

Research Article

Existence of Conformal Metrics with Prescribed Q-Curvature

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We consider the problem of existence of conformal metrics with prescribed Q-curvature on standard sphere S^n , $n \geq 5$. Under the assumption that the order of flatness at critical points of prescribed Q-curvature function $K(x)$ is $\beta \in]1, n - 4]$, we give precise estimates on the losses of the compactness, and we prove new existence and multiplicity results through an Euler-Hopf type formula.

1. Introduction and Main Result

The Paneitz operator on n -Riemannian manifold (M, g_0) is a fourth-order differential operator which arises in conformal geometry and satisfies a certain covariance property (see [1]). For $n \geq 5$ it is defined by

$$P_{g_0}^n u = \Delta_{g_0}^2 u - \operatorname{div}_{g_0} (a_n S_{g_0} g_0 + b_n \operatorname{Ric}_{g_0}) du + \frac{n-4}{2} Q_{g_0}^n u, \quad (1)$$

where

$$a_n = \frac{(n-2)^2 + 4}{2(n-1)(n-2)}, \quad b_n = -\frac{4}{n-2},$$

$$Q_{g_0}^n = -\frac{1}{2(n-1)} \Delta_{g_0} S_{g_0} + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} S_{g_0}^2 - \frac{2}{(n-2)^2} |\operatorname{Ric}_{g_0}|^2. \quad (2)$$

Such a $Q_{g_0}^n$ is a fourth-order invariant called Q-curvature or Paneitz curvature. See [2, 3] for details about the properties of $P_{g_0}^n$.

If $g = u^{4/(n-4)} g_0$ is a conformal metric to g_0 , where u is a smooth positive function, then the conformal covariance property of the Paneitz operator reads as follows:

$$\forall \psi \in C^\infty(M) : P_{g_0}^n (u\psi) = u^{(n+4)/(n-4)} P_g^n (\psi). \quad (3)$$

If one prescribes the Q-curvature for the metric g by a function K , this leads to the following equation:

$$P_{g_0}^n (u) = u^{(n+4)/(n-4)} P_g^n (1) = \frac{n-4}{2} K u^{(n+4)/(n-4)}. \quad (4)$$

The literature for the existence of solutions of the prescribed Q-curvature problem on compact manifolds is considerably bigger. In [4–7], existence results for the constant Q-curvature problem in 4-dimensional manifolds are given. On manifolds of dimension greater than 4, existence results were given for Einstein manifolds in [3]. On the sphere S^n , we refer to results of [8–14] and the references therein.

In this paper we continue to study the problem of prescribing Q-curvature on the standard sphere (S^n, g_0) , $n \geq 5$. The Paneitz operator in this case is coercive on the Sobolev space $H_2^2(S^n)$ and has the following expression:

$$\mathcal{P}(u) := P_{g_0}^n (u) = \Delta_{g_0}^2 u - c_n \Delta_{g_0} u + d_n u, \quad (5)$$

where

$$c_n = \frac{1}{2}(n^2 - 2n - 4), \quad d_n = \frac{n-4}{16}n(n^2 - 4). \quad (6)$$

According to problem (4), the problem can be formulated as follows. Given a smooth function K on S^n , we look for solutions of

$$\begin{aligned} \mathcal{P}(u) &= \frac{n-4}{2}Ku^{(n+4)/(n-4)}, \\ u &> 0 \quad \text{on } S^n. \end{aligned} \quad (7)$$

The special case where the manifold is a sphere endowed with its standard metric deserves particular attention. Indeed due to Kazdan-Warner type obstructions, see [3], conditions have to be imposed on the function K .

To state our result we set up the following conditions and notation.

$(f)_\beta$ Assume that $K : S^n \rightarrow \mathbb{R}$, $n \geq 5$ be a C^1 positive function such that for any critical point y of K , there exists some real number $\beta = \beta(y)$ such that in some geodesic normal coordinate system centered at y (through the exponential map), we have

$$K(x) = K(y) + \sum_{k=1}^n b_k |(x-y)_k|^\beta + R(x-y), \quad (8)$$

where $b_k = b_k(y) \neq 0$, for all $k = 1, \dots, n$, $\sum_{k=1}^n b_k \neq 0$ and $\sum_{s=0}^{[\beta]} |\nabla^s R(x-y)| |x-y|^{-\beta+s} = o(1)$ as x tends to y . Here ∇^s denotes all possible derivatives of order s and $[\beta]$ is the integer part of β . For each critical point y of K , we denote

$$\tilde{i}(y) =: \#\{b_k, k = 1, \dots, n, \text{ such that } b_k < 0\}. \quad (9)$$

Let,

$$\begin{aligned} \mathcal{K} &= \{y \in S^n, \nabla_{g_0} K(y) = 0\} \\ \mathcal{K}^+ &= \left\{y \in \mathcal{K}, -\sum_{k=1}^n b_k > 0\right\}, \end{aligned} \quad (10)$$

$$\mathcal{K}_{n-4} = \{y \in \mathcal{K}, \beta = \beta(y) = n-4\}.$$

For each p -tuple, $p \geq 1$ of distinct points $\tau_p := (y_1, \dots, y_p)$ such that for all $i = 1, \dots, p$, $y_i \in \mathcal{K}^+ \cap \mathcal{K}_{n-4}$, we define a $p \times p$ symmetric matrix $M(\tau_p) = (m_{ij})$ by

$$\begin{aligned} m_{ii} &= \frac{n-4}{n} c_1 \frac{-\sum_{k=1}^n b_k(y_i)}{K(y_i)^{n/4}}, \\ m_{ij} &= 2^{(n-4)/2} c_1 \frac{-G(y_i, y_j)}{[K(y_i)K(y_j)]^{(n-4)/8}}, \end{aligned} \quad (11)$$

where

$$\begin{aligned} c_1 &= \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{(n+4)/2}}, \\ \tilde{c}_1 &= c_0^{2n/(n-4)} \int_{\mathbb{R}^n} \frac{|x_1|^{n-4}}{(1+|x|^2)^n} dx, \end{aligned} \quad (12)$$

and G is the Green function for the operator $\mathcal{P}(u)$ on S^n . Here x_1 is the first component of x in some geodesic normal coordinate system.

Let $\rho(\tau_p)$ be the least eigenvalue of $M(\tau_p)$. It was first pointed out by Bahri [15] that when the interaction between the different bubbles is of the same order as the self interaction, the least eigenvalue of some matrices like (11) plays a fundamental role in the existence of solutions to problems like (7). Regarding problem (7), such kind of phenomenon appears under $(f)_\beta$ condition in \mathcal{K}_{n-4} ; see [11].

(A_1) Assume that $\rho(\tau_p) \neq 0$ for each distinct point $y_{l_1}, \dots, y_{l_p} \in \mathcal{K}^+ \cap \mathcal{K}_{n-4}$.

Now, we introduce the following set:

$$\begin{aligned} \mathcal{E}_\infty &:= \left\{ \tau_p = (y_1, \dots, y_{l_p}) \in (\mathcal{K}^+)^p, p \geq 1, \right. \\ &\quad \text{s.t. } y_i \neq y_j \quad \forall i \neq j, \text{ and if} \\ &\quad \left. \{y_{l_1}, \dots, y_{l_p}\} \cap \mathcal{K}_{n-4} \neq \emptyset, \right. \\ &\quad \text{we denote by } y_{i_1}, \dots, y_{i_q} \text{ all elements of} \\ &\quad \left. \{y_{l_1}, \dots, y_{l_p}\} \text{ with } \beta(y_{i_j}) = n-4 \right. \\ &\quad \left. \text{for each } j = 1, \dots, q \text{ then} \right. \\ &\quad \left. \rho(y_{i_1}, \dots, y_{i_q}) > 0 \right\}. \end{aligned} \quad (13)$$

The main result of this paper is the following.

Theorem 1. Assume that K satisfies (A_1) and $(f)_\beta$, with

$$1 < \beta \leq n-4. \quad (14)$$

If

$$d = \sum_{(y_1, \dots, y_p) \in \mathcal{E}_\infty} (-1)^{p-1+\sum_{k=1}^p n-\tilde{i}(y_k)} - 1 \neq 0. \quad (15)$$

Then (7) has at least one solution.

Moreover if $(n-4)/2 < \beta \leq n-4$, for generic K one has

$$\#\mathcal{S} \geq |d|, \quad (16)$$

where \mathcal{S} denotes the set of solutions to (7).

Our argument uses a careful analysis of the lack of compactness of the Euler Lagrange functional J associated with problem (7). Namely, we study the noncompact orbits of the gradient flow of J the so-called critical points at infinity following the terminology of Bahri [15]. These critical points at infinity can be treated as usual critical points once a Morse lemma at infinity is performed from which we can derive just as in the classical Morse theory the difference of topology induced by these noncompact orbits and compute their Morse index. Such a Morse lemma at infinity is obtained through the construction of suitable pseudogradient for which the Palais-Smale condition is satisfied along the decreasing flow lines, as long as these flow lines do

not enter the neighborhood of a finite number y_1, \dots, y_p of critical points of K such that $(y_1, \dots, y_p) \in \mathcal{C}_\infty$.

Similar Morse lemma at infinity has been established for the problem (7) on the sphere S^n , $n \geq 5$, under the hypothesis that the order of flatness at critical points of K is $\beta \in [n-4, n]$; see [11].

The rest of this paper is organized as follows. In Section 2, we set up the variational problem and we recall the expansion of the gradient of the associated Euler-Lagrange functional near infinity. In Section 3, we characterize the critical points at infinity of the associated variational problem. Section 4 is devoted to the proof of the main result Theorem 1, while we give in Section 5 a more general statement than Theorem 1.

2. General Framework and Some Known Facts

2.1. Variational Structure and Lack of Compactness. Our problem (7) enjoys a variational structure. Indeed, solutions to (7) correspond to positive critical points of the functional

$$J(u) = \frac{\int_{S^n} \mathcal{P}uudv_{g_0}}{\left(\int_{S^n} K|u|^{2n/(n-4)} dv_{g_0}\right)^{(n-4)/2n}}, \quad (17)$$

defined on

$$\Sigma = \left\{ u \in H_2^2(S^n), \|u\|^2 := \int_{S^n} \mathcal{P}uudv_{g_0} = 1 \right\}. \quad (18)$$

However, it is delicate from a variational viewpoint because the functional J does not satisfy the Palais-Smale condition. This means that there exist sequences along which J is bounded, its gradient goes to zero and which not converge. The analysis of sequences failing Palais-Smale condition can be analyzed along the ideas introduced in [16, pages 325 and 334]. In order to describe such a characterization in our case, we need to introduce some notations. For $a \in S^n$ and $\lambda > 0$, let

$$\begin{aligned} \delta_{(a,\lambda)}(x) &= c_0 \frac{1}{2^{(n-4)/2}} \\ &\times \frac{\lambda^{(n-4)/2}}{\left(1 + ((\lambda^2 - 1)/2)(1 - \cos(d(x, a)))\right)^{(n-4)/2}}, \end{aligned} \quad (19)$$

where d is the geodesic distance on (S^n, g) and c_0 is chosen so that $\delta_{(a,\lambda)}$ is the family of solutions of the problem

$$\mathcal{P}(u) = u^{(n+4)/(n-4)}, \quad u > 0, \quad \text{on } S^n \quad (20)$$

(see [17]).

We define now the set of potential critical points at infinity associated with the function J . For $\varepsilon > 0$ and $p \in \mathbb{N}^*$, let us define

$$\begin{aligned} &V(p, \varepsilon) \\ &= \left\{ \begin{array}{l} u \in \Sigma / \exists a_1, \dots, a_p \in S^n, \exists \lambda_1, \dots, \lambda_p > \varepsilon^{-1}, \\ \exists \alpha_1, \dots, \alpha_p > 0 \text{ with } \left\| u - \sum_{i=1}^p \alpha_i \delta_{(a_i, \lambda_i)} \right\| < \varepsilon, \\ \left| J(u)^{n/(n-4)} \alpha_i^{8/(n-4)} K(a_i) - 1 \right| < \varepsilon \forall i, \varepsilon_{ij} < \varepsilon \forall i \neq j. \end{array} \right. \end{aligned} \quad (21)$$

Here, $\varepsilon_{ij} = [(\lambda_i/\lambda_j) + (\lambda_j/\lambda_i) + (\lambda_i\lambda_j/2)(1 - \cos d(a_i, a_j))]^{-(n-4)/2}$.

For w a solution to (7) we also define $V(p, \varepsilon, w)$ as

$$V(p, \varepsilon, w) = \left\{ \begin{array}{l} u \in \Sigma / \exists \alpha_0 > 0 \text{ s.t. } u - \alpha_0 w \in V(p, \varepsilon), \\ \left| \alpha_0^{8/(n-4)} J(u)^{n/(n-4)} - 1 \right| < \varepsilon. \end{array} \right. \quad (22)$$

If u is a function in $V(p, \varepsilon, w)$, one can find an optimal representation, following the ideas introduced in Proposition 5.2 of [15] and [16, pages 348–350]. Namely, we have the following.

Proposition 2. *For any $p \in \mathbb{N}^*$, there is $\varepsilon_p > 0$ such that if $\varepsilon \leq \varepsilon_p$ and $u \in V(p, \varepsilon, w)$, then the following minimization problem*

$$\min_{\substack{\alpha_i > 0, \lambda_i > 0, a_i \in S^n, \\ h \in T_w(W_u(w))}} \left\| u - \sum_{i=1}^p \alpha_i \delta_{(a_i, \lambda_i)} - \alpha_0(w + h) \right\| \quad (23)$$

has a unique solution (α, λ, a, h) , up to a permutation.

In particular, we can write u as follows:

$$u = \sum_{i=1}^p \alpha_i \delta_{(a_i, \lambda_i)} + \alpha_0(w + h) + v, \quad (24)$$

where v belongs to $H_2^2(S^n) \cap T_w(W_s(w))$ and it satisfies (V_0) , $T_w(W_u(w))$ and $T_w(W_s(w))$ are the tangent spaces at w of the unstable and stable manifolds of w for a decreasing pseudogradient of J , and (V_0) is the following:

$$\begin{aligned} \langle v, \psi \rangle &= 0 \quad \text{for } \psi \in \left\{ \delta_i, \frac{\partial \delta_i}{\partial \lambda_i}, \frac{\partial \delta_i}{\partial a_i}, i = 1, \dots, p \right\} \\ \langle v, w \rangle &= 0, \\ \langle v, h \rangle &= 0 \quad \forall h \in T_w W_u(w). \end{aligned} \quad (V_0)$$

Here, $\delta_i = \delta_{(a_i, \lambda_i)}$ and $\langle \cdot, \cdot \rangle$ denotes the scalar product defined on $H_2^2(S^n)$ by

$$\begin{aligned} \langle u, v \rangle &= \int_{S^n} \Delta_{g_0} u \Delta_{g_0} v dv_{g_0} + c_n \int_{S^n} \nabla_{g_0} u \nabla_{g_0} v dv_{g_0} \\ &+ d_n \int_{S^n} uv dv_{g_0}. \end{aligned} \quad (25)$$

Notice that Proposition 2 is also true if we take $w = 0$ and, therefore, $h = 0$ and u in $V(p, \varepsilon)$.

We are ready now to state the characterization of the sequences failing the Palais-Smale condition. For technical reasons, we introduce the following subsets of Σ . Let $\Sigma^+ = \{u \in \Sigma, u \geq 0\}$ and let for η positive very small,

$$V_\eta(\Sigma^+) = \left\{ u \in \Sigma, J(u)^{(2n-4)/(n-4)} e^{2J(u)} \|u^-\|_{L^{2n/(n-4)}}^{8/(n-4)} < \eta \right\}, \quad (26)$$

where $u^- = \max(0, -u)$. Using the idea introduced in [16, 18], see also [19], we have the following proposition.

Proposition 3. Let (u_k) be a sequence in $V_\eta(\Sigma^+)$ such that $J(u_k)$ is bounded and $\partial J(u_k)$ goes to zero. Then there exists an integer $p \in \mathbb{N}^*$, a sequence $(\varepsilon_k) > 0$, ε_k tends to zero, and an extracted subsequence of u_k 's, again denoted (u_k) , such that $u_k \in V(p, \varepsilon_k, w)$ where w is zero or a solution to (7).

Now arguing as in [16] (pages 326, 327, and 334), we have the following Morse lemma which completely gets rid of the v -contributions and shows that it can be neglected with respect to the concentration phenomenon.

Proposition 4. There is a \mathcal{C}^1 -map which to each $(\alpha_i, a_i, \lambda_i, h)$ such that $\sum_{i=1}^p \alpha_i \delta_{(a_i, \lambda_i)} + \alpha_0(w + h)$ belongs to $V(p, \varepsilon, w)$ associates $\bar{v} = \bar{v}(\alpha, a, \lambda, h)$ such that \bar{v} is unique and satisfies

$$J\left(\sum_{i=1}^p \alpha_i \delta_{(a_i, \lambda_i)} + \alpha_0(w + h) + \bar{v}\right) = \min_{v \in (V_0)} \left\{ J\left(\sum_{i=1}^p \alpha_i \delta_{(a_i, \lambda_i)} + \alpha_0(w + h) + v\right) \right\}. \tag{27}$$

Moreover, there exists a change of variables $v - \bar{v} \rightarrow V$ such that

$$J\left(\sum_{i=1}^p \alpha_i \delta_{(a_i, \lambda_i)} + \alpha_0(w + h) + v\right) = J\left(\sum_{i=1}^p \alpha_i \delta_{(a_i, \lambda_i)} + \alpha_0(w + h) + \bar{v}\right) + \|V\|^2. \tag{28}$$

The following proposition gives precise estimates of \bar{v} .

Proposition 5 (see [11, Lemma 3.1]). Let $u = \sum_{i=1}^p \alpha_i \delta_i + \alpha_0(w + h) \in V(p, \varepsilon, w)$ and let \bar{v} be defined in Proposition 4. One has the following estimates: there exists $c > 0$ independent of u such that the following holds:

$$\|\bar{v}\| \leq c \sum_{i=1}^p \left[\frac{1}{\lambda_i^{n/2}} + \frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{(\log \lambda_i)^{(n+4)/2n}}{\lambda_i^{(n+4)/2}} \right] + c \begin{cases} \sum_{k \neq r} \varepsilon_{kr}^{(n+4)/2(n-4)} (\log \varepsilon_{kr}^{-1})^{(n+4)/2n}, & \text{if } n \geq 12 \\ \sum_{k \neq r} \varepsilon_{kr} (\log \varepsilon_{kr}^{-1})^{(n-4)/n}, & \text{if } n < 12. \end{cases} \tag{29}$$

At the end of this subsection, we give the following definition.

Definition 6 (see [16, 18]). A critical point at infinity of J on $V_\eta(\Sigma^+)$ is a limit of a flow line $u(s)$ of the equation

$$\frac{\partial u}{\partial s} = -\partial J(u(s)), \tag{30}$$

$$u(0) = u_0,$$

such that $u(s)$ remains in $V(p, \varepsilon(s), w)$ for $s \geq s_0$. Here w is either zero or a solution to (7) and $\varepsilon(s)$ is some

positive function tending to zero when $s \rightarrow +\infty$. Using Proposition 2, $u(s)$ can be written as

$$u(s) = \sum_{i=1}^p \alpha_i(s) \delta_{(a_i(s), \lambda_i(s))} + \alpha_0(s) (w + h(s)) + v(s). \tag{31}$$

Denoting $\tilde{\alpha}_i := \lim_{s \rightarrow +\infty} \alpha_i(s)$, $\tilde{y}_i := \lim_{s \rightarrow +\infty} a_i(s)$, we denote by

$$\sum_{i=1}^p \tilde{\alpha}_i \delta_{(\tilde{y}_i, \infty)} + \tilde{\alpha}_0 w \quad \text{or} \quad (\tilde{y}_1, \dots, \tilde{y}_p, w)_\infty \tag{32}$$

such a critical point at infinity. If $w \neq 0$, it is called of w -type or mixed type.

Notice that $V_\eta(\Sigma^+)$ remains invariant under the flow generated by $(-\partial J)$ (see Lemma 5.1 of [20]; see also Lemma 4.1 of [18]).

2.2. Expansion of the Gradient of the Functional. In this subsection, we recall the expansion of the gradient of the functional J in $V(p, \varepsilon)$, $p \geq 1$.

Proposition 7 (see [11]). For any $u = \sum_{j=1}^p \alpha_j \delta_j$ in $V(p, \varepsilon)$, the following expansion holds

$$(i) \quad \left\langle \partial J(u), \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \right\rangle = -2c_2 J(u) \sum_{i \neq j} \alpha_j \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + o\left(\sum_{i \neq j} \varepsilon_{ij}\right) + o\left(\frac{1}{\lambda_i}\right), \tag{33}$$

where $c_2 = c_0^{2n/(n-4)} \int_{\mathbb{R}^n} dy / (1 + |y|^2)^{(n+4)/2}$.

(ii) If $a_i \in B(y_j, \rho)$, $y_j \in \mathcal{X}$ and ρ is a positive constant small enough, one has

$$\left\langle \partial J(u), \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \right\rangle = 2J(u) \left[-c_2 \sum_{j \neq i} \alpha_j \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{n-4}{2n} c_0^{2n/(n-4)} \beta \frac{\alpha_i}{K(a_i)} \times \frac{\sum_{k=1}^n b_k}{\lambda_i^\beta} \int_{\mathbb{R}^n} \text{signe}(x_k + \lambda_i(a_i - y_j)_k) \times |x_k + \lambda_i(a_i - y_j)_k|^{\beta-1} \times \frac{x_k}{(1 + |x|^2)^n} dx + o\left(\sum_{j \neq i} \varepsilon_{ij} + \sum_{j=1}^p \frac{1}{\lambda_j^\beta}\right) \right]. \tag{34}$$

(iii) Furthermore if $\lambda_i |a_i - y_j| < \delta$, for δ very small, one then has

$$\begin{aligned} & \left\langle \partial J(u), \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \right\rangle \\ &= 2J(u) \times \left[\frac{n-4}{2n} \beta c_3 \frac{\alpha_i}{K(a_i)} \frac{\sum_{k=1}^n b_k}{\lambda_i^\beta} - c_2 \sum_{j \neq i} \alpha_j \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \right. \\ & \quad \left. + o \left(\sum_{j \neq i} \varepsilon_{ij} + \sum_{j=1}^p \frac{1}{\lambda_j^\beta} \right) \right], \end{aligned} \tag{35}$$

where $c_3 = c_0^{2n/(n-4)} \int_{\mathbb{S}^n} |x_1|^\beta / (1 + |x|^2)^n dx$.

Proposition 8 (see [11]). Letting $u = \sum_{j=1}^p \alpha_j \delta_j \in V(p, \varepsilon)$, one has

$$\begin{aligned} & (i) \quad \left\langle \partial J(u), \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial a_i} \right\rangle \\ &= -c_5 (J(u))^{2(n-2)/(n-4)} \alpha_i^{(n+4)/(n-4)} \frac{\nabla K(a_i)}{\lambda_i} \\ & \quad + O \left(\sum_{i \neq j} \frac{1}{\lambda_i} \left| \frac{\partial \varepsilon_{ij}}{\partial a_i} \right| \right) + o \left(\sum_{i \neq j} \varepsilon_{ij} + \frac{1}{\lambda_i} \right), \end{aligned} \tag{36}$$

where $c_5 = \int_{\mathbb{R}^n} dy / (1 + |y|^2)^n$.

(ii) If $a_i \in B(y_j, \rho)$, $y_j \in \mathcal{X}$, one has

$$\begin{aligned} & \left\langle \partial J(u), \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_k} \right\rangle \\ &= -2(n-4) c_0^{2n/(n-4)} \alpha_i^{(n+4)/(n-4)} (J(u))^{2(n-2)/(n-4)} \frac{1}{\lambda_i^\beta} \\ & \quad \times \int_{\mathbb{R}^n} b_k |x_k + \lambda_i (a_i - y_j)_k|^\beta \frac{x_k}{(1 + |x|^2)^{n+1}} dy \\ & \quad + o \left(\sum_{i \neq j} \varepsilon_{ij} \right) + o \left(\sum_{i=1}^p \frac{1}{\lambda_i^\beta} \right) + O \left(\sum_{i \neq j} \frac{1}{\lambda_i} \left| \frac{\partial \varepsilon_{ij}}{\partial a_i} \right| \right), \end{aligned} \tag{37}$$

where $k = 1, \dots, n$ and $(a_i)_k$ is the k th component of a_i in some geodesic normal coordinates system.

3. Characterization of the Critical Points at Infinity

This section is devoted to the characterization of the critical points at infinity in $V(p, \varepsilon)$, $p \geq 1$, under β -flatness condition with $1 < \beta \leq n - 4$. This characterization is obtained through the construction of a suitable pseudogradient at infinity for which the Palais-Smale condition is satisfied along the decreasing flow-lines as long as these flow lines do not

enter in the neighborhood of finite number of critical points $y_i, i = 1, \dots, p$ of K such that $(y_1, \dots, y_p) \in \mathcal{C}_\infty$.

Now we introduce the following main result.

Theorem 9. Assume that K satisfies (A_1) and $(f)_\beta$, $1 < \beta \leq n - 4$.

Let $\beta := \max\{\beta(y) / y \in \mathcal{X}\}$. For $p \geq 1$, there exists a pseudogradient W in $V(p, \varepsilon)$ so that the following holds.

There exists a constant $c > 0$ independent of $u = \sum_{i=1}^p \alpha_i \delta_{(a_i, \lambda_i)} \in V(p, \varepsilon)$ such that

$$\begin{aligned} & (i) \quad \langle \partial J(u), W(u) \rangle \\ & \leq -c \left(\sum_{i=1}^p \frac{1}{\lambda_i^\beta} + \sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{j \neq i} \varepsilon_{ij} \right). \\ & (ii) \quad \left\langle \partial J(u + \bar{v}), W(u) + \frac{\partial \bar{v}}{\partial (\alpha_i, a_i, \lambda_i)} (W(u)) \right\rangle \\ & \leq -c \left(\sum_{i=1}^p \frac{1}{\lambda_i^\beta} + \sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{j \neq i} \varepsilon_{ij} \right). \end{aligned} \tag{38}$$

Furthermore $|W|$ is bounded and the only case where the maximum of the λ_i 's is not bounded is when $a_i \in B(y_i, \rho)$ with $y_i \in \mathcal{X}$, for all $i = 1, \dots, p$, $(y_1, \dots, y_p) \in \mathcal{C}_\infty$.

We will prove Theorem 9 at the end of the section. Now we state two results which deal with two specific cases of Theorem 9. Let

$$\begin{aligned} V_1(p, \varepsilon) &= \left\{ u = \sum_{i=1}^p \alpha_i \delta_i \in V(p, \varepsilon) \text{ s.t., } a_i \in B(y_i, \rho), \right. \\ & \quad \left. y_i \in \mathcal{X} \setminus \mathcal{X}_{n-4} \quad \forall i = 1, \dots, p \right\}, \\ V_2(p, \varepsilon) &= \left\{ u = \sum_{i=1}^p \alpha_i \delta_i \in V(p, \varepsilon) \text{ s.t., } a_i \in B(y_i, \rho), \right. \\ & \quad \left. y_i \in \mathcal{X}_{n-4} \quad \forall i = 1, \dots, p \right\}. \end{aligned} \tag{39}$$

We then have the following.

Proposition 10 (see [11], Proposition 3.7). For $p \geq 1$ there exists a pseudogradient W_2 in $V_2(p, \varepsilon)$ such that for all $u = \sum_{i=1}^p \alpha_i \delta_i \in V_2(p, \varepsilon)$, one has

$$\langle \partial J(u), W_2(u) \rangle \leq -c \left(\sum_{i=1}^p \frac{1}{\lambda_i^{n-4}} + \sum_{i \neq j} \varepsilon_{ij} + \sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} \right), \tag{40}$$

where c is a positive constant independent of u . Furthermore, one has $|W_2|$ which is bounded and the only case where the maximum of λ_i 's is not bounded is when $a_i \in B(y_i, \rho)$, $y_i \in \mathcal{X}^+$, for all $i = 1, \dots, p$, with $\rho(y_1, \dots, y_p) > 0$.

Proposition 11. For $p \geq 1$, there exists a pseudogradient W_1 in $V_1(p, \varepsilon)$ so that the following holds.

There exists $c > 0$ independent of $u = \sum_{i=1}^p \alpha_i \delta_i \in V_1(p, \varepsilon)$ such that

$$\langle \partial J(u), W_1(u) \rangle \leq -c \left(\sum_{i=1}^p \frac{1}{\lambda_i^\beta} + \sum_{i \neq j} \varepsilon_{ij} + \sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} \right). \quad (41)$$

Furthermore $|W_1|$ is bounded and the only case where the maximum of the λ_i 's is not bounded is when $a_i \in B(y_i, \rho)$ with $y_i \in \mathcal{X}^+$, for all $i = 1, \dots, p$, and $y_i \neq y_j$ for all $i \neq j$.

Observe that in $V_1(p, \varepsilon)$ the interaction of two bubbles is negligible with respect to the self-interaction. Similar phenomena occur for the scalar curvature problem, see [21], so the proof of Proposition 11 is similar to the corresponding statement in [21].

Before giving the proof of Theorem 9, we state the following notations extracted from [11].

Let M_1 be a fixed positive constant large enough and let $u = \sum_{i=1}^p \alpha_i \delta_{(a_i, \lambda_i)} \in V(p, \varepsilon)$ such that $a_i \in B(a_i, \rho)$, $y_i \in \mathcal{X}$, for all $i = 1, \dots, p$. For any index i , $1 \leq i \leq p$, we define the following vector fields:

$$\begin{aligned} Z_i(u) &= \alpha_i \lambda_i \frac{\partial \delta_{(a_i, \lambda_i)}}{\partial \lambda_i}, \\ X_i &= \alpha_i \sum_{k=1}^n \frac{1}{\lambda_i} \frac{\partial \delta_{(a_i, \lambda_i)}}{\partial (a_i)_k} \int_{\mathbb{R}^n} b_k \frac{|x_k + \lambda_i (a_i - y_i)_k|^{\beta_i}}{(1 + \lambda_i |(a_i - y_i)_k|)^{\beta_i - 1}} \\ &\quad \times \frac{x_k}{(1 + |x|^2)^{n+1}} dx. \end{aligned} \quad (42)$$

We claim that X_i is bounded. Indeed, the claim is trivial if $\lambda_i |a_i - y_i| \leq M_1$. If $\lambda_i |a_i - y_i| > M_1$, for any k , $1 \leq k \leq n$, such that $\lambda_i |(a_i - y_i)_k| > M_1 / \sqrt{n}$, we have the following estimate:

$$\begin{aligned} &\int_{\mathbb{R}^n} \frac{|x_k + \lambda_i (a_i - y_i)_k|^{\beta_i} x_k}{(1 + |x|^2)^{n+1}} dx \\ &= c (\text{signe } \lambda_i (a_i - y_i)_k) (\lambda_i |(a_i - y_i)_k|)^{\beta_i - 1} \\ &\quad \times (1 + o(1)). \end{aligned} \quad (43)$$

Hence our claim follows. Next, we will say that

$$\begin{aligned} i \in L_1 &\quad \text{if } \lambda_i |a_i - y_i| \leq M_1, \\ i \in L_2 &\quad \text{if } \lambda_i |a_i - y_i| > M_1, \end{aligned} \quad (44)$$

and we will denote by k_i the index satisfying

$$|(a_i - y_i)_{k_i}| = \max_{1 \leq k \leq n} |(a_i - y_i)_k|. \quad (45)$$

It easy to see that if $i \in L_2$, then $\lambda_i |(a_i - y_i)_{k_i}| > M_1 / \sqrt{n}$.

Now, we introduce the following two lemmas.

Lemma 12. Let $u = \sum_{i=1}^p \alpha_i \delta_{(a_i, \lambda_i)} \in V(p, \varepsilon)$, such that $a_i \in B(y_i, \rho)$, $y_i \in \mathcal{X}$, for all $i = 1, \dots, p$. One then has

$$\begin{aligned} \langle \partial J(u), Z_i(u) \rangle &= -2c_2 J(u) \sum_{j \neq i} \alpha_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + O\left(\frac{1}{\lambda_i^{\beta_i}}\right) \\ &\quad + \left[O\left(\frac{|(a_i - y_i)_{k_i}|^{\beta_i - 2}}{\lambda_i^2}\right), \text{ if } i \in L_2 \right] \\ &\quad + o\left(\sum_{j \neq i} \varepsilon_{ij}\right) + o\left(\sum_{j=1}^p \frac{1}{\lambda_j^{\beta_j}}\right), \end{aligned} \quad (46)$$

where k_i is defined in (45).

Proof. Using Proposition 7, we have

$$\begin{aligned} \langle \partial J(u), Z_i(u) \rangle &= -2c_2 J(u) \sum_{j \neq i} \alpha_j \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{n-4}{2n} c_0^{2n/(n-4)} \beta \frac{\alpha_i^2}{K(a_i)} \\ &\quad \times \int_{\mathbb{R}^n} \text{signe}(x_k + \lambda_i (a_i - y_i)_k) |x_k + \lambda_i (a_i - y_i)_k|^{\beta_i - 1} \\ &\quad \times \frac{x_k}{(1 + |x|^2)^n} dx + o\left(\sum_{j \neq i} \varepsilon_{ij}\right) + o\left(\sum_{j=1}^p \frac{1}{\lambda_j^{\beta_j}}\right). \end{aligned} \quad (47)$$

Observe that for $k \in \{1, \dots, n\}$, if $\lambda_i |(a_i - y_i)_k| > M_1 / \sqrt{n}$, we have

$$\begin{aligned} &\int_{\mathbb{R}^n} \text{signe}(x_k + \lambda_i (a_i - y_i)_k) \\ &\quad \times \frac{|x_k + \lambda_i (a_i - y_i)_k|^{\beta_i - 1} x_k}{(1 + |x|^2)^n} dx \\ &= c \text{signe}(\lambda_i (a_i - y_i)_k) \\ &\quad \times (\lambda_i |(a_i - y_i)_k|)^{\beta_i - 2} (1 + o(1)), \end{aligned} \quad (48)$$

taking M_1 large enough. If not, we have

$$\int_{\mathbb{R}^n} \frac{|x_k + \lambda_i (a_i - y_i)_k|^{\beta_i - 1} |x_k|}{(1 + |x|^2)^n} dx = O(1). \quad (49)$$

Using the fact that k_i defined in (45) satisfies $\lambda_i |(a_i - y_i)_{k_i}| > M_1 / \sqrt{n}$, if $i \in L_2$, Lemma 12 follows. \square

Lemma 13. For $u = \sum_{i=1}^p \alpha_i \delta_{(a_i, \lambda_i)} \in V(p, \varepsilon)$, such that $a_i \in B(y_i, \rho)$, $y_i \in \mathcal{X}$, for all $i = 1, \dots, p$, one has

$$\begin{aligned} & \langle \partial J(u), X_i(u) \rangle \\ & \leq O\left(\sum_{j \neq i} \frac{1}{\lambda_j} \left| \frac{\partial \varepsilon_{ij}}{\partial a_i} \right|\right) + O\left[\left(\frac{1}{\lambda_i^{\beta_i}}\right), \text{ if } i \in L_1\right] \\ & + \left[-c\left(\frac{1}{\lambda_i^{\beta_i}} + \frac{|(a_i - y_i)_{k_i}|^{\beta_i - 1}}{\lambda_i}\right), \text{ if } i \in L_2\right] \quad (50) \\ & + o\left(\sum_{j=1}^p \frac{1}{\lambda_j^{\beta_j}}\right), \end{aligned}$$

where k_i is defined in (45).

Proof. Using Proposition 8, we have

$$\begin{aligned} & \langle \partial J(u), X_i(u) \rangle \\ & = -2(n-4) c_0^{2n/(n-4)} \alpha_i^{2n/(n-4)} \frac{1}{\lambda_i^{\beta_i}} \\ & \times \sum_{k=1}^n \left(\int_{\mathbb{R}^n} b_k \frac{|x_k + \lambda_i(a_i - y_i)_{k_i}|^{\beta_i}}{(1 + \lambda_i |(a_i - y_i)_{k_i}|)^{(\beta_i - 1)/2}} \right. \\ & \quad \left. \times \frac{x_k}{(1 + |x|^2)^{n+1}} dx \right)^2 \\ & + O\left(\sum_{j \neq i} \frac{1}{\lambda_j} \left| \frac{\partial \varepsilon_{ij}}{\partial a_i} \right|\right) + o\left(\sum_{j=1}^p \frac{1}{\lambda_j^{\beta_j}}\right) \quad (51) \\ & \leq -c \frac{1}{\lambda_i^{\beta_i}} \left(\int_{\mathbb{R}^n} b_{k_i} \frac{|x_{k_i} + \lambda_i(a_i - y_i)_{k_i}|^{\beta_i}}{(1 + \lambda_i |(a_i - y_i)_{k_i}|)^{(\beta_i - 1)/2}} \right. \\ & \quad \left. \times \frac{x_{k_i}}{(1 + |x|^2)^{n+1}} dx \right)^2 \\ & + O\left(\sum_{j \neq i} \frac{1}{\lambda_j} \left| \frac{\partial \varepsilon_{ij}}{\partial a_i} \right|\right) + o\left(\sum_{j=1}^p \frac{1}{\lambda_j^{\beta_j}}\right). \end{aligned}$$

Using (43) and the fact that $\lambda_i |(a_i - y_i)_{k_i}| > M_1/\sqrt{n}$, if $i \in L_2$, Lemma 13 follows. \square

Proof of Theorem 9. In order to complete the construction of the pseudogradient W suggested in Theorem 9, it only

remains (using Propositions 10 and 11) to focus attention on the two following subsets of $V(p, \varepsilon)$.

Subset 1. We consider here the case of $u = \sum_{i=1}^p \alpha_i \delta_{(a_i, \lambda_i)} = \sum_{i \in I_1} \alpha_i \delta_{(a_i, \lambda_i)} + \sum_{i \in I_2} \alpha_i \delta_{(a_i, \lambda_i)}$ such that

$$\begin{aligned} & I_1 \neq \emptyset, \quad I_2 \neq \emptyset, \quad \sum_{i \in I_1} \alpha_i \delta_{(a_i, \lambda_i)} \in V_1 \quad (\#I_1, \varepsilon), \\ & \sum_{i \in I_2} \alpha_i \delta_{(a_i, \lambda_i)} \in V_2 \quad (\#I_2, \varepsilon). \end{aligned} \quad (52)$$

Without loss of generality, we can assume that

$$\lambda_1 \leq \dots \leq \lambda_p. \quad (53)$$

We distinguish three cases.

Case 1. $u_1 := \sum_{i \in I_1} \alpha_i \delta_{(a_i, \lambda_i)} \notin V_1^1(\#I_1, \varepsilon) = \{u = \sum_{j=1}^{\#I_1} \alpha_j \delta_{(a_j, \lambda_j)}, a_j \in B(y_j, \rho), y_j \in \mathcal{X}^+$ for all $j = 1, \dots, \#I_1$ and $y_j \neq y_k$ for all $j \neq k\}$.

Let \widetilde{W}_1 be the pseudogradient on $V(p, \varepsilon)$ defined by $\widetilde{W}_1(u) = W_1(u_1)$, where W_1 is the vector field defined by Proposition 11 in $V_1(\#I_1, \varepsilon)$. Note that if $u_1 \notin V_1^1(\#I_1, \varepsilon)$, then the pseudo-gradient $W_1(u_1)$ does not increase the maximum of the λ_i 's, $i \in I_1$. Using Proposition 11, we have

$$\begin{aligned} & \langle \partial J(u), \widetilde{W}_1(u) \rangle \\ & \leq -c \left(\sum_{i \in I_1} \frac{1}{\lambda_i^{\beta_i}} + \sum_{j \neq i, j \in I_1} \varepsilon_{ij} + \sum_{i \in I_1} \frac{|\nabla K(a_i)|}{\lambda_i} \right) \\ & + O\left(\sum_{i \in I_1, j \in I_2} \varepsilon_{ij}\right). \end{aligned} \quad (54)$$

An easy calculation yields

$$\varepsilon_{ij} = O\left(\frac{1}{(\lambda_i \lambda_j)^{(n-4)/2}}\right) = o\left(\frac{1}{\lambda_i^{\beta_i}}\right) + o\left(\frac{1}{\lambda_j^{\beta_j}}\right), \quad (55)$$

Fix $i_0 \in I_1$, we denote by

$$J_1 = \left\{ i \in I_2, \text{ s.t. } \lambda_i^{n-4} \geq \frac{1}{2} \lambda_{i_0}^{\beta_{i_0}} \right\}, \quad J_2 = I_2 \setminus J_1. \quad (56)$$

Using (54) and (55), we find that

$$\begin{aligned} & \langle \partial J(u), \widetilde{W}_1(u) \rangle \\ & \leq -c \left(\sum_{i \in I_1 \cup J_1} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i \in I_1} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{j \neq i \in I_1} \varepsilon_{ij} \right) \\ & + o\left(\sum_{i=1}^p \frac{1}{\lambda_i^{\beta_i}}\right). \end{aligned} \quad (57)$$

From another part, by Lemma 12 we have

$$\begin{aligned} & \left\langle \partial J(u), \sum_{i \in J_1} -2^i Z_i(u) \right\rangle \\ & \leq c \sum_{j \neq i, i \in J_1} 2^i \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + O\left(\sum_{i \in J_1} \frac{1}{\lambda_i^{\beta_i}}\right) \\ & + O\left(\sum_{i \in J_1 \cap L_2} \frac{|(a_i - y_i)_{k_i}|^{\beta_i - 2}}{\lambda_i^2}\right). \end{aligned} \tag{58}$$

Observe that using a direct calculation, we have

$$\begin{aligned} \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} & \leq -c \varepsilon_{ij}, \quad \text{if } \lambda_i \geq \lambda_j \text{ or } \lambda_i \sim \lambda_j \\ & \text{or } |a_i - a_j| \geq \delta_0 > 0. \end{aligned} \tag{59}$$

Since for $i < j$, we have

$$2^i \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + 2^j \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \leq -c \varepsilon_{ij}, \tag{60}$$

and for $i \in J_1$ and $j \in J_2$ we have $\lambda_j \leq \lambda_i$, we obtain $\lambda_i (\partial \varepsilon_{ij} / \partial \lambda_i) \leq -c \varepsilon_{ij}$. These estimates yield

$$\begin{aligned} & \left\langle \partial J(u), \sum_{i \in J_1} -2^i Z_i(u) \right\rangle \\ & \leq -c \sum_{j \neq i, i \in J_1, j \in J_1 \cup J_2} \varepsilon_{ij} + O\left(\sum_{i \in J_1} \frac{1}{\lambda_i^{\beta_i}}\right) \\ & + O\left(\sum_{i \in J_1 \cap L_2} \frac{|(a_i - y_i)_{k_i}|^{\beta_i - 2}}{\lambda_i^2}\right) \\ & + O\left(\sum_{i \in J_1, j \in I_1} \varepsilon_{ij}\right). \end{aligned} \tag{61}$$

It is easy to see that for any index $i \in L_2$, we have

$$\frac{|(a_i - y_i)_{k_i}|^{\beta_i - 2}}{\lambda_i^2} \leq \frac{\sqrt{n} |(a_i - y_i)_{k_i}|^{\beta_i - 1}}{M_1 \lambda_i}, \tag{62}$$

where k_i is defined in (45) and M_1 is large enough. Thus, we derive that

$$\frac{|(a_i - y_i)_{k_i}|^{\beta_i - 2}}{\lambda_i^2} = o\left(\frac{|(a_i - y_i)_{k_i}|^{\beta_i - 1}}{\lambda_i}\right), \tag{63}$$

for any $i \in L_2$.

Let $m_1 > 0$ be small enough; using Lemma 13, (63), and (55), we get

$$\begin{aligned} & \left\langle \partial J(u), \sum_{i \in J_1} -2^i Z_i(u) + m_1 \sum_{i \in J_1 \cap L_2} X_i(u) \right\rangle \\ & \leq -c \left(\sum_{j \neq i, i \in J_1, j \in J_1 \cup J_2} \varepsilon_{ij} + \sum_{i \in J_1} \frac{|\nabla K(a_i)|}{\lambda_i} \right) \\ & + O\left(\sum_{i \in J_1} \frac{1}{\lambda_i^{\beta_i}}\right) + o\left(\sum_{i=1}^p \frac{1}{\lambda_i^{\beta_i}}\right), \end{aligned} \tag{64}$$

and by (57) we obtain

$$\begin{aligned} & \left\langle \partial J(u), \widetilde{W}_1(u) + m_1 \left(\sum_{i \in J_1} -2^i Z_i(u) + m_1 \sum_{i \in J_1 \cap L_2} X_i(u) \right) \right\rangle \\ & \leq -c \left(\sum_{i \in I_1 \cup J_1} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i \neq j \in I_1} \varepsilon_{ij} \right. \\ & \quad \left. + \sum_{j \neq i, i \in J_1, j \in J_1 \cup J_2} \varepsilon_{ij} + \sum_{i \in I_1 \cup J_1} \frac{|\nabla K(a_i)|}{\lambda_i} \right) \\ & + o\left(\sum_{i=1}^p \frac{1}{\lambda_i^{\beta_i}}\right). \end{aligned} \tag{65}$$

We need to add the remainder indices $i \in J_2$. Note that $\tilde{u} := \sum_{j \in J_2} \alpha_j \delta_j \in V_2(\#J_2, \varepsilon)$. Thus using Proposition 10, we apply the associated vector field which we will denote by \widetilde{W}_2 . We then have the following estimate:

$$\begin{aligned} & \left\langle \partial J(u), \widetilde{W}_2(u) \right\rangle \\ & \leq -c \left(\sum_{j \in J_2} \frac{1}{\lambda_j^{\beta_j}} + \sum_{i \neq j, i, j \in J_2} \varepsilon_{ij} + \sum_{j \in J_2} \frac{|\nabla K(a_j)|}{\lambda_j} \right) \\ & + O\left(\sum_{j \in J_2, i \in J_1} \varepsilon_{ij}\right) + o\left(\sum_{i=1}^p \frac{1}{\lambda_i^{\beta_i}}\right), \end{aligned} \tag{66}$$

since $|a_i - a_j| \geq \rho$ for $j \in J_2$ and $i \in I_1$.

Let in this case $W = \widetilde{W}_1 + m_1(\widetilde{W}_2 + \sum_{i \in J_1} -2^i Z_i + m_1 \sum_{i \in J_1 \cap L_2} X_i)$.

From (65) and (66) we obtain

$$\left\langle \partial J(u), W(u) \right\rangle \leq -c \left(\sum_{i=1}^p \frac{1}{\lambda_i^{\beta_i}} + \sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} \right). \tag{67}$$

Case 2. $u_1 := \sum_{i \in I_1} \alpha_i \delta_{(a_i, \lambda_i)} \in V_1^1(\#I_1, \varepsilon)$ and $u_2 := \sum_{i \notin I_2} \alpha_i \delta_{(a_i, \lambda_i)} \notin V_2^1(\#I_2, \varepsilon) := \{u = \sum_{j=1}^{\#I_2} \alpha_j \delta_{(a_j, \lambda_j)}, a_j \in B(y_j, \rho), y_j \in \mathcal{X}^+, \text{ for all } j = 1, \dots, \#I_2 \text{ and } \rho(y_1, \dots, y_{\#I_2}) > 0\}$.

Since $u_2 \in V_2(\#I_2, \varepsilon)$, by Proposition 10, we can apply the associated vector field which we will denote by V_1 . We get

$$\begin{aligned} & \langle \partial J(u), V_1(u) \rangle \\ & \leq -c \left(\sum_{i \in I_2} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i \in I_2} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \neq j, i, j \in I_2} \varepsilon_{ij} \right) \\ & \quad + O \left(\sum_{i \in I_2, j \in I_1} \varepsilon_{ij} \right). \end{aligned} \quad (68)$$

Observe that $V_1(u)$ does not increase the maximum of the λ_i 's, $i \in I_2$, since $u_2 \notin V_2^1(\#I_2, \varepsilon)$. Fix $i_0 \in I_2$ and let

$$\tilde{I}_1 = \left\{ i \in I_1, \text{ s.t. } \lambda_i^{\beta_i} \geq \frac{1}{2} \lambda_{i_0}^{n-4} \right\}, \quad \tilde{I}_2 = I_1 \setminus \tilde{I}_1. \quad (69)$$

Using (68) and (55), we get

$$\begin{aligned} & \langle \partial J(u), V_1(u) \rangle \\ & \leq -c \left(\sum_{i \in I_2 \cup \tilde{I}_1} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i \in I_2} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \neq j, i, j \in I_2} \varepsilon_{ij} \right) \\ & \quad + o \left(\sum_{i=1}^p \frac{1}{\lambda_i^{\beta_i}} \right). \end{aligned} \quad (70)$$

We need to add the indices $i, i \in \tilde{I}_2$. Letting $\tilde{u} := \sum_{j \in \tilde{I}_2} \alpha_j \delta_{(a_j, \lambda_j)}$, since $\tilde{u} \in V_1(\#\tilde{I}_2, \varepsilon)$, we can apply the associated vector field given by Proposition 10. Let V_2 be this vector field. By Proposition 11 we have

$$\begin{aligned} & \langle \partial J(u), V_2(u) \rangle \\ & \leq -c \left(\sum_{j \in \tilde{I}_2} \frac{1}{\lambda_j^{\beta_j}} + \sum_{j \in \tilde{I}_2} \frac{|\nabla K(a_j)|}{\lambda_j} + \sum_{i \neq j, i, j \in \tilde{I}_2} \varepsilon_{ij} \right) \\ & \quad + O \left(\sum_{j \in \tilde{I}_2, i \notin \tilde{I}_2} \varepsilon_{ij} \right). \end{aligned} \quad (71)$$

Observe that $I_1 = \tilde{I}_1 \cup \tilde{I}_2$ and we are in the case where for all $i \neq j \in I_1$, we have $|a_i - a_j| \geq \rho$. Thus by (55), we get

$$O \left(\sum_{j \in \tilde{I}_2, i \notin \tilde{I}_2} \varepsilon_{ij} \right) = o \left(\sum_{i=1}^p \frac{1}{\lambda_i^{\beta_i}} \right), \quad (72)$$

and hence

$$\begin{aligned} & \langle \partial J(u), V_1(u) + V_2(u) \rangle \\ & \leq -c \left(\sum_{i=1}^p \frac{1}{\lambda_i^{\beta_i}} + \sum_{i \in I_2 \cup \tilde{I}_2} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} \right). \end{aligned} \quad (73)$$

Let in this case $W = V_1 + V_2 + m_1 \sum_{i \in \tilde{I}_1} X_i(u)$, m_1 small enough.

Using the previous estimate and Lemma 13, we find that

$$\langle \partial J(u), W(u) \rangle \leq -c \left(\sum_{i=1}^p \frac{1}{\lambda_i^{\beta_i}} + \sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} \right). \quad (74)$$

Case 3. $u_1 \in V_1^1(\#I_1, \varepsilon)$ and $u_2 \in V_2^1(\#I_2, \varepsilon)$.

Let \tilde{V}_1 (resp., \tilde{V}_2) be the pseudo-gradient in $V(p, \varepsilon)$ defined by $\tilde{V}_1(u) = W_1(u_1)$ (resp., $\tilde{V}_2(u) = W_2(u_2)$) where W_1 (resp., W_2) is the vector field defined by Proposition 11 (resp., Proposition 10) in $V_1^1(\#I_1, \varepsilon)$ (resp., $V_2^1(\#I_2, \varepsilon)$) and let in this case

$$W = \tilde{V}_1 + \tilde{V}_2. \quad (75)$$

Using Proposition 10, Proposition 11, and (55) we get

$$\langle \partial J(u), W(u) \rangle \leq -c \left(\sum_{i=1}^p \frac{1}{\lambda_i^{\beta_i}} + \sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} \right). \quad (76)$$

Note that in the first and second cases, the maximum of the λ_i 's, $1 \leq i \leq p$, is a bounded function and hence the Palais-Smale condition is satisfied along the flow lines of W . However in the third case all the λ_i 's, $1 \leq i \leq p$, will increase and go to $+\infty$ along the flow lines generated by W .

Subset 2. We consider the case of $u = \sum_{i=1}^p \alpha_i \delta_i \in V(p, \varepsilon)$, such that there exist a_i satisfying $a_i \notin \cup_{y \in \mathcal{R}} B(y, \rho)$.

In this region, the construction of the pseudo-gradient W proceeds exactly as the proof of Theorem 3.2, of subset 2, of [11].

Finally, observe that our pseudogradient W in $V(p, \varepsilon)$ satisfies claim (i) of Theorem 9 and it is bounded, since $\|\lambda_i(\partial \delta_{(a_i, \lambda_i)} / \partial \lambda_i)\|$ and $\|(1/\lambda_i)(\partial \delta_{(a_i, \lambda_i)} / \partial a_i)\|$ are bounded. From the definition of W , the λ_i 's, $1 \leq i \leq p$, decrease along the flow lines of W as long as these flow lines do not enter in the neighborhood of finite number of critical points y_i , $i = 1, \dots, p$, of \mathcal{R} such that $(y_1, \dots, y_p) \in \mathcal{C}_\infty$.

Now, arguing as in Appendix 2 of [16], see also Appendix B of [18], claim (ii) of Theorem 9 follows from (i) and Proposition 5. This completes the proof of Theorem 9. \square

Corollary 14. *Let $p \geq 1$. The critical points at infinity of J in $V(p, \varepsilon)$ correspond to*

$$(y_1, \dots, y_p)_\infty := \sum_{i=1}^p \frac{1}{K(y_i)^{(n-4)/2}} \delta_{(y_i, \infty)}, \quad (77)$$

where $(y_1, \dots, y_p) \in \mathcal{C}_\infty$. Moreover, such a critical point at infinity has an index equal to $i(y_1, \dots, y_p)_\infty = p - 1 + \sum_{i=1}^p n - \tilde{i}(y)$.

4. Proof of Theorem 1

We prove the existence result by contradiction. Assume that J has no critical point in $V_\eta(\Sigma^+)$. It follows from Corollary 14

that the critical points at infinity of the associated variational problem are in one to one correspondence with the elements of \mathcal{C}_∞ defined in (13).

Notice that, just like for usual critical points, it is associated with each critical point at infinity w_∞ of J stable and unstable manifolds $W_s^\infty(w_\infty)$ and $W_u^\infty(w_\infty)$ (see [16, pages 356-357]). These manifolds can be easily described once a finite-dimensional reduction like the one we performed in Section 3 is established.

For any $w_\infty = (y_{i_1}, \dots, y_{i_p}) \in \mathcal{C}_\infty$, let $c(w)_\infty = S_n(\sum_{j=1}^p 1/K(y_{i_j})^{(n-4)/2})^{4/n}$ denote the associated critical value. Here we choose to consider a simplified situation, where for any $w_\infty \neq w'_\infty$, $c(w)_\infty \neq c(w')_\infty$, and thus order the $c(w)_\infty$'s, $w_\infty \in \mathcal{C}_\infty$ as

$$c(w_1)_\infty < \dots < c(w_{k_0})_\infty. \tag{78}$$

By using a deformation lemma (see [22, proposition 7.24 and Theorem 8.2]), we know that if $c(w_{k-1})_\infty < a < c(w_k)_\infty < b < c(w_{k+1})_\infty$, then

$$J_b \simeq J_a \cup W_u^\infty(w_k)_\infty, \tag{79}$$

where $J_b = \{u \in V_\eta(\Sigma^+), J(u) \leq b\}$ and \simeq denotes retracts by deformation.

We apply the Euler-Poincaré characteristic of both sides of (79); we find that

$$\chi(J_b) = \chi(J_a) + (-1)^{i(w_k)_\infty}, \tag{80}$$

where $i(w_k)_\infty$ denotes the index of the critical point at infinity $(w_k)_\infty$. Let

$$b_1 < c(w_1)_\infty = \min_{u \in V_\eta(\Sigma^+)} J(u) < b_2 < c(w_2)_\infty < \dots < b_{k_0} < c(w_{k_0})_\infty < b_{k_0+1}. \tag{81}$$

Since we have assumed that (4) has no solution, $J_{b_{k_0+1}}$ is a retract by deformation of $V_\eta(\Sigma^+)$. Therefore $\chi(J_{b_{k_0+1}}) = 1$, since $V_\eta(\Sigma^+)$ is a contractible set. Now using (80), we derive after recalling that $\chi(J_{b_1}) = \chi(\emptyset) = 0$

$$1 = \sum_{j=1}^{k_0} (-1)^{i(w_j)_\infty}. \tag{82}$$

Hence if (82) is violated, J has a critical point in $V_\eta(\Sigma^+)$.

Now, arguing as in the proof of theorem of [18, pages 659-660], we prove that such a critical point is positive.

To prove the multiplicity part of the statement, we observe that it follows from Sard-Smale theorem that for generic K 's, the solutions to (7) are all nondegenerate, in the sense that the associated linearized operator does not admit zero as eigenvalue. We need to introduce the following lemma extracted from [11].

Lemma 15 (see [11, Section 3.2]). *Let w be a solution to (7). Assume that the function K satisfies condition $(f)_\beta$, with $(n-4)/2 < \beta \leq n-4$; then for each $p \in \mathbb{N}^*$, there are neither critical points nor critical points at infinity in $V(p, \varepsilon, w)$.*

Once the existence of mixed critical points at infinity is ruled out, it follows from the previous arguments that

$$V_\eta(\Sigma^+) \simeq \bigcup_{j=1}^{k_0} W_u^\infty(w_j)_\infty \cup \bigcup_{w, \partial J(w)=0} W_u(w). \tag{83}$$

Now using the Euler-Poincaré theorem, we derive that

$$1 = \sum_{j=1}^{k_0} (-1)^{i(w_j)_\infty} + \sum_{w, \partial J(w)=0} (-1)^{\text{morse}(w)}. \tag{84}$$

Hence our theorem follows.

5. A General Existence Result

In this last section of this paper, we give a generalization of Theorem 1. Namely, in view of the result of Theorem 1, one may think about the situation where the degree d given in Theorem 1 is equal to zero; that is, the total sum in d is equal to 1, but a partial one is not equal to 1. A natural question arises: is it possible in this case to use such information to derive an existence result? In the following theorem we give a partial answer to this question.

Theorem 16. *Let K be a function satisfying (A_1) and $(f)_\beta$, with $1 < \beta \leq n-4$. If there exists $k \in \mathbb{N}$ such that*

- (1) $\forall w_\infty \in \mathcal{C}_\infty, i(w)_\infty \neq k+1$
- (2) $\sum_{w_\infty \in \mathcal{C}_\infty, i(w_\infty) \leq k} (-1)^{i(w_\infty)} \neq 1,$

then the problem (7) has at least one solution.

Moreover for generic K 's, if $(n-4)/2 < \beta \leq n-4$, then the number of solutions is lower bounded by

$$\left| 1 - \sum_{w_\infty \in \mathcal{C}_\infty, i(w_\infty) \leq k} (-1)^{i(w_\infty)} \right|. \tag{86}$$

Let $l_\#$ be the maximal index over all elements of \mathcal{C}_∞ . Please observe that the integer $k = l_\#$ satisfies (1) of Theorem 16; it follows that Theorem 1 is a corollary of Theorem 16.

Proof of Theorem 16. We set

$$X_k^\infty = \bigcup_{x_\infty \in \mathcal{C}_\infty, i(x_\infty) \leq k} \overline{W_u^\infty(x_\infty)}, \tag{87}$$

where $\overline{W_u^\infty(x_\infty)}$ is the closure of the unstable manifolds of x_∞ . Observe that X_k^∞ is a stratified set of top dimension k , which is contractible set in $V_\eta(\Sigma^+)$, since $V_\eta(\Sigma^+)$ is a contractible set. Let U denote the image of such a contraction. To prove the first part of Theorem 16, arguing by contradiction, we assume that (7) has no solution. Using the pseudo-gradient constructed in Theorem 9, we can deform U . By transversality arguments, we can assume that such a deformation avoids all critical points at infinity of index

greater or equal to $k + 1$. Note that, from assumption (1) of Theorem 16, there is no critical point at infinity with index $k + 1$. It follows then from a theorem of Bahri and Rabinowitz [22] that

$$U \simeq X_k^\infty. \quad (88)$$

Hence from the Euler-Poincaré theorem, we get

$$1 = \chi(X_k^\infty) = \sum_{w_\infty \in C_\infty, i(w_\infty) \leq k} (-1)^{i(w_\infty)}, \quad (89)$$

which is a contradiction with assumption (2) of Theorem 16.

Regarding the multiplicity result, we observe that for generic K 's the functional J admits only nondegenerate critical points. Hence by Lemma 15, the set U will be deformed into

$$U \simeq X_k^\infty \cup \bigcup_{w \in C_k} W_u(w), \quad (90)$$

where C_k denotes the set of the critical points of J of Morse index less than or equal to k , which are dominated by U . Finally by using the Euler-Poincaré theorem, the proof of Theorem 16 follows. \square

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