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Research Article

Modeling a Microstretch Thermoelastic Body with Two Temperatures

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We consider a theory of thermoelasticity constructed by taking into account the heat conduction in deformable bodies which depends on two temperatures. The first one is the conductive temperature, the second is the thermodynamic temperature, and the difference between them is proportional to the heat supply.

1. Introduction

In the classical uncoupled theory of thermoelasticity the equation of the heat conduction does not contain any elastic term. On the other hand, the heat equation is of parabolic type, which leads to infinite speeds of propagation for heat waves.

In view of eliminating these two phenomena which are not compatible with physical observations, some researchers proposed different generalizations of the classical theory. We restrict our attention to two such extensions. According to the model proposed by Lord and Shulman [1], the classical Fourier's law of heat conduction is replaced by a wave type heat equation. This new equation ensures finite speeds of propagation for the heat and elastic waves.

It is important to remark that in this model the equation of motion and constitutive equations remain the same as those for the coupled and uncoupled theories.

Another generalization is known as the theory of temperature-rate-dependent thermoelasticity or the thermoelasticity with two relaxation times. This theory contains two constants that act as relaxation times and modify the heat equation and, also, the equation of motion and constitutive equations. This theory was first proposed by Green and Lindsay [2] and has aroused much interest in recent years. Unlike the coupled thermoelasticity theory, this theory includes temperature rate among the constitutive variables and consequently predicts a finite speed for the propagation of thermal signals. Since thermal signals propagating with finite speeds have actually been observed in solids, the theory of temperature-rate-dependent thermoelasticity is more general and physically more realistic than the coupled theory.

In [3], Chen et al. gave a theory of thermodynamics of nonsimple elastic materials with two temperatures For the linearized form, Iesan establish in [4] some general theorems.

For time-dependent problems, in particular for wave propagation, the conductive temperature is different from thermodynamic temperature, regardless of presence of a heat supply.

The two temperatures have representation in the form a travelling wave plus a response which occurs instantaneously through the body.

First studies dedicated to the theory of microstretch elastic bodies were published by Eringen [5, 6]. This theory is a generalization of the micropolar theory and a special case of the micromorphic theory. In the context of this theory each material point is endowed with three deformable directors. A body is a microstretch continuum if the directors are constrained to have only breathing-type microdeformations. Also, the material points of a microstretch solid can stretch and contract independently of their translations and rotations.

The purpose of this theory is to eliminate discrepancies between the classical elasticity and experiments, since the classical elasticity failed to present acceptable results when the effects of material microstructure were known to contribute significantly to the body's overall deformations, for example, in the case of granular bodies with large molecules (e.g., polymers), graphite, or human bones (see [6]).

These cases are becoming increasingly important in the design and manufacture of modern day advanced materials, as small-scale effects become paramount in the prediction of the overall mechanical behaviour of these materials.

Other intended applications of this theory are to composite materials reinforced with chopped fibers and various porous materials.

In [7, 8], we find some basic results regarding thermoelastic microstretch bodies.

In [9], the governing equations are modified in the context of Lord and Shulman's theory of generalized thermoelasticity to include the two temperatures.

2. Basic Equations

Let us summarize the basic equations of the theory of thermoelasticity of microstretch bodies with two temperatures. Let B be a bounded regular region of three-dimensional Euclidian space R^3 occupied by a microstretch elastic body, referred to the reference configuration (at time $t = 0$). Let $\overline{B}$ denote the closure of B and call ∂B the boundary of the domain B. We consider ∂B a piecewise smooth surface designated by n_i the components of the outward unit normal to the surface ∂B. Letters in boldface stand for vector fields. We use the notation v_i to designate the components of the vector $\mathbf{v}$ in the underlying rectangular Cartesian coordinates frame. A superposed dot stands for the material time derivative. We will employ the usual summation and differentiation conventions: the subscripts are understood to range over integer $(1, 2, 3)$. Summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate.

The spatial argument and time argument of a function will be omitted when there is no likelihood of confusion. We refer the motion of the body to a fixed system of rectangular Cartesian axes Ox_i, $i = 1, 2, 3$, and to the reference configuration.

The behaviour of the thermoelastic microstretch body is characterized by the following kinematic variables:

$$
\begin{aligned}
u_i &= u_i(x,t), \qquad \varphi_i = \varphi_i(x,t), \\
\psi &= \psi(x,t), \qquad (x,t) \in B \times [0, t_0),
\end{aligned}
\tag{1}
$$

where u_i are the components of the displacement field, φ_i are the components of the microrotation field, and ψ is the microstretch function.

The the components of the strain tensors ε_{ij}, k_{ij}, and γ_i are defined by means of the geometric equations:

$$
\varepsilon_{ij} = u_{j,i} + \varepsilon_{ijk}\varphi_k, \qquad \eta_{ij} = \varphi_{j,i}, \qquad \gamma_i = \psi_{,i}, \tag{2}
$$

where ε_{ijk} is the alternating symbol.

As usual, we denote by t_{ij} the components of the stress tensor and by m_{ij} the components of the couple stress tensor over B. Also, we denote by λ_i the components of the internal hypertraction vector and by q_i the components of the heat flux vector.

Now, we define the surface traction t_i, the surface couple m_i, the microstretch traction p, and the heat flux q at regular points of the surface ∂B by means of the components of the outward unit normal to the surface ∂B and components of stress tensors:

$$
t_i = t_{ji}n_j, \qquad m_i = m_{ji}n_j, \qquad p = \lambda_i n_i, \qquad q = q_i n_i. \tag{3}
$$

If we refer the motion relative to a stress-free and undistorted reference state, then the basic equations for the theory of thermoelasticity of microstretch bodies with two temperatures are

(i) the equations of motion:

$$
\begin{aligned}
t_{ij,j} + F_i &= \varrho \ddot{u}_i, \\
m_{ij,j} + \varepsilon_{ijk}t_{jk} + G_i &= I_{ij}\ddot{\varphi}_j,
\end{aligned}
\tag{4}
$$

(ii) the balance of the equilibrated forces:

$$
\lambda_{i,i} + L = J\ddot{\psi}, \tag{5}
$$

(iii) the energy equation:

$$
\varphi_0 \dot{\psi} = -q_{i,i} + r. \tag{6}
$$

For an isotropic and homogeneous microstretch elastic material, the constitutive equations have the following form:

$$
\begin{aligned}
t_{ij} &= (\lambda\varepsilon_{kk} + \lambda_0\psi - \beta\theta)\,\delta_{ij} + (\mu + \nu)\,\varepsilon_{ij} + \mu\varepsilon_{ji}, \\
m_{ij} &= \alpha\eta_{kk}\delta_{ij} + \beta\eta_{ji} + \gamma\eta_{ij} + b_0\varepsilon_{kji}\gamma_k, \\
\lambda_i &= a_0\gamma_i + \varepsilon_{ijk}\eta_{jk}, \\
q_i &= -k\varphi_{,i}, \\
\eta &= \beta\varepsilon_{rr} + \frac{c}{\varphi_0}\theta,
\end{aligned}
\tag{7}
$$

where δ_{ij} is the Kronecker symbol.

We define the thermodynamic temperature θ measured from a constant reference temperature φ_0 by means of formula

$$
\theta = \varphi - a\varphi_{,kk} \tag{8}
$$

in which a is the two-temperature parameter, $a > 0$.

In the above equations we have used the following notations:

(i) F_i are the components of body force;

(ii) G_i are the components of body couple;

(iii) L is the generalized external body load;

(iv) ϱ is the reference mass density;

(v) $I_{ij} = I_{ji}$ and J are the coefficients of inertia;

(vi) η is the entropy per unit volume measured from the entropy of the reference state;

(vii) r is heat supply per unit mass;

(viii) φ is the conductive temperature measured from a constant reference temperature φ_0.

Also, the constants λ, λ_0, μ, β, ν, α, γ, a_0, b_0, k, a and c, from the above relations, are the characteristic coefficients of the material and full characterize the mechanical properties of the body.

As usual in mechanics of continua, we assume that the internal energy density is a positive definite quadratic form and this hypothesis implies (see Eringen [6])

$$\begin{gathered} 3\lambda + 2\mu + \nu > \frac{\lambda_0^2}{\lambda_1}, \quad 2\mu + \nu > 0, \ \nu > 0, \\ 3\alpha + \beta + \gamma > 0, \quad \gamma + \beta > 0, \ \gamma - \beta > 0, \\ \alpha_0 > 0, \quad \lambda_1 > 0, \quad c > 0, \quad k > 0, \quad a \geq 0. \end{gathered} \tag{9}$$

To the system of field equations (4)–(7) we adjoin the following initial conditions:

$$\begin{aligned} u_i(x,0) &= a_i^0(x), & \dot{u}_i(x,0) &= a_i^1(x), \\ \varphi_i(x,0) &= b_i^0(x), & \dot{\varphi}_i(x,0) &= b_i^1(x), \\ \psi(x,0) &= \psi^0(x), & \dot{\psi}(x,0) &= \psi^1(x), \\ \eta(x,0) &= \eta^0(x), & x &= (x_1,x_2,x_3) \in \overline{B}. \end{aligned} \tag{10}$$

and the following prescribed boundary conditions:

$$\begin{aligned} u_i &= \widetilde{u}_i \quad \text{on } \partial B_1 \times [0,t_0), & t_i &= \widetilde{t}_i \quad \text{on } \partial B_1^c \times [0,t_0), \\ \varphi_i &= \widetilde{\varphi}_i \quad \text{on } \partial B_2 \times [0,t_0), & m_i &= \widetilde{m}_i \quad \text{on } \partial B_2^c \times [0,t_0), \\ \psi &= \widetilde{\psi} \quad \text{on } \partial B_3 \times [0,t_0), & p &= \widetilde{p} \quad \text{on } \partial B_3^c \times [0,t_0), \\ \varphi &= \widetilde{\varphi} \quad \text{on } \partial B_4 \times [0,t_0), & q &= \widetilde{q} \quad \text{on } \partial B_4^c \times [0,t_0), \end{aligned} \tag{11}$$

where a_i^0, a_i^1, b_i^0, b_i^1, ψ^0, ψ^1, η^0, $\widetilde{u}_i$, $\widetilde{t}_i$, $\widetilde{\varphi}_i$, $\widetilde{m}_i$, $\widetilde{\psi}$, $\widetilde{\varphi}$, and $\widetilde{q}$ are prescribed functions in their domains of definition. t_0 is some instant that may be infinite.

Also, $\partial B_1, \partial B_2, \partial B_3$, and ∂B_4 with respective complements $\partial B_1^c, \partial B_2^c, \partial B_3^c$, and ∂B_4^c are subsets of the surface ∂B such that

$$\begin{gathered} \partial B_1 \cap \partial B_1^c = \partial B_2 \cap \partial B_2^c = \partial B_3 \cap \partial B_3^c = \partial B_4 \cap \partial B_4^c = \emptyset, \\ \partial B_1 \cup \partial B_1^c = \partial B_2 \cup \partial B_2^c = \partial B_3 \cup \partial B_3^c = \partial B_4 \cup \partial B_4^c = \partial B. \end{gathered} \tag{12}$$

For a function f defined on the cylinder $\overline{B} \times [0,t_0)$, we will use the notations $f \in C^{M,N}$ which means that the derivative,

$$\frac{\partial^m}{\partial x_i \partial x_j \cdots \partial x_k}\left(\frac{\partial^n f}{\partial t^n}\right), \tag{13}$$

exists and is continuous on $B \times [0,t_0)$ for $m = 0,1,2,\ldots,M$, $n = 0,1,2,\ldots,N$, and $m + n \leq \max\{M,N\}$.

We say that the ordered array,

$$S = \left\{u_i, \varphi_i, \psi, q_i, \varepsilon_{ij}, \eta_{ij}, \gamma_i, \eta, t_{ij}, m_{ij}, \lambda_i, \varphi\right\}, \tag{14}$$

is an *admissible state* if

$$\begin{gathered} u_i \in C^{1,2}, \quad \varphi_i \in C^{1,2}, \quad \psi \in C^{1,2}, \\ \varepsilon_{ij} \in C^{0,0}, \quad \eta_{ij} \in C^{0,0}, \quad \gamma_i \in C^{0,0}, \quad \eta \in C^{0,1}, \\ t_{ij} \in C^{1,0}, \quad m_{ij} \in C^{1,0}, \quad \lambda_i \in C^{1,0}, \quad \varphi \in C^{2,0}. \end{gathered} \tag{15}$$

It easy to verify that the set of all admissible states is a linear space with regard to the addition of two admissible states and multiplication of a admissible state by a scalar

$$\begin{aligned} S_1 + S_2 = {} & \left\{u_i^1 + u_i^2, \varphi_i^1 + \varphi_i^2, \psi^1 + \psi^2, q_i^1 + q_i^2,\right. \\ & \varepsilon_{ij}^1 + \varepsilon_{ij}^2, \eta_{ij}^1 + \eta_{ij}^2, \gamma_i^1 + \gamma_i^2, \eta^1 + \eta^2, t_{ij}^1 + t_{ij}^2, \\ & \left. m_{ij}^1 + m_{ij}^2, \lambda_i^1 + \lambda_i^2, \varphi^1 + \varphi^2\right\}, \\ \lambda S = {} & \left\{\lambda u_i, \lambda\varphi_i, \lambda\psi, \lambda q_i, \lambda\varepsilon_{ij}, \lambda\eta_{ij}, \lambda\gamma_i, \lambda\eta, \lambda t_{ij}, \lambda m_{ij}, \lambda\lambda_i, \lambda\varphi\right\}. \end{aligned} \tag{16}$$

By a solution of the mixed initial boundary value problem of the theory of thermoelasticity of microstretch bodies with two temperatures in the cylinder $\Omega_0 = B \times [0,t_0)$ we mean an admissible state which satisfies (4)–(7), the initial conditions (10) and the boundary conditions (11) for all $(x,t) \in \Omega_0$.

3. Main Results

Let us denote by ε the specific internal energy. Using a usual procedure, we can write

$$\dot{\varepsilon} = t_{ij}\dot{\varepsilon}_{ij} + m_{ij}\dot{\eta}_{ij} + \lambda_i\dot{\gamma}_i + \theta\dot{\eta}. \tag{17}$$

Theorem 1. *For an admissible state one has the following equality:*

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_B \left(2\varepsilon + \varrho\dot{u}_i\dot{u}_i + I_{ij}\dot{\varphi}_i\dot{\varphi}_j + J\dot{\psi}^2\right)dV \\ = \int_B \left(F_i\dot{u}_i + G_i\dot{\varphi}_i + J\dot{\psi} + \frac{1}{\varphi_0}r\theta\right)dV \\ - \frac{k}{\varphi_0}\int_B \left(\varphi_{,k}\varphi_{,k} + a\varphi_{,ii}\varphi_{,jj}\right)dV \\ + \int_{\partial B}\left(t_i\dot{u}_i + m_i\dot{\varphi}_i + p\dot{\psi} - \frac{q}{\varphi_0}\varphi\right)dA. \end{aligned} \tag{18}$$

Proof. Taking into account, first the geometric equations (2) and then (4)–(6), we can write

$$\begin{aligned}
&t_{ij}\dot{\varepsilon}_{ij} + m_{ij}\dot{\eta}_{ij} + \lambda_i\dot{\gamma}_i + \theta\dot{\eta} \\
&\quad= t_{ij}\left(\dot{u}_{i,j} + \varepsilon_{jik}\dot{\varphi}_k\right) \\
&\qquad+ m_{ij}\dot{\varphi}_{j,i} + \lambda_i\dot{\psi}_{,i} - \frac{1}{\varphi_0}q_{i,i}\left(\varphi - a\varphi_{,kk}\right) + \frac{1}{\varphi_0}r\theta \\
&\quad= \left(t_{ij}\dot{u}_i\right)_{,} - t_{ij,j}\dot{u}_i + \varepsilon_{jik}t_{ij}\dot{\varphi}_k \\
&\qquad+ \left(m_{ij}\dot{\varphi}_j\right)_{,i} - m_{ij,i}\dot{\varphi}_j + \left(\lambda_i\dot{\psi}\right)_{,i} \\
&\qquad- \lambda_{i,i}\dot{\psi} + \frac{1}{\varphi_0}r\theta - \frac{1}{\varphi_0}\left(q_i\varphi\right)_{,i} + \frac{1}{\varphi_0}q_i\varphi_{,i} \\
&\quad= \left(t_{ij}\dot{u}_i\right)_{,j} + F_i - \varrho\ddot{u}_i + F_i\dot{u}_i - \varrho\dot{u}_i\ddot{u}_i \\
&\qquad+ \varepsilon_{jik}t_{ij}\dot{\varphi}_k + \left(m_{ij}\dot{\varphi}_j\right)_{,i} \\
&\qquad+ \varepsilon_{ijk}t_{jk}\dot{\varphi}_i + G_i\dot{\varphi}_i - I_{ij}\ddot{\varphi}_j\dot{\varphi}_i + \left(\lambda_i\dot{\psi}\right)_{,i} + L\dot{\psi} - J\dot{\psi}\ddot{\psi} \\
&\qquad+ \frac{1}{\varphi_0}r\theta - \frac{1}{\varphi_0}\left(q_i\varphi\right)_{,i} + \frac{1}{\varphi_0}q_i\varphi_{,i} + \frac{a}{\varphi_0}q_{i,i}\varphi_{,kk} \\
&\quad= \left(t_{ij}\dot{u}_i\right)_{,j} + \left(m_{ij}\dot{\varphi}_j\right)_{,i} + \left(\lambda_i\dot{\psi}\right)_{,i} - \frac{1}{\varphi_0}\left(q_i\varphi\right)_{,i} \\
&\qquad+ F_i\dot{u}_i + G_i\dot{\varphi}_i + L\dot{\psi} + \frac{1}{\varphi_0}r\theta \\
&\qquad- \varrho\dot{u}_i\ddot{u}_i - I_{ij}\dot{\varphi}_i\ddot{\varphi}_j - J\dot{\psi}\ddot{\psi} - \frac{k}{\varphi_0}\varphi_{,i}\varphi_{,i} - \frac{ak}{\varphi_0}\varphi_{,ii}\varphi_{,kk}.
\end{aligned} \tag{19}$$

Using (17) and (19), we can write

$$\begin{aligned}
&\dot{\varepsilon} + \varrho\dot{u}_i\ddot{u}_i + I_{ij}\dot{\varphi}_i\ddot{\varphi}_j + J\dot{\psi}\ddot{\psi} \\
&\quad= F_i\dot{u}_i + G_i\dot{\varphi}_i + L\dot{\psi} + \frac{1}{\varphi_0}r\theta \\
&\qquad+ \left(t_{ij}\dot{u}_i\right)_{,j} + \left(m_{ij}\dot{\varphi}_j\right)_{,i} + \left(\lambda_i\dot{\psi}\right)_{,i} \\
&\qquad- \frac{1}{\varphi_0}\left(q_i\varphi\right)_{,i} - \frac{k}{\varphi_0}\left(\varphi_{,i}\varphi_{,i} + a\varphi_{,ii}\varphi_{,kk}\right).
\end{aligned} \tag{20}$$

Now, we integrate equality (20) over B and by using the divergence theorem, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_B\left(2\varepsilon + \varrho\dot{u}_i\dot{u}_i + I_{ij}\dot{\varphi}_i\dot{\varphi}_j + J\dot{\psi}^2\right)dV \\
&\quad= \int_B\left(F_i\dot{u}_i + G_i\dot{\varphi}_i + L\dot{\psi} + \frac{1}{\varphi_0}r\theta\right)dV \\
&\qquad- \frac{k}{\varphi_0}\int_B\left(\varphi_{,i}\varphi_{,i} + a\varphi_{,ii}\varphi_{,kk}\right)dV \\
&\qquad+ \int_{\partial B}\left(t_{ij}n_j\dot{u}_i + m_{ij}n_j\dot{\varphi}_i + \lambda_i n_i\dot{\psi} - \frac{1}{\varphi_0}q_i n_i\varphi\right)dA.
\end{aligned} \tag{21}$$

On the last integral in the right-side of equality (21) we take into account the boundary condition (11) such that we are led to

$$\begin{aligned}
&\int_{\partial B}\left(t_{ij}n_j\dot{u}_i + m_{ij}n_j\dot{\varphi}_i + \lambda_i n_i\dot{\psi} - \frac{1}{\varphi_0}q_i n_i\varphi\right)dA \\
&\quad= \int_{\partial B}\left(t_i\dot{u}_i + m_i\dot{\varphi}_i + p\dot{\psi} - \frac{1}{\varphi_0}q\varphi\right)dA.
\end{aligned} \tag{22}$$

Finally, from (21) and (22) we deduce the desired equality (18) and the proof of theorem is complete. □

Based on equality (18) we can prove the uniqueness result of the solution in the following theorem.

Theorem 2. *The mixed problem of the thermoelasticity of microstretch materials with two temperatures has at most one solution.*

Proof. Suppose, by contrast, that our problem admits two solutions:

$$u_i^{(\alpha)},\ \varphi_i^{(\alpha)},\ \psi_i^{(\alpha)},\ \varphi^{(\alpha)}, \tag{23}$$

where $\alpha = 1, 2$ and denote by $\overline{u}_i$, $\overline{\varphi}_i$, $\overline{\psi}_i$, and $\overline{\varphi}$ the difference of two solutions; that is

$$\begin{aligned}
\overline{u}_i &= u_i^{(1)} - u_i^{(2)}, \qquad \overline{\varphi}_i = \varphi_i^{(1)} - \varphi_i^{(2)}, \\
\overline{\psi} &= \psi^{(1)} - \psi^{(2)}, \qquad \overline{\varphi} = \varphi^{(1)} - \varphi^{(2)}.
\end{aligned} \tag{24}$$

Of course, because of the linearity of the problem, the differences (24) satisfy the equations and conditions of the problem, but in their homogeneous form. If we write relation (18) for the differences (24) and take into account the hypothesis (9) we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_B\left(2\overline{\varepsilon} + \varrho\dot{\overline{u}}_i\dot{\overline{u}}_i + I_{ij}\dot{\overline{\varphi}}_i\dot{\overline{\varphi}}_j + J\overline{\dot{\psi}^2}\right)dV \\
&\quad= -\frac{k}{\varphi_0}\int_B\left(\overline{\varphi}_{,k}\overline{\varphi}_{,k} + a\overline{\varphi}_{,ii}\overline{\varphi}_{,jj}\right)dV,
\end{aligned} \tag{25}$$

where $\overline{\varepsilon}$ is the specific internal energy corresponding to the differences (24).

Clarly, from (25) taking into account that $k > 0$, $\varphi_0 > 0$, and $a > 0$ we deduce

$$\frac{d}{dt}\int_B\left(2\overline{\varepsilon} + \varrho\dot{\overline{u}}_i\dot{\overline{u}}_i + I_{ij}\dot{\overline{\varphi}}_i\dot{\overline{\varphi}}_j + J\overline{\dot{\psi}^2}\right)dV \le 0 \tag{26}$$

and this inequality assures that the function,

$$\int_B\left(2\overline{\varepsilon} + \varrho\dot{\overline{u}}_i\dot{\overline{u}}_i + I_{ij}\dot{\overline{\varphi}}_i\dot{\overline{\varphi}}_j + J\overline{\dot{\psi}^2}\right)dV, \tag{27}$$

is decreasing.

According to the homogeneous initial conditions for differences (24), this integral is null at time $t = 0$. On the other hand, this integral cannot be negative, such that we deduce that the integral vanishes for all time $t > 0$.

Thus we obtain

$$\overline{u}_i = 0, \qquad \overline{\varphi}_i = 0, \qquad \overline{\psi} = 0,$$
$$\overline{\varphi} = 0, \quad \overline{\theta} = \overline{\varphi} - a\overline{\varphi}_{,kk} = 0. \tag{28}$$

It is easy to see that

$$\int_B \overline{\varphi}\left(\overline{\varphi} - a\overline{\varphi}_{,kk}\right) dV = \int_B \left(\overline{\varphi}^2 + a\overline{\varphi}_{,i}\overline{\varphi}_{,i}\right) dV + \frac{a}{k}\int_{\partial B} \overline{q}\,\overline{\varphi}\, dA. \tag{29}$$

Using (28) and (29) and taking into account the fact that $\overline{q}$ and $\overline{\varphi}$ satisfy the homogeneous boundary conditions, we obtain $\overline{\varphi} = 0$ and Theorem 2 is concluded. □

This theorem generalizes Iesan's uniqueness result from the classical thermoelasticity with two temperatures.

Now, we give an alternative form of our mixed problem, by using the convolution of two functions.

If α and β are two functions defined on $\overline{B} \times [0, \infty)$, assumed be continuous on $[0, \infty)$, with respect to t, for each $x \in \overline{B}$, then we define the convolution $\alpha * \beta$ by

$$(\alpha * \beta)(x, t) = \int_0^t \alpha(x, t - \tau)\beta(x, \tau)\, d\tau \tag{30}$$

which has the known properties:

$$\alpha * \beta = \beta * \alpha,$$
$$(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma) = \alpha * \beta * \gamma,$$
$$\alpha * (\beta + \gamma) = (\alpha + \beta) * (\alpha + \gamma),$$
$$\alpha * \beta = 0 \Longrightarrow \alpha = 0 \quad \text{or} \quad \beta = 0. \tag{31}$$

Let us consider the functions h and g defined on $[0, \infty)$ by

$$h(t) = 1, \qquad g(t) = t, \tag{32}$$

and the functions f_i, g_i, l, and W defined on $B \times [0, \infty)$ by

$$f_i = h * F_i + \varrho\left(ta_i^0 + a_i^1\right),$$
$$g_i = h * G_i + I_{ij}\left(tb_j^0 + b_j^1\right),$$
$$l = h * L + J\left(t\psi^0 + \psi^1\right),$$
$$W = h * r + \varphi_0\eta^0. \tag{33}$$

Then (4)–(6) received the form

$$h * t_{ij,j} + f_i = \varrho u_i,$$
$$h * m_{ij,j} + \varepsilon_{ijk} h * t_{jk} + g_i = I_{ij}\varphi_j, \tag{34}$$
$$h * \lambda_{i,i} + l = J\psi,$$

$$\varphi_0\eta = -l * q_{i,i} + W. \tag{35}$$

In this way, we obtain the following result.

Theorem 3. *An admissible state*

$$S = \left\{u_i, \varphi_i, \psi, q_i, \varepsilon_{ij}, \eta_{ij}, \gamma_i, \eta, t_{ij}, m_{ij}, \lambda_i, \varphi\right\} \tag{36}$$

is a solution of the mixed problem of thermoelasticity of microstretch bodies with two temperatures if and only if it satisfies (34), (35), (7), *and* (8) *and the boundary conditions* (11).

We must outline that in this form of the mixed problem, the initial conditions are included in the field of equations.

In the following, we propose to find a result of Betti's type regarding our mixed problem.

Assume that the thermoelastic body with two temperatures is subjected to two systems of thermoelastic loadings:

$$L^{(\alpha)} = \left\{F_i^{(\alpha)}, G_i^{(\alpha)}, L^{(\alpha)}, r^{(\alpha)}, \widetilde{u}_i^{(\alpha)}, \widetilde{\varphi}_i^{(\alpha)}, \widetilde{\psi}^{(\alpha)}, \widetilde{\varphi}^{(\alpha)}, \widetilde{t}_i^{(\alpha)}, \widetilde{m}_i^{(\alpha)}\right.$$
$$\left.\widetilde{p}^{(\alpha)}, \widetilde{q}^{(\alpha)}, a_i^{0(\alpha)}, a_i^{1(\alpha)}, b_i^{0(\alpha)}, b_i^{1(\alpha)}, \psi^{0(\alpha)}, \psi^{1(\alpha)}, \eta_0^{(\alpha)}\right\}, \tag{37}$$

where $\alpha = 1, 2$.

These loadings correspond to the thermoelastic configurations:

$$C^{(\alpha)} = \left\{u_i^{(\alpha)}, \varphi_i^{(\alpha)}, \psi^{(\alpha)}, \varphi^{(\alpha)}\right\}. \tag{38}$$

Theorem 4. *Suppose that the thermoelastic body with two temperatures is subjected to two systems of loadings $L^{(\alpha)}$, $\alpha = 1, 2$. Then between the corresponding thermoelastic configurations $C^{(\alpha)}$, $\alpha = 1, 2$, there is the following reciprocity relation:*

$$\begin{aligned}
&\int_B \left[f_i^{(1)} * u_i^{(2)} + g_i^{(1)} * \varphi_i^{(2)} + l^{(1)} * \psi^{(2)} - \frac{1}{\varphi_0} g * W^{(1)} * \theta^{(2)}\right] dV \\
&\quad + \int_{\partial B} g * \left[t_i^{(1)} * u_i^{(2)} + m_i^{(1)} * \varphi_i^{(2)} + p^{(1)} * \psi^{(2)} + \frac{1}{\varphi_0} h * q^{(1)} * \varphi^{(2)}\right] dA \\
&= \int_B \left[f_i^{(2)} * u_i^{(1)} + g_i^{(2)} * \varphi_i^{(1)} + l^{(2)} * \psi^{(1)} - \frac{1}{\varphi_0} g * W^{(2)} * \theta^{(1)}\right] dV \\
&\quad + \int_{\partial B} g * \left[t_i^{(2)} * u_i^{(1)} + m_i^{(2)} * \varphi_i^{(1)} + p^{(2)} * \psi^{(1)} + \frac{1}{\varphi_0} h * q^{(2)} * \varphi^{(1)}\right] dA,
\end{aligned} \tag{39}$$

where $f_i^{(\alpha)}$, $g_i^{(\alpha)}$, $l^{(\alpha)}$, and $W^{(\alpha)}$ are given by relations of the form (33).

Proof. Taking into account the properties of the convolution, from the constitutive equations (7) and relation (8), we get

$$\begin{gathered}\left[t_{ij}^{(1)}+\beta\theta^{(1)}\delta_{ij}\right]*e_{ij}^{(2)}=\left[t_{ij}^{(2)}+\beta\theta^{(2)}\delta_{ij}\right]*e_{ij}^{(1)},\\ m_{ij}^{(1)}*\eta_{ij}^{(2)}=m_{ij}^{(2)}*\eta_{ij}^{(1)},\\ \lambda_i^{(1)}*\gamma_i^{(2)}=\lambda_i^{(2)}*\gamma_i^{(1)},\\ \left[\beta\varepsilon_{kk}^{(1)}-\eta^{(1)}\right]*\theta^{(2)}=\left[\beta\varepsilon_{kk}^{(2)}-\eta^{(2)}\right]*\theta^{(1)}.\end{gathered}\tag{40}$$

Adding these relations we deduce that

$$\begin{aligned}&t_{ij}^{(1)}*e_{ij}^{(2)}+m_{ij}^{(1)}*\eta_{ij}^{(2)}+\lambda_i^{(1)}*\gamma_i^{(2)}-\eta^{(1)}*\theta^{(2)}\\&\quad=t_{ij}^{(2)}*e_{ij}^{(1)}+m_{ij}^{(2)}*\eta_{ij}^{(1)}+\lambda_i^{(2)}*\gamma_i^{(1)}-\eta^{(2)}*\theta^{(1)}.\end{aligned}\tag{41}$$

Now, we introduce the following notation:

$$\begin{aligned}L_{\alpha\beta}=\int_B g*\Big[&t_{ij}^{(\alpha)}*e_{ij}^{(\beta)}+m_{ij}^{(\alpha)}*\eta_{ij}^{(\beta)}\\&+\lambda_i^{(\alpha)}*\gamma_i^{(\beta)}-\eta^{(\alpha)}*\theta^{(\beta)}\Big]dV.\end{aligned}\tag{42}$$

Thus, from (41) it is easy to deduce that

$$L_{21}=L_{12}.\tag{43}$$

Taking into account the constitutive equations (7), relation (8), the geometric equations (2), and (34) and (35), we obtain

$$\begin{aligned}&g*\left[t_{ij}^{(\alpha)}*e_{ij}^{(\beta)}+m_{ij}^{(\alpha)}*\eta_{ij}^{(\beta)}+\lambda_i^{(\alpha)}*\gamma_i^{(\beta)}-\eta^{(\alpha)}*\theta^{(\beta)}\right]\\
&\quad=g*t_{ij}^{(\alpha)}*u_{i,j}^{(\beta)}+\varepsilon_{jik}g*t_{ij}^{(\alpha)}*\varphi_k^{(\beta)}\\
&\qquad+g*m_{ij}^{(\alpha)}*\varphi_{j,i}^{(\beta)}+g*\lambda_i^{(\alpha)}*\psi_{,i}^{(\beta)}\\
&\qquad+\frac{1}{\varphi_0}g*h*q_{i,i}^{(\alpha)}*\left(\varphi^{(\beta)}-a\varphi_{,kk}^{(\beta)}\right)\\
&\qquad-\frac{1}{\varphi_0}g*W^{(\alpha)}*\theta^{(\beta)}\\
&\quad=g*\left(t_{ij}^{(\alpha)}*u_i^{(\beta)}\right)_{,j}+g*\left(m_{ij}^{(\alpha)}*\varphi_i^{(\beta)}\right)_{,j}\\
&\qquad+g*\left(\lambda_i^{(\alpha)}*\psi^{(\beta)}\right)_{,i}\\
&\qquad-g*t_{ij,j}^{(\alpha)}*u_i^{(\beta)}-g*m_{ij,j}^{(\alpha)}*\varphi_i^{(\beta)}-g*\lambda_{i,i}^{(\alpha)}*\psi^{(\beta)}\\
&\qquad+\frac{1}{\varphi_0}g*h*\left(q_i^{(\alpha)}*\varphi^{(\beta)}\right)_{,i}\\
&\qquad-\frac{1}{\varphi_0}g*h*q_i^{(\alpha)}*\varphi_{,i}^{(\beta)}-\frac{a}{\varphi_0}g*h*q_{i,i}^{(\alpha)}*\varphi_{,kk}^{(\beta)}\\
&\qquad-\frac{1}{\varphi_0}g*W^{(\alpha)}*\theta^{(\beta)}\\
&\quad=g*\left(t_{ij}^{(\alpha)}*u_i^{(\beta)}\right)_{,j}+g*\left(m_{ij}^{(\alpha)}*\varphi_i^{(\beta)}\right)_{,j}\\
&\qquad+g*\left(\lambda_i^{(\alpha)}*\psi^{(\beta)}\right)_{,i}\\
&\qquad+f_i^{(\alpha)}*u_i^{(\beta)}+g_i^{(\alpha)}*\varphi_i^{(\beta)}+l^{(\alpha)}*\psi^{(\beta)}\\
&\qquad-\frac{1}{\varphi_0}g*W^{(\alpha)}*\theta^{(\beta)}\\
&\qquad-\varrho u_i^{(\alpha)}*u_i^{(\beta)}-I_{ij}\varphi_i^{(\alpha)}*\varphi_j^{(\beta)}-J\psi^{(\alpha)}*\psi^{(\beta)}\\
&\qquad+\frac{k}{\varphi_0}g*h*\varphi_{,i}^{(\alpha)}*\varphi_{,i}^{(\beta)}+\frac{ka}{\varphi_0}g*h*\varphi_{,ii}^{(\alpha)}*\varphi_{,kk}^{(\beta)}.\end{aligned}\tag{44}$$

If we introduce this result in (42) we are led to

$$\begin{aligned}L_{\alpha\beta}=&\int_B\Big[f_i^{(\alpha)}*u_i^{(\beta)}+g_i^{(\alpha)}*\varphi_i^{(\beta)}\\
&\qquad+l^{(\alpha)}*\psi^{(\beta)}-\frac{1}{\varphi_0}g*W^{(\alpha)}*\theta^{(\beta)}\Big]dV\\
&+\int_{\partial B}g*\Big[t_i^{(\alpha)}*u_i^{(\beta)}+m_i^{(\alpha)}*\varphi_i^{(\beta)}\\
&\qquad+p^{(\alpha)}*\psi^{(\beta)}+\frac{1}{\varphi_0}h*q^{(\alpha)}*\varphi^{(\beta)}\Big]dA\\
&-\int_B\Big[\varrho u_i^{(\alpha)}*u_i^{(\beta)}+I_{ij}\varphi_i^{(\alpha)}*\varphi_j^{(\beta)}-J\psi^{(\alpha)}*\psi^{(\beta)}\\
&\qquad-\frac{k}{\varphi_0}g*h*\varphi_{,i}^{(\alpha)}*\varphi_{,i}^{(\beta)}\\
&\qquad-\frac{ka}{\varphi_0}g*h*\varphi_{,ii}^{(\alpha)}*\varphi_{,kk}^{(\beta)}\Big]dV.\end{aligned}\tag{45}$$

Using form (45) of the functional $L_{\alpha\beta}$, taking into account the symmetry relation (43), we obtain the desired result (39) that concludes the proof of Theorem 4. □

Let us make some considerations on the uncoupled problem of the thermoelasticity with two temperatures. So, instead of relation $(7)_5$ we assume that

$$\eta=\frac{c}{\varphi_0}\theta.\tag{46}$$

In this way, we can obtain the relation between the thermoelastic configurations $C^{(1)}$ and $C^{(2)}$ corresponding to the coupled problem, say $P^{(1)}$, and the uncoupled problem $P^{(2)}$, respectively.

Suppose that the above systems of thermoelastic loadings $L^{(\alpha)}$ correspond to the problems $P^{(\alpha)}$, $\alpha=1,2$.

Using the same procedure as in the proof of Theorem 4, we obtain the following result.

Theorem 5. *Assume that the microstretch thermoelastic body with two temperatures is subjected to two thermoelastic bodies*

$L^{(\alpha)}$, $\alpha = 1, 2$; *then between the thermoelastic configurations* $C^{(\alpha)}$, $\alpha = 1, 2$, *which correspond to the problems* $P^{(\alpha)}$, $\alpha = 1, 2$, *there is the following reciprocity relation:*

$$\begin{aligned}
&\int_B \left[f_i^{(1)} * u_i^{(2)} + g_i^{(1)} * \varphi_i^{(2)} \right. \\
&\qquad \left. + l^{(1)} * \psi^{(2)} - \frac{1}{\varphi_0} g * W^{(1)} * \theta^{(2)} \right] dV \\
&\qquad + \int_{\partial B} g * \left[t_i^{(1)} * u_i^{(2)} + m_i^{(1)} * \varphi_i^{(2)} \right. \\
&\qquad\qquad \left. + p^{(1)} * \psi^{(2)} + \frac{1}{\varphi_0} g * q^{(1)} * \varphi^{(2)} \right] dA \\
&= \int_B \left[f_i^{(2)} * u_i^{(1)} + g_i^{(2)} * \varphi_i^{(1)} \right. \\
&\qquad \left. + l^{(2)} * \psi^{(1)} - \frac{1}{\varphi_0} g * W^{(2)} * \theta^{(1)} \right] dV \\
&\qquad + \int_{\partial B} g * \left[t_i^{(2)} * u_i^{(1)} + m_i^{(2)} * \varphi_i^{(1)} \right. \\
&\qquad\qquad \left. + p^{(2)} * \psi^{(1)} + \frac{1}{\varphi_0} g * q^{(2)} * \varphi^{(1)} \right] dA \\
&\qquad - \int_B g * \varepsilon_{kk}^{(2)} * \theta^{(1)} dV.
\end{aligned} \tag{47}$$

As an immediately consequence of Theorem 5, we indicate the following particular application. In fact, using the reciprocity relation (47) the problem of coupled thermoelasticity will be reduced to an associated problem of uncoupled thermoelasticity and to an integral equation.

Indeed, we take

$$\begin{gathered}
r^{(2)} = \delta(x - \xi)\,\delta(t), \qquad F_i^{(\alpha)} = G_i^{(\alpha)} = L^{(\alpha)} = 0, \\
\widetilde{u}_i^{(\alpha)} = \widetilde{\varphi}_i^{(\alpha)} = \widetilde{\psi}^{(\alpha)} = \widetilde{\varphi}^{(\alpha)} = 0, \quad \alpha = 1, 2, \\
\widetilde{t}_i^{(\alpha)} = \widetilde{m}_i^{(\alpha)} = \widetilde{p}^{(\alpha)} = \widetilde{q}^{(\alpha)} = 0,
\end{gathered} \tag{48}$$

$$a_i^{0(\alpha)} = a_i^{1(\alpha)} = b_i^{0(\alpha)} = b_i^{1(\alpha)} = \psi^{0(\alpha)} = \psi^{1(\alpha)} = \eta_0^{(\alpha)} = 0,$$

where δ is the Dirac's distribution.

Using the properties of convolution and taking into account definition (33) of the functions f_i, g_i, l and W, the reciprocity relation (47) permits to reduce the problem $P^{(1)}$ of coupled thermoelasticity to solving the problem $P^{(2)}$ of uncoupled thermoelasticity and to find the solution of the following integral equation:

$$\theta^{(1)}(\xi, t) = \int_B r^{(1)} * \theta^{(2)} dV - \varphi_0 \beta \int_B \dot{e}_{kk} r^{(2)} * \theta^{(1)} dV. \tag{49}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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