimov's theorem and the two new modification rules. Some conclusions and discussions are given in Section 5.

2. Preliminaries

In this section, we briefly present the main notations and theorems [5–7] used in this paper. Consider an sth-order nonlinear evolution equation

$$F\left(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(s)}\right) = 0, \tag{3}$$

with n independent variables $x = (x_1, x_2, \ldots, x_n)$ and a dependent variable u, where $u_{(1)}, u_{(2)}, \ldots, u_{(s)}$ denote the collection of all first-, second-, …, sth-order partial derivatives. $u_i = D_i(u)$, $u_{ij} = D_j D_i(u), \ldots$. Here

$$D_i = \frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots, \quad i = 1, 2, \ldots, n, \tag{4}$$

is the total differential operator with respect to x_i.

Definition 1. The adjoint equation of (3) is defined by

$$F^*\left(x, u, v, u_{(1)}, v_{(1)}, u_{(2)}, v_{(2)}, \ldots, u_{(s)}, v_{(s)}\right) = 0, \tag{5}$$

with

$$F^*\left(x, u, v, u_{(1)}, v_{(1)}, u_{(2)}, v_{(2)}, \ldots, u_{(s)}, v_{(s)}\right) = \frac{\delta(vF)}{\delta u}, \tag{6}$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{m=1}^{\infty} (-1)^m D_{i_1} \cdots D_{i_2} \frac{\partial}{\partial u_{i_1 i_2 \cdots i_m}} \tag{7}$$

denotes the Euler-Lagrange operator, v is a new dependent variable, and $v = v(x)$.

Theorem 2. *The system consisting of* (3) *and its adjoint equation* (5),

$$\begin{gathered} F\left(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(s)}\right) = 0, \\ F^*\left(x, u, v, u_{(1)}, v_{(1)}, u_{(2)}, v_{(2)}, \ldots, u_{(s)}, v_{(s)}\right) = 0, \end{gathered} \tag{8}$$

has a formal Lagrangian; namely,

$$L = vF\left(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(s)}\right). \tag{9}$$

In the following we recall the "new conservation theorem" given by Ibragimov in [5].

Theorem 3. *Any Lie point, Lie-Backlund, and nonlocal symmetries,*

$$V = \xi^i \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial u}, \tag{10}$$

of (3) *provide a conservation law* $D_i(T^i) = 0$ *for the system* (8). *The conserved vector is given by*

$$\begin{aligned} T^i = \xi^i L &+ W\left(\frac{\partial L}{\partial u_i} - D_j\left(\frac{\partial L}{\partial u_{ij}}\right) + D_j D_k \left(\frac{\partial L}{\partial u_{ijk}}\right) - D_j D_k D_r \left(\frac{\partial L}{\partial u_{ijkr}}\right) + \cdots\right) \\ &+ D_j W \left(\frac{\partial L}{\partial u_{ij}} - D_k\left(\frac{\partial L}{\partial u_{ijk}}\right) + D_k D_r \left(\frac{\partial L}{\partial u_{ijkr}}\right) - \cdots\right) \\ &+ D_j D_k W \left(\frac{\partial L}{\partial u_{ijk}} - D_r\left(\frac{\partial L}{\partial u_{ijkr}}\right) + \cdots\right) + \cdots, \end{aligned} \tag{11}$$

where L is determined by (9), *W is the Lie characteristic function, and*

$$W = \eta - \xi^j u_j. \tag{12}$$

3. Two Modification Rules and Conservation Laws for the ANNV Equation

The asymmetric Nizhnik-Novikov-Veselov (ANNV) equation (1) is equivalent to the ANNV system [17, 18]

$$p_t + p_{xxx} - 3q_x p - 3q p_x = 0, \qquad p_x = q_y, \tag{13}$$

by the transformation $q = u_x$, $p = u_y$. A series of new double periodic solutions to the system (13) were derived in [17], and the variable separation solutions of (13) have been given in [18]. The Lie symmetry, reductions, and new exact solutions of the ANNV equation (1) have been studied by us from the point of Lax pair [19]. Optimal system of group-invariant solutions and conservation laws of (1) have been studied by Wang et al. [20]. In the following, we will study the conservation laws of (1) by Theorem 3.

3.1. Two Modification Rules and Formulas of Conservation Laws for the ANNV Equation.

To search for conservation laws of (1) by Theorem 3, Lie symmetry, formal Lagrangian, and adjoint equation of (1) must be known. According to Definition 1, the adjoint equation of (1) is

$$v_{yt} + v_{yxxx} - 6u_{xy} v_x - 3u_y v_{xx} - 3u_x v_{xy} = 0, \tag{14}$$

where v is a new dependent variable with respect to x, y, and t.

According to Theorem 2, the formal Lagrangian for the system consisting of (1) and (14) is

$$L = \left(u_{yt} + u_{yxxx} - 3u_{xx} u_y - 3u_x u_{xy}\right) v, \tag{15}$$

where v is a solution of (14).

Suppose that the Lie symmetry for the ANNV equation (1) is as follows:

$$V = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \tau \frac{\partial}{\partial t} + \phi \frac{\partial}{\partial u}. \tag{16}$$

From Theorem 3, we get the general formula of conservation laws for the system consisting of (1) and (14):

$$X = \xi L + W\left(\frac{\partial L}{\partial u_x} - D_x\left(\frac{\partial L}{\partial u_{xx}}\right) - D_y\left(\frac{\partial L}{\partial u_{xy}}\right) - D_{xxy}\left(\frac{\partial L}{\partial u_{xxxy}}\right)\right) + D_x(W)\left(\frac{\partial L}{\partial u_{xx}} + D_{xy}\left(\frac{\partial L}{\partial u_{xxxy}}\right)\right) + D_y(W)\left(\frac{\partial L}{\partial u_{xy}} + D_{xx}\left(\frac{\partial L}{\partial u_{xxxy}}\right)\right) + D_{xx}(W)\left(-D_y\left(\frac{\partial L}{\partial u_{xxxy}}\right)\right) + D_{xy}(W)\left(-D_x\left(\frac{\partial L}{\partial u_{xxxy}}\right)\right) + D_{xxy}(W)\left(\frac{\partial L}{\partial u_{xxxy}}\right),$$

$$Y = \eta L + W\left(\frac{\partial L}{\partial u_y} - D_x\left(\frac{\partial L}{\partial u_{xy}}\right) - D_t\left(\frac{\partial L}{\partial u_{yt}}\right) - D_{xxx}\left(\frac{\partial L}{\partial u_{xxxy}}\right)\right) + D_x(W)\left(\frac{\partial L}{\partial u_{xy}} + D_{xx}\left(\frac{\partial L}{\partial u_{xxxy}}\right)\right) + D_t(W)\left(\frac{\partial L}{\partial u_{yt}}\right) + D_{xx}(W)\left(-D_x\left(\frac{\partial L}{\partial u_{xxxy}}\right)\right) + D_{xxx}(W)\left(\frac{\partial L}{\partial u_{xxxy}}\right),$$

$$T = \tau L + W\left(-D_y\left(\frac{\partial L}{\partial u_{yt}}\right)\right) + D_y(W)\left(\frac{\partial L}{\partial u_{yt}}\right), \quad (17)$$

where W is the Lie characteristic function, $W = \phi - \xi u_x - \eta u_y - \tau u_t$, and L is the formal Lagrangian determined by (15).

In fact, because of the existence of the mixed derivative terms u_{xy}, u_{yt}, and u_{xxxy}, the general formula of conservation laws must be modified; otherwise the previous X, Y, and T do not satisfy

$$\left(D_x X + D_y Y + D_t T\right)\Big|_{u_{xxxy}=3u_{xx}u_y+3u_x u_{xy}-u_{yt}} = 0. \quad (18)$$

The rules of modifications are as follows.

(1) In one conservation vector (X, Y, or T), the time that one derivative with respect to a mixed derivative term appears is determined by the order of the derivative with respect to its independent variables. For example, whether in X or in Y, $\partial L/\partial u_{xy}$ can only appear once; $\partial L/\partial u_{xxxy}$ can only appear once in Y and can appear three times in X; $\partial L/\partial u_{xxxt}$ can only appear once in T and can appear three times in X; $\partial L/\partial u_{xxt}$ can only appear once in T and can appear two times in X.

(2) The location that one derivative with respect to a mixed derivative term appears at cannot be the same in different conservation vectors. That is to say, if there is $W(-D_y(\partial L/\partial u_{xy}))$ in X, then the term appears in Y can only be $D_x(W)(\partial L/\partial u_{xy})$ and the term $W(-D_x(\partial L/\partial u_{xy}))$ cannot appear in Y at the same time. And if there is $W(-D_{xxx}(\partial L/\partial u_{xxxy}))$ in Y, then the terms that appear in X contain $D_{xxy}(w)(\partial L/\partial u_{xxxy})$ and first and second total derivatives of $\partial L/\partial u_{xxxy}$.

Applying the two rules to the general conservation laws formula in Theorem 3, we can get the following results.

Theorem 4. *Suppose that the Lie symmetry of the ANNV equation* (1) *is expressed as* (16). *According to the different locations of* $\partial L/\partial u_{xy}$, $\partial L/\partial u_{yt}$, *and* $\partial L/\partial u_{xxxy}$, *the symmetry provides sixteen different conservation laws for the system consisting of* (1) *and* (14). *The conserved vectors are given as follows:*

$$\left(X_{ij}, Y_{ij}, T_{ij}\right) = \left(X^i, Y^i, T^i\right) + \left(B_j^X, B_j^Y, 0\right), \quad i, j = 1, 2, 3, 4, \quad (19)$$

with

$$X^1 = \xi L + W\left(\frac{\partial L}{\partial u_x} - D_x\left(\frac{\partial L}{\partial u_{xx}}\right)\right) + D_x(W)\left(\frac{\partial L}{\partial u_{xx}}\right) + W\left(-D_y\left(\frac{\partial L}{\partial u_{xy}}\right)\right),$$

$$Y^1 = \eta L + W\left(\frac{\partial L}{\partial u_y} - D_t\left(\frac{\partial L}{\partial u_{yt}}\right)\right) + D_x(W)\left(\frac{\partial L}{\partial u_{xy}}\right),$$

$$T^1 = \tau L + D_y(W)\left(\frac{\partial L}{\partial u_{yt}}\right),$$

$$X^2 = \xi L + W\left(\frac{\partial L}{\partial u_x} - D_x\left(\frac{\partial L}{\partial u_{xx}}\right)\right) + D_x(W)\left(\frac{\partial L}{\partial u_{xx}}\right) + W\left(-D_y\left(\frac{\partial L}{\partial u_{xy}}\right)\right),$$

$$Y^2 = \eta L + W\left(\frac{\partial L}{\partial u_y}\right) + D_x(W)\left(\frac{\partial L}{\partial u_{xy}}\right) + D_t(W)\left(\frac{\partial L}{\partial u_{yt}}\right),$$

$$T^2 = \tau L + W\left(-D_y\left(\frac{\partial L}{\partial u_{yt}}\right)\right),$$

$$X^3 = \xi L + W\left(\frac{\partial L}{\partial u_x} - D_x\left(\frac{\partial L}{\partial u_{xx}}\right)\right) + D_x(W)\left(\frac{\partial L}{\partial u_{xx}}\right) + D_y(W)\left(\frac{\partial L}{\partial u_{xy}}\right),$$

$$Y^3 = \eta L + W\left(\frac{\partial L}{\partial u_y} - D_x\left(\frac{\partial L}{\partial u_{xy}}\right) - D_t\left(\frac{\partial L}{\partial u_{yt}}\right)\right),$$

$$T^3 = \tau L + D_y(W)\left(\frac{\partial L}{\partial u_{yt}}\right),$$

$$X^4 = \xi L + W\left(\frac{\partial L}{\partial u_x} - D_x\left(\frac{\partial L}{\partial u_{xx}}\right)\right) + D_x(W)\left(\frac{\partial L}{\partial u_{xx}}\right) + D_y(W)\left(\frac{\partial L}{\partial u_{xy}}\right),$$

$$Y^4 = \eta L + W\left(\frac{\partial L}{\partial u_y} - D_x\left(\frac{\partial L}{\partial u_{xy}}\right)\right) + D_t(W)\left(\frac{\partial L}{\partial u_{yt}}\right),$$

$$T^4 = \tau L + W\left(-D_y\left(\frac{\partial L}{\partial u_{yt}}\right)\right),$$

$$B_1^X = D_{xxy}(W)\left(\frac{\partial L}{\partial u_{xxxy}}\right) + D_{xy}(W)\left(-D_x\left(\frac{\partial L}{\partial u_{xxxy}}\right)\right) + D_y(W)\left(D_{xx}\left(\frac{\partial L}{\partial u_{xxxy}}\right)\right),$$

$$B_1^Y = W\left(-D_{xxx}\left(\frac{\partial L}{\partial u_{xxxy}}\right)\right),$$

$$B_2^X = W\left(-D_{xxy}\left(\frac{\partial L}{\partial u_{xxxy}}\right)\right) + D_{xxy}(W)\left(\frac{\partial L}{\partial u_{xxxy}}\right) + D_{xy}(W)\left(-D_x\left(\frac{\partial L}{\partial u_{xxxy}}\right)\right),$$

$$B_2^Y = D_x(W)\left(D_{xx}\left(\frac{\partial L}{\partial u_{xxxy}}\right)\right),$$

$$B_3^X = W\left(-D_{xxy}\left(\frac{\partial L}{\partial u_{xxxy}}\right)\right) + D_{xxy}(W)\left(\frac{\partial L}{\partial u_{xxxy}}\right) + D_x(W)\left(D_{xy}\left(\frac{\partial L}{\partial u_{xxxy}}\right)\right),$$

$$B_3^Y = D_{xx}(W)\left(-D_x\left(\frac{\partial L}{\partial u_{xxxy}}\right)\right),$$

$$B_4^X = W\left(-D_{xxy}\left(\frac{\partial L}{\partial u_{xxxy}}\right)\right) + D_{xx}(W)\left(-D_y\left(\frac{\partial L}{\partial u_{xxxy}}\right)\right) + D_x(W)\left(D_{xy}\left(\frac{\partial L}{\partial u_{xxxy}}\right)\right),$$

$$B_4^Y = D_{xxx}(W)\left(\frac{\partial L}{\partial u_{xxxy}}\right), \tag{20}$$

where W is the Lie characteristic function and $W = \phi - \xi u_x - \eta u_y - \tau u_t$, L is the formal Lagrangian determined by (15).

3.2. Explicit Conservation Laws of the ANNV Equation. Now, conservation laws of (1) can be derived by Theorem 4 if Lie symmetries of (1) are known. In fact, Lie symmetries of (1) have been obtained in [19] and they are as follows:

$$V_1 = g(y)\frac{\partial}{\partial y}, \qquad V_2 = -F(t)\frac{\partial}{\partial u},$$
$$V_3 = f(t)\frac{\partial}{\partial x} - \frac{x f_t}{3}\frac{\partial}{\partial u}, \tag{21}$$
$$V_4 = \frac{x h_t}{3}\frac{\partial}{\partial x} + h(t)\frac{\partial}{\partial t} - \left(\frac{h_t}{3}u + \frac{h_{tt}}{18}x^2\right)\frac{\partial}{\partial u},$$

where $g(y)$, $F(t)$, $f(t)$, and $h(t)$ are arbitrary functions.

Using the Lie symmetry V_1 and Theorem 4, we can get sixteen conservation laws for the system consisting of (1) and (14). They are listed as follows:

$$X_{111} = -3g(y)u_y^2 v_x - 3g(y)u_y u_x v_y - v g_y u_{xxy} - v g(y) u_{xxyy} - v_{xx} g_y u_y - v_{xx} g(y) u_{yy} + v_x g_y u_{xy} + v_x g(y) u_{xyy},$$

$$Y_{111} = g(y) v u_{ty} + g(y) v u_{xxxy} + g(y) u_y v_t + g(y) u_y v_{xxx},$$

$$T_{111} = -v g_y u_y - v g(y) u_{yy},$$

$$X_{112} = -3g(y)u_y^2 v_x - 3g(y)u_y u_x v_y - v g_y u_{xxy} - v g(y) u_{xxyy} + v_{xxy} g(y) u_y + v_x g_y u_{xy} + g(y) v_x u_{xyy},$$

$$Y_{112} = g(y) v u_{ty} + g(y) v u_{xxxy} + g(y) u_y v_t - g(y) u_{xy} v_{xx},$$

$$T_{112} = -v g_y u_y - v g(y) u_{yy},$$

$$X_{113} = -3g(y)u_y^2v_x - 3g(y)u_yu_xv_y - vg_yu_{xxy} - vg(y)u_{xxyy} + v_{xxy}g(y)u_y - g(y)u_{xy}v_{xy},$$
$$Y_{113} = g(y)vu_{ty} + g(y)vu_{xxxy} + g(y)u_yv_t + g(y)u_{xxy}v_x,$$
$$T_{113} = -vg_yu_y - vg(y)u_{yy},$$
$$X_{114} = -3g(y)u_y^2v_x - 3g(y)u_yu_xv_y + v_{xxy}g(y)u_y + g(y)u_{xxy}v_y - g(y)u_{xy}v_{xy},$$
$$Y_{114} = g(y)vu_{ty} + g(y)u_yv_t,$$
$$T_{114} = -vg_yu_y - vg(y)u_{yy},$$
$$X_{211} = -3g(y)u_y^2v_x - 3g(y)u_yu_xv_y - vg_yu_{xxy} - vg(y)u_{xxyy} - v_{xx}g_yu_y - v_{xx}g(y)u_{yy} + v_xg_yu_{xy} + v_xg(y)u_{xyy},$$
$$Y_{211} = g(y)vu_{xxxy} + g(y)u_yv_{xxx},$$
$$T_{211} = g(y)u_yv_y,$$
$$X_{221} = -3g(y)u_y^2v_x - 3g(y)u_yu_xv_y - vg_yu_{xxy} - vg(y)u_{xxyy} + g(y)u_yv_{xxy} + v_xg_yu_{xy} + v_xg(y)u_{xyy},$$
$$Y_{221} = g(y)vu_{xxxy} - g(y)u_{xy}v_{xx},$$
$$T_{221} = g(y)u_yv_y,$$
$$X_{231} = -3g(y)u_y^2v_x - 3g(y)u_yu_xv_y - vg_yu_{xxy} - vg(y)u_{xxyy} + g(y)u_yv_{xxy} - g(y)u_{xy}v_{xy},$$
$$Y_{231} = g(y)vu_{xxxy} + g(y)u_{xxy}v_x,$$
$$T_{231} = g(y)u_yv_y,$$
$$X_{241} = -3g(y)u_y^2v_x - 3g(y)u_yu_xv_y + g(y)u_yv_{xxy} + g(y)u_{xxy}v_y - g(y)u_{xy}v_{xy},$$
$$Y_{241} = 0,$$
$$T_{241} = g(y)u_yv_y,$$
$$X_{311} = -3g(y)u_y^2v_x + 3g(y)u_{xy}u_yv + 3u_xvg_yu_y + 3u_xvg(y)u_{yy} - vg_yu_{xxy} - vg(y)u_{xxyy} - v_{xx}g_yu_y - v_{xx}g(y)u_{yy} + v_xg_yu_{xy} + v_xg(y)u_{xyy},$$
$$Y_{311} = g(y)vu_{ty} + g(y)vu_{xxxy} - 3g(y)vu_{xx}u_y - 3g(y)vu_xu_{xy} - 3g(y)u_yu_xv_x + g(y)u_yv_t + g(y)u_yv_{xxx},$$
$$T_{311} = -vg_yu_y - vg(y)u_{yy},$$
$$X_{321} = -3g(y)u_y^2v_x + 3g(y)u_{xy}u_yv + 3u_xvg_yu_y + 3u_xvg(y)u_{yy} - vg_yu_{xxy} - vg(y)u_{xxyy} + g(y)u_yv_{xxy} + v_xg_yu_{xy} + v_xg(y)u_{xyy},$$
$$Y_{321} = g(y)vu_{ty} + g(y)vu_{xxxy} - 3g(y)vu_{xx}u_y - 3g(y)vu_xu_{xy} - 3g(y)u_yu_xv_x + g(y)u_yv_t - g(y)u_{xy}v_{xx},$$
$$T_{321} = -vg_yu_y - vg(y)u_{yy},$$
$$X_{331} = -3g(y)u_y^2v_x + 3g(y)u_{xy}u_yv + 3u_xvg_yu_y + 3u_xvg(y)u_{yy} - vg_yu_{xxy} - vg(y)u_{xxyy} + g(y)u_yv_{xxy} - g(y)u_{xy}v_{xy},$$
$$Y_{331} = g(y)vu_{ty} + g(y)vu_{xxxy} - 3g(y)vu_{xx}u_y - 3g(y)vu_xu_{xy} - 3g(y)u_yu_xv_x + g(y)u_yv_t + g(y)u_{xxy}v_x,$$
$$T_{331} = -vg_yu_y - vg(y)u_{yy},$$
$$X_{341} = -3g(y)u_y^2v_x + 3g(y)u_{xy}u_yv + 3u_xvg_yu_y + 3u_xvg(y)u_{yy} + g(y)u_yv_{xxy} + g(y)u_{xxy}v_y - g(y)u_{xy}v_{xy},$$
$$Y_{341} = g(y)vu_{ty} - 3g(y)vu_{xx}u_y - 3g(y)vu_xu_{xy} - 3g(y)u_yu_xv_x + g(y)u_yv_t,$$
$$T_{341} = -vg_yu_y - vg(y)u_{yy},$$
$$X_{411} = -3g(y)u_y^2v_x + 3g(y)u_{xy}u_yv + 3u_xvg_yu_y + 3u_xvg(y)u_{yy} - vg_yu_{xxy} - vg(y)u_{xxyy} - v_{xx}g_yu_y - v_{xx}g(y)u_{yy} + v_xg_yu_{xy} + v_xg(y)u_{xyy},$$
$$Y_{411} = g(y)vu_{xxxy} - 3g(y)vu_{xx}u_y - 3g(y)vu_xu_{xy} - 3g(y)u_yu_xv_x + g(y)u_yv_{xxx},$$
$$T_{411} = g(y)u_yv_y,$$

$$\begin{aligned}
X_{421} =& -3g(y)u_y^2v_x + 3g(y)u_{xy}u_yv + 3u_xvg_yu_y \\
&+ 3u_xvg(y)u_{yy} - vg_yu_{xxy} - vg(y)u_{xxyy} \\
&+ g(y)u_yv_{xxy} + v_xg_yu_{xy} + v_xg(y)u_{xyy}, \\
Y_{421} =& g(y)vu_{xxxy} - 3g(y)vu_{xx}u_y - 3g(y)vu_xu_{xy} \\
&- 3g(y)u_yu_xv_x - g(y)u_{xy}v_{xx}, \\
T_{421} =& g(y)u_yv_y, \\
X_{431} =& -3g(y)u_y^2v_x + 3g(y)u_{xy}u_yv + 3u_xvg_yu_y \\
&+ 3u_xvg(y)u_{yy} - vg_yu_{xxy} - vg(y)u_{xxyy} \\
&+ g(y)u_yv_{xxy} - g(y)u_{xy}v_{xy}, \\
Y_{431} =& g(y)vu_{xxxy} - 3g(y)vu_{xx}u_y - 3g(y)vu_xu_{xy} \\
&- 3g(y)u_yu_xv_x + g(y)u_{xxy}v_x, \\
T_{431} =& g(y)u_yv_y, \\
X_{441} =& -3g(y)u_y^2v_x + 3g(y)u_{xy}u_yv + 3u_xvg_yu_y \\
&+ 3u_xvg(y)u_{yy} + g(y)u_yv_{xxy} + g(y)u_{xxy}v_y \\
&- g(y)u_{xy}v_{xy}, \\
Y_{441} =& -3g(y)vu_{xx}u_y - 3g(y)vu_xu_{xy} - 3g(y)u_yu_xv_x, \\
T_{441} =& g(y)u_yv_y.
\end{aligned} \tag{22}$$

For the Lie symmetry V_2, we can also get sixteen conservation laws by Theorem 4. For example, making use of

$$(X_{21}, Y_{21}, T_{21}) = (X^2, Y^2, T^2) + (B_1^X, B_1^Y, 0), \tag{23}$$

we can get

$$\begin{aligned}
X_{212} &= -3F(t)u_yv_x - 3F(t)u_{xy}v - 3F(t)u_xv_y, \\
Y_{212} &= 3F(t)u_{xx}v - F_tv + F(t)v_{xxx}, \\
T_{212} &= F(t)v_y.
\end{aligned} \tag{24}$$

For the Lie symmetry V_3, we can also get sixteen conservation laws by Theorem 4. For example, making use of

$$(X_{42}, Y_{42}, T_{42}) = (X^4, Y^4, T^4) + (B_2^X, B_2^Y, 0), \tag{25}$$

we can get

$$\begin{aligned}
X_{423} =& f(t)vu_{ty} - xf_tu_yv_x - 3f(t)u_yv_xu_x + u_yvf_t \\
&+ \frac{1}{3}xf_tv_{xxy} + f(t)u_xv_{xxy} + f(t)u_{xxy}v_x, \\
Y_{423} =& -xf_tu_xv_x - 3f(t)u_x^2v_x - \frac{1}{3}xvf_{tt} - vf_tu_x \\
&- vf(t)u_{tx} - \frac{1}{3}f_tv_{xx} - f(t)v_{xx}u_{xx}, \\
T_{423} =& \frac{1}{3}xf_tv_y + f(t)v_yu_x.
\end{aligned} \tag{26}$$

Using the Lie symmetry V_4 and Theorem 4, sixteen conservation laws for (1) can be obtained. For example, making use of

$$(X_{13}, Y_{13}, T_{13}) = (X^1, Y^1, T^1) + (B_3^X, B_3^Y, 0), \tag{27}$$

we can get

$$\begin{aligned}
X_{134} =& -h(t)vu_{xxyt} + \frac{1}{3}uh_tv_{xxy} - h(t)u_{tx}v_{xy} - h_tvu_{xxy} \\
&- xh_tu_xu_yv_x - \frac{1}{9}xh_{tt}v_{xy} + h(t)v_{xxy}u_t - \frac{2}{3}h_tu_xv_{xy} \\
&+ \frac{1}{18}x^2h_{tt}v_{xxy} - 2xh_tvu_xu_{xy} - h_tuu_yv_x - h_tuu_{xy}v \\
&- h_tuu_xv_y - \frac{1}{6}x^2h_{tt}u_yv_x - \frac{1}{6}x^2h_{tt}u_{xy}v \\
&- \frac{1}{6}x^2h_{tt}u_xv_y - xh_tu_x^2v_y - 3h(t)u_tu_yv_x \\
&- 3h(t)u_tu_{xy}v - 3h(t)u_tu_xv_y + 2h_tvu_yu_x \\
&+ \frac{1}{3}xvu_yh_{tt} + 3h(t)vu_yu_{tx} + \frac{1}{3}xh_tv_{xxy}u_x \\
&- \frac{1}{3}xh_tv_{xy}u_{xx} + \frac{1}{3}xh_tvu_{ty}, \\
Y_{134} =& h_tvuu_{xx} + \frac{1}{6}x^2vh_{tt}u_{xx} + 2xh_tvu_{xx}u_x \\
&+ 3h(t)vu_{xx}u_t + \frac{1}{3}uv_th_t + \frac{1}{18}x^2v_th_{tt} \\
&+ \frac{1}{3}xh_tv_tu_x + h(t)v_tu_t + 2h_tvu_x^2 \\
&+ \frac{1}{3}xh_{tt}vu_x + 3h(t)u_xvu_{tx} + h_tu_{xx}v_x \\
&+ \frac{1}{9}v_xh_{tt} + \frac{1}{3}xh_tv_xu_{xxx} + h(t)v_xu_{txx}, \\
T_{134} =& h(t)vu_{xxxy} - 3h(t)vu_{xx}u_y - 3h(t)vu_xu_{xy} \\
&- \frac{1}{3}vu_yh_t - \frac{1}{3}vxu_{xy}h_t.
\end{aligned} \tag{28}$$

In the previous expressions of conservation laws, v is a solution of (14). If we can find an exact solution v of (14), explicit conservation laws of the ANNV equation (1) can be obtained by substituting it with the previous expressions. For example,

$$v = m(y) + n(t) \tag{29}$$

is a solution of (14) with $m(y)$ and $n(t)$ being two arbitrary functions. By that, nontrivial conservation laws of (1) can be obtained.

Remark 5. It is pointed out that the previous conservation laws are all nontrivial. The accuracy of them has been checked by Maple software.

Remark 6. The conservation laws of (1) obtained in this paper are different from each other and are all different from those in [20].

4. Formulas of Conservation Laws and Explicit Conservation Laws for the KP-BBM Equation

The solutions of the KP-BBM equation (2) have been studied by Wazwaz in [21, 22] who used the sine-cosine method, the tanh method, and the extended tanh method. Abdou [23] used the extended mapping method with symbolic computation to obtain some periodic solutions, solitary wave solution, and triangular wave solution. Exact solutions and conservation laws of (2) have been studied by Adem and Khalique using the Lie group analysis and the simplest equation method [24].

4.1. Formulas of Conservation Laws of the KP-BBM Equation. To search for conservation laws of (2) by Theorem 3, Lie symmetry, formal Lagrangian, and adjoint equation of (2) must be known. According to Definition 1, the adjoint equation of (2) is

$$v_{xt} + v_{xx} - 2\alpha u v_{xx} - \beta v_{xxxt} + \gamma v_{yy} = 0, \tag{30}$$

where v is a new dependent variable with respect to x, y, and t.

According to Theorem 2, the formal Lagrangian for the system consisting of (2) and (30) is

$$L = \left(u_{xt} + u_{xx} - 2\alpha u_x^2 - 2\alpha u u_{xx} - \beta u_{xxxt} + \gamma u_{yy}\right) v. \tag{31}$$

Since there are a higher-order mixed derivative u_{xxxt} and a mixed derivative u_{xt} in (2), the two modification rules must be used if we want to get conservation laws of (2) by Theorem 3. Therefore, we can get the following statement.

Theorem 7. *Suppose that the Lie symmetry of the KP-BBM equation is as follows:*

$$V = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u}. \tag{32}$$

According to the different locations of $\partial L/\partial u_{xt}$ and $\partial L/\partial u_{xxxt}$, the symmetry provides eight different conservation laws for the system consisting of (2) *and* (30)*. The conserved vectors are given as follows:*

$$\left(X_{ij}, Y_{ij}, T_{ij}\right) = \left(X^i, Y^i, T^i\right) + \left(A_j^X, 0, A_j^T\right), \quad i = 1,2, \; j = 1,2,3,4 \tag{33}$$

with

$$\begin{aligned}
X^1 &= \xi L + W\left(\frac{\partial L}{\partial u_x} - D_x\left(\frac{\partial L}{\partial u_{xx}}\right)\right) + D_x(W)\left(\frac{\partial L}{\partial u_{xx}}\right) + W\left(-D_t\left(\frac{\partial L}{\partial u_{xt}}\right)\right),\\
Y^1 &= \eta L + W\left(-D_y\left(\frac{\partial L}{\partial u_{yy}}\right)\right) + D_y(W)\left(\frac{\partial L}{\partial u_{yy}}\right),\\
T^1 &= \tau L + D_x(W)\left(\frac{\partial L}{\partial u_{xt}}\right),\\
X^2 &= \xi L + W\left(\frac{\partial L}{\partial u_x} - D_x\left(\frac{\partial L}{\partial u_{xx}}\right)\right) + D_x(W)\left(\frac{\partial L}{\partial u_{xx}}\right) + D_t(W)\left(\frac{\partial L}{\partial u_{xt}}\right),\\
Y^2 &= Y^1,\\
T^2 &= \tau L + W\left(-D_x\left(\frac{\partial L}{\partial u_{xt}}\right)\right),\\
A_1^X &= D_{xxt}(W)\left(\frac{\partial L}{\partial u_{xxxt}}\right) + D_{xt}(W)\left(-D_x\left(\frac{\partial L}{\partial u_{xxxt}}\right)\right) + D_t(W)\left(D_{xx}\left(\frac{\partial L}{\partial u_{xxxt}}\right)\right),\\
A_1^T &= W\left(-D_{xxx}\left(\frac{\partial L}{\partial u_{xxxt}}\right)\right),\\
A_2^X &= D_{xxt}(W)\left(\frac{\partial L}{\partial u_{xxxt}}\right) + D_{xt}(W)\left(-D_x\left(\frac{\partial L}{\partial u_{xxxt}}\right)\right) + W\left(-D_{xxt}\left(\frac{\partial L}{\partial u_{xxxt}}\right)\right),\\
A_2^T &= D_x(W)\left(D_{xx}\left(\frac{\partial L}{\partial u_{xxxt}}\right)\right),\\
A_3^X &= D_{xxt}(W)\left(\frac{\partial L}{\partial u_{xxxt}}\right) + D_x(W)\left(D_{xt}\left(\frac{\partial L}{\partial u_{xxxt}}\right)\right) + W\left(-D_{xxt}\left(\frac{\partial L}{\partial u_{xxxt}}\right)\right),\\
A_3^T &= D_{xx}(W)\left(-D_x\left(\frac{\partial L}{\partial u_{xxxt}}\right)\right),
\end{aligned}$$

$$\begin{aligned}A_4^X &= D_{xx}(W)\left(-D_t\left(\frac{\partial L}{\partial u_{xxxt}}\right)\right)\\&\quad + D_x(W)\left(D_{xt}\left(\frac{\partial L}{\partial u_{xxxt}}\right)\right)\\&\quad + W\left(-D_{xxt}\left(\frac{\partial L}{\partial u_{xxxt}}\right)\right),\\A_4^T &= D_{xxx}(W)\left(\frac{\partial L}{\partial u_{xxxt}}\right),\end{aligned} \tag{34}$$

where W is the Lie characteristic function, $W = \phi - \xi u_x - \eta u_y - \tau u_t$*, and L is the formal Lagrangian determined by* (31).

4.2. Explicit Conservation Laws of the KP-BBM Equation. Lie symmetries of (2) have been derived in [24] and are listed as follows:

$$\begin{aligned}&V_1 = \frac{\partial}{\partial x}, \qquad V_2 = \frac{\partial}{\partial y}, \qquad V_3 = \frac{\partial}{\partial t},\\&V_4 = -\alpha y\frac{\partial}{\partial y} - 2\alpha t\frac{\partial}{\partial t} + (2\alpha u - 1)\frac{\partial}{\partial u}.\end{aligned} \tag{35}$$

Applying **Theorem** 7, we can obtain conservation laws for the system consisting of (2) and (30). For the symmetry V_1, we can get the following eight conservation laws:

$$\begin{aligned}X_{111} &= vu_{tx} + \gamma vu_{yy} - 2\alpha u_x v_x u + u_x v_x + u_x v_t\\&\quad + \beta u_{tx} v_{xx} - \beta u_{txx} v_x,\\T_{111} &= -vu_{xx} - \beta u_x v_{xxx},\\X_{121} &= vu_{tx} + \gamma vu_{yy} - 2\alpha u_x v_x u + u_x v_x + u_x v_t\\&\quad - \beta v_{txx} u_x - \beta u_{txx} v_x,\\T_{121} &= -vu_{xx} + \beta u_{xx} v_{xx},\\X_{131} &= vu_{tx} + \gamma vu_{yy} - 2\alpha u_x v_x u + u_x v_x + u_x v_t\\&\quad - \beta v_{txx} u_x + \beta u_{xx} v_{tx},\\T_{131} &= -vu_{xx} - \beta u_{xxx} v_x,\\X_{141} &= vu_{tx} - \beta u_{txxx} v + \gamma vu_{yy} - 2\alpha u_x v_x u + u_x v_x\\&\quad + u_x v_t - \beta v_{txx} u_x + \beta u_{xx} v_{tx} - \beta u_{xxx} v_t,\\T_{141} &= -vu_{xx} + \beta u_{xxxx} v,\\X_{211} &= \gamma vu_{yy} - 2\alpha u_x v_x u + u_x v_x + \beta u_{tx} v_{xx} - \beta u_{txx} v_x,\\T_{211} &= u_x v_x - \beta u_x v_{xxx},\\X_{221} &= \gamma vu_{yy} - 2\alpha u_x v_x u + u_x v_x - \beta v_{txx} u_x - \beta u_{txx} v_x,\\T_{221} &= u_x v_x + \beta u_{xx} v_{xx},\\X_{231} &= \gamma vu_{yy} - 2\alpha u_x v_x u + u_x v_x - \beta v_{txx} u_x + \beta u_{xx} v_{tx},\\T_{231} &= u_x v_x - \beta u_{xxx} v_x,\\X_{241} &= -\beta u_{txxx} v + \gamma vu_{yy} - 2\alpha u_x v_x u + u_x v_x\\&\quad - \beta v_{txx} u_x + \beta u_{xx} v_{tx} - \beta u_{xxx} v_t,\\T_{241} &= u_x v_x + \beta u_{xxxx} v,\end{aligned} \tag{36}$$

where $Y_{ij1} = \gamma u_x v_y - \gamma u_{xy} v$, $i = 1, 2$, $j = 1, 2, 3, 4$.

For the symmetry V_2, we can also get eight conservation laws for the system of (2) and (30). For example, making use of

$$\left(X_{13}, Y_{13}, T_{13}\right) = \left(X^1, Y^1, T^1\right) + \left(A_3^X, 0, A_3^T\right), \tag{37}$$

we can get

$$\begin{aligned}X_{132} &= 2\alpha u_y u_x v - 2\alpha u_y v_x u + u_y v_x - u_{xy} v + 2\alpha u_{xy} vu\\&\quad + u_y v_t - \beta v_{txx} u_y + \beta u_{txxy} v + \beta u_{xy} v_{tx},\\Y_{132} &= vu_{tx} + vu_{xx} - 2\alpha vu_x^2 - 2\alpha vuu_{xx} - \beta vu_{txxx} + \gamma u_y v_y,\\T_{132} &= -u_{xy} v - \beta u_{xxy} v_x.\end{aligned} \tag{38}$$

Similarly, for the symmetry V_3, we can get eight conservation laws for the system of (2) and (30). For example, making use of

$$\left(X_{22}, Y_{22}, T_{22}\right) = \left(X^2, Y^2, T^2\right) + \left(A_2^X, 0, A_2^T\right), \tag{39}$$

we can get

$$\begin{aligned}X_{223} &= 2\alpha u_t u_x v - 2\alpha u_t v_x u + u_t v_x - u_{tx} v + 2\alpha u_{tx} vu\\&\quad - u_{tt} v - \beta v_{txx} u_t + \beta u_{ttxx} v - \beta u_{ttx} v_x,\\Y_{223} &= \gamma u_t v_y - \gamma u_{ty} v,\\T_{223} &= u_{tx} v + vu_{xx} - 2\alpha vu_x^2 - 2\alpha vuu_{xx} - \beta vu_{txxx}\\&\quad + \gamma vu_{yy} + u_t v_x + \beta u_{tx} v_{xx}.\end{aligned} \tag{40}$$

For the symmetry V_4, we only list the conservation laws derived by

$$\left(X_{14}, Y_{14}, T_{14}\right) = \left(X^1, Y^1, T^1\right) + \left(A_4^X, 0, A_4^T\right), \tag{41}$$

and they are as follows:

$$\begin{aligned} X_{144} = {} & 4\alpha u_x v + 4\alpha^2 u^2 v_x - 2\alpha v_t u - 4\alpha v_x u - \beta\alpha y v_{tx} u_{xy} \\ & + v_x + \beta\alpha y v_t u_{xxy} - \beta v_{txx} + v_t - 8\alpha^2 u u_x v \\ & - \alpha y u_y v_x - 2\alpha t u_t v_x + \alpha y u_{xy} v + 2\alpha t u_{tx} v - \alpha y v_t u_y \\ & - 2\alpha t v_t u_t + 2\beta\alpha v_{txx} u - 2\beta\alpha v_{tx} u_x \\ & - 2\alpha^2 y u_y u_x v + 2\beta\alpha v_t u_{xx} + 2\alpha^2 y u_y v_x u, \\ Y_{144} = {} & - 4\alpha^2 t u_t u_x v + 4\alpha^2 t u_t v_x u - 2\alpha^2 y u_{xy} v u - 4\alpha^2 t u_{tx} v u \\ & + \beta\alpha y v_{txx} u_y + 2\beta\alpha t v_{txx} u_t - 2\beta\alpha t v_{tx} u_{tx} \\ & + 2\beta\alpha t v_t u_{txx}, \\ T_{144} = {} & - 2\alpha t v u_{xx} + 4\alpha^2 t v u_x^2 + 4\alpha^2 t v u u_{xx} - 2\alpha\gamma t v u_{yy} \\ & + 2\alpha u_x v + \alpha y u_{xy} v - 2\beta\alpha v u_{xxx} - \beta\alpha y v u_{xxxy}. \end{aligned} \tag{42}$$

In the previous expressions of conservation laws, v is a solution of the adjoint equation (30). If we can find an exact solution v of (30), explicit conservation laws for the KP-BBM equation (2) can be obtained by substituting it with the previous expressions. For example,

$$v = (x + M(t))\, y + N(t) \tag{43}$$

is a solution of (30) with $M(t)$ and $N(t)$ being two arbitrary functions. By that we can get many infinite conservation laws for (2). Furthermore, the conservation laws are nontrivial and time dependent.

Remark 8. The correctness of the conservation laws of (2) obtained here has been checked by Maple software. The conservation laws obtained here for (2) are much more than those in [24] and different from them.

5. Concluding Remarks

Recently, conservation laws of nonlinear evolution equations with mixed derivatives have attracted the interest of mathematical and physical researchers. As shown in [16], when applying Noether's theorem and partial Noether's theorem to obtain conservation laws of nonlinear evolution equations with higher-order mixed derivatives, the obtained conservation laws must be adjusted to satisfy the definition of conservation laws. We face the same problem when applying Ibragimov's new conservation theorem to find conservation laws of nonlinear evolution equations with mixed derivatives. In this paper, we propose two modification rules which ensure that Ibragimov's theorem can be applied to nonlinear evolution equations with higher-order and lower-order mixed derivatives. The two modification rules given in this paper are a generalization of those proposed in [9]. The results are used to study the conservation laws of two partial differential equations with higher-order mixed derivatives: the ANNV equation and the KP-BBM equation. Many infinite explicit and nontrivial conservation laws are obtained for the two equations. Based on the two modification rules, Ibragimov's new conservation theorem can be used to find conservation laws of other partial differential equations with any mixed derivatives.

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