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Research Article

Stochastic Optimization Theory of Backward Stochastic Differential Equations Driven by G-Brownian Motion

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We consider the stochastic optimal control problems under G-expectation. Based on the theory of backward stochastic differential equations driven by G-Brownian motion, which was introduced in Hu et al. (2012), we can investigate the more general stochastic optimal control problems under G-expectation than that were constructed in Zhang (2011). Then we obtain a generalized dynamic programming principle, and the value function is proved to be a viscosity solution of a fully nonlinear second-order partial differential equation.

1. Introduction

Nonlinear BSDEs in the framework of linear expectation were introduced by Pardoux and Peng [1] in 1990. Then, a lot of researches were studied by many authors, and they provided various applications of BSDEs in stochastic control, finance, stochastic differential games, and second-order partial differential equations theory; see [2–9].

The notion of sublinear expectation space was introduced by Peng [10–12], which is a generalization of classical probability space. The G-expectation, a type of sublinear expectation, has played an important role in the researches of sublinear expectation space recently. It can be regarded as a counterpart of the Wiener probability space in the linear case. Within this G-expectation framework, the G-Brownian motion is the canonical process. Besides, the notions of the G-martingales and the Itô integral with respect to G-Brownian motion were also derived. There are some new structures in these notions and some new applications in the financial models with volatility uncertainty; see Peng [12, 13].

In the G-expectation framework, thanks to a series of studies [14–17], the complete representation theorem for G-martingales has been obtained by Peng et al. [18]. Due to this contribution, a natural formulation of BSDEs driven by G-Brownian motion was found by Hu et al. [19]. In addition, the existence and uniqueness of the solution to the BSDEs driven by G-Brownian motion have been proved. They also have given the comparison theorem, Feynman-Kac formula and Girsanov transformation for BSDEs driven by G-Brownian motion in [20]. So the complete theory of BSDEs driven by G-Brownian motion has been established.

An important application of BSDEs is that we can define the recursive utility functions from BSDEs, which can index scaling risks in the study of economics and finance [21–24]. Based on these results, a type of significant stochastic optimal control problems under linear expectation with a BSDE as cost function was studied [2, 4, 7–9]. Under G-expectation, the similar problems will be useful in the future studies of finance models with volatility uncertainty. So we arise a natural question: can we construct the similar results in G-expectation framework? In [25, 26], Zhang have given the study about the stochastic control problems under G-expectation based on the preliminary theory of BSDEs driven by G-Brownian motion. When the complete results about BSDEs driven by G-Brownian motion were established in [19, 20], we tried to prove the complete results of stochastic optimization theory of BSDEs driven by G-Brownian motion in this paper.

In this paper, we investigate the stochastic optimal control problems with a BSDE driven by G-Brownian motion constructed in [19, 20] as cost function. Based on the results in [19, 20], we obtain the dynamic programming principle under G-expectation. Besides, the value function is proved to be a viscosity solution of a fully nonlinear second-order partial differential equation.

The rest of the paper is organized as follows. In Section 2, we recall the G-expectation framework and adapt it according to our objective. Besides, we give the related properties of forward and backward stochastic differential equations driven by G-Brownian motion, which will be needed in the sequel sections. In Section 3, the stochastic optimal control problems with a BSDE driven by G-Brownian motion as cost function are investigated and a dynamic programming principle under G-expectation is obtained. In Section 4, The value function is proved to be a viscosity solution of a fully nonlinear second-order partial differential equation.

2. Preliminaries

In this section, we recall the G-expectation framework established by Peng [10–12, 27]. Besides, we give some results about forward and backward stochastic differential equations driven by G-Brownian motion, which we need in the following sections. Some details can be found in [19, 20].

2.1. G-Expectation and G-Martingales

Definition 1. Let Ω be a given set, and let $\mathscr{H}$ be a linear space of real valued functions defined on Ω; namely, $c \in \mathscr{H}$ for each constant c and $|X| \in \mathscr{H}$ if $X \in \mathscr{H}$. The space $\mathscr{H}$ can be considered as the space of random variables. A sublinear expectation $\mathbb{E}$ is a functional $\mathbb{E} : \mathscr{H} \to \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathscr{H}$, we have

(i) monotonicity: $\mathbb{E}[X] \geq \mathbb{E}[Y]$ if $X \geq Y$;

(ii) constant preservation: $\mathbb{E}[c] = c$, for $c \in \mathbb{R}$;

(iii) subadditivity: $\mathbb{E}[X+Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$;

(iv) positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$, for $\lambda \geq 0$.

The triple $(\Omega, \mathscr{H}, \mathbb{E})$ is called a sublinear expectation space.

Definition 2 (G-normal distribution). A d-dimensional random vector $X = (X_1, \ldots, X_d)$ on a sublinear expectation space $(\Omega, \mathscr{H}, \mathbb{E})$ is called G-normally distributed if for each $a, b \geq 0$, we have

$$X + b\overline{X} \overset{d}{=} \sqrt{a^2 + b^2} X, \tag{1}$$

where $\overline{X}$ is an independent copy of X; that is, $\overline{X}$ and X are identically distributed, and $\overline{X}$ is independent of X. Here, the letter G denotes the function

$$G(A) := \frac{1}{2}\mathbb{E}\left[(AX, X)\right] : \mathbb{S}_d \longmapsto \mathbb{R}, \tag{2}$$

where $\mathbb{S}_d$ denotes the collection of all $d \times d$ symmetric matrices.

Proposition 3. *Let X be G-normal distributed. The distribution of X is characterized by*

$$u(t,x) = \mathbb{E}\left[\varphi\left(x + \sqrt{t}X\right)\right], \quad \varphi \in C_{b,L_{ip}}\left(\mathbb{R}^d\right). \tag{3}$$

In particular, $\mathbb{E}[\varphi(X)] = u(1,0)$, where u is the unique viscosity solution of the following parabolic PDE defined on $[0,\infty)\times\mathbb{R}^d$:

$$\partial_t u - G\left(D^2 u\right) = 0, \quad u|_{t=0} = \varphi, \tag{4}$$

where G is defined by (2).

Remark 4. It is easy to check that G is a monotonic sublinear function defined on $\mathbb{S}(d)$ and $G(A) := (1/2)\mathbb{E}[(AX, X)] \leq (1/2)|A|\mathbb{E}[|X|^2] = (1/2)|A|\overline{\sigma}^2$ implies that there exists a bounded, convex, and closed subset $\Gamma \subset \mathbb{S}_d^+$ such that

$$G(A) = \frac{1}{2}\sup_{\gamma\in\Gamma} \operatorname{Tr}(\gamma A), \tag{5}$$

where $\mathbb{S}_d^+$ denotes the collection of nonnegative elements in $\mathbb{S}_d$. If there exist some $\beta > 0$ such that $G(A) - G(B) \geq \beta \operatorname{tr}[A - B]$ for any $A \geq B$, we call the G-normal distribution nondegenerate, which is the case we consider throughout this paper.

Definition 5. Let $\Omega = C_0^d([0,T])$, that is, the space of all $\mathbb{R}^d$-valued continuous paths $(\omega_t)_{t\in[0,T]}$ with $\omega_0 = 0$. The corresponding canonical process is $B_t(\omega) = \omega_t$, $t \in [0,T]$. P_0 is wiener measure. $\mathbb{F} = \{\mathscr{F}_t^B\}_{t\geq 0}$ is the filtration generated by B. We let $\mathscr{H} := L_{ip}(\Omega_T)$ be a linear space of random variables for each fixed $T \geq 0$, where $L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, \ldots, B_{t_n}) : n \geq 1, t_1, \ldots, t_n \in [0,T], \varphi \in C_{b,L_{ip}}(\mathbb{R}^{d\times n})\}$.

(i) The G-expectation $\widehat{E}$ is a sublinear expectation defined by

$$\widehat{E}[X] := \widetilde{E}\left[\varphi\left(\sqrt{t_1 - t_0}\xi_1, \ldots, \sqrt{t_n - t_{n-1}}\xi_n\right)\right], \tag{6}$$

for each $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})$, where $(\xi_i)_{i=1}^n$ are identically distributed d-dimensional G-normally distributed random vectors in a sublinear expectation space $(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{E})$ such that ξ_{i+1} is independent of $(\xi_1, \ldots, \xi_i)$ for each $i = 1, 2, \ldots, n-1$. $(\Omega, \mathscr{H}, \widehat{E})$ is called G-expectation space and the canonical process $\{B_t\}_{t\in[0,T]}$ in the sublinear space $(\Omega, \mathscr{H}, \widehat{E})$ is called a G-Brownian motion.

(ii) The conditional G-expectation $\widehat{E}_t$ of $X \in L_{ip}(\Omega_T)$ is defined by

$$\begin{aligned}\widehat{E}_{t_j}&\left[\varphi\left(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}\right)\right] \\ &:= \psi\left(B_{t_1} - B_{t_0}, \ldots, B_{t_j} - B_{t_{j-1}}\right),\end{aligned} \tag{7}$$

where $\psi(x_1, \ldots, x_j) = \widetilde{E}[\varphi(x_1, \ldots, x_j, \sqrt{t_{j+1} - t_j}\xi_{j+1}, \ldots, \sqrt{t_n - t_{n-1}}\xi_n)]$.

We denote by $L_G^p(\Omega_T)$, $p \geq 1$, the completion of G-expectation space $L_{ip}(\Omega_T)$ under the norm

$\|X\|_{p,G} := (\widehat{E}[|X|^p])^{1/p}$. For all $t \in [0,T]$, $\widehat{E}[\cdot]$ and $\widehat{E}_t[\cdot]$ are continuous mappings on $L_{ip}(\Omega_T)$ endowed with the norm $\|\cdot\|_{1,G}$. Therefore, it can be extended continuously to $L_G^p(\Omega_T)$.

Definition 6. A process $\{M_t\}_{t\geq 0}$ is called a G-martingale if for each $t \in [0,T]$, $M_t \in L_G^1(\Omega_t)$, and for each $s \in [0,t]$, we have $\widehat{E}_s[M_t] = M_s$.

Now we introduce the Itô integral and quadratic variation process with respect to G-Brownian motion in G-expectation space.

Definition 7. Let $p \geq 1$ be fixed. For a given partition $\pi_T = \{t_0, \ldots, t_N\}$ of $[0,T]$, we denote $M_G^{p,0}(0,T)$ as the collection of the following type of simple processes:

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) 1_{[t_k,t_{k+1})}(t), \tag{8}$$

where $\xi_k \in L_{ip}(\Omega_{t_k})$, $k = 0,1,2,\ldots,N-1$. We denote by $M_G^p(0,T)$ the completion of $M_G^{p,0}(0,T)$ under the norm $\|\cdot\|_{M_G^p(0,T)} := \{\widehat{E}[\int_0^T |\cdot|^p dt]\}^{1/p}$.

Definition 8. For each $\eta \in M_G^{2,0}(0,T)$, we define

$$I(\eta) = \int_0^T \eta_t dB_t := \sum_{j=0}^{N-1} \xi_j \left(B_{t_{j+1}} - B_{t_j}\right). \tag{9}$$

The mapping $I : M_G^{2,0}(0,T) \to L_G^2(\Omega_T)$ is continuous and thus can be continuously extended to $M_G^2(0,T)$.

Definition 9. The quadratic variation process of G-Brownian motion is defined by

$$\langle B\rangle_t := B_t^2 - 2\int_0^t B_s dB_s, \tag{10}$$

which is a continuous, nondecreasing process.

Definition 10. We now define the integral of a process $\eta \in M_G^1(0,T)$ with respect to $\langle B\rangle$ as follows:

$$\begin{aligned} Q_{0,T}(\eta) &= \int_0^T \eta_t d\langle B\rangle_t \\ &:= \sum_{j=0}^{N-1} \xi_j \left(\langle B\rangle_{t_{j+1}} - \langle B\rangle_{t_j}\right) : M_G^{1,0}(0,T) \\ &\longrightarrow L_G^1(\Omega_T). \end{aligned} \tag{11}$$

The mapping is continuous and can be extended to $M_G^1(0,T)$ uniquely.

Then, we detail some results about the quasianalysis theory constructed in [27].

Theorem 11. *There exists a weakly compact probability measures family $\mathscr{P}$ on $(\Omega, \mathscr{B}(\Omega))$ such that*

$$\widehat{E}[X] = \sup_{P\in\mathscr{P}} E_P[X], \quad \forall X \in L_{ip}(\Omega_T), \tag{12}$$

$\mathscr{P}$ is called a set of probability measures that represents $\widehat{E}$.

Definition 12. We define the capacity associated to $\mathscr{P}$, which is a weakly compact family of probability measure that represents $\widehat{E}$, as follows:

$$\widehat{c}(A) := \sup_{P\in\mathscr{P}} P(A), \quad A \in \mathscr{B}(\Omega_T); \tag{13}$$

$\widehat{c}$ is also called the capacity induced by $\widehat{E}$.

Let $(\Omega^0, \mathscr{F}^0 = \{\mathscr{F}_t^0\}, \mathscr{F}, P^0)$ be a filtered probability space, and let $\{W_t\}$ be a d-dimensional Brownian motion under P^0. [27] proved that $\mathscr{P}_M := \{P_0 \circ X^{-1} \mid X_t = \int_0^t h_s dW_s, h \in L_{\mathscr{F}^0}^2([0,T]; \Gamma^{1/2})\}$ represents G-expectation $\widehat{E}$, where $\Gamma^{1/2} := \{\gamma^{1/2} \mid \gamma \in \Gamma\}$ and Γ is the set in the representation of $G(\cdot)$ of the formula (5).

Definition 13. (i) Let $\widehat{c}$ be the capacity induced by $\widehat{E}$. A set $A \subset \Omega$ is polar if $\widehat{c}(A) = 0$. A property holds "quasi-surely" (q.s. for short) if it holds outside a polar set.

(ii) Let X and Y be two random variables; we say that X is a version of Y if $X = Y$ q.s.

Let $\|\psi\|_{p,G} = [\widehat{E}(|\psi|^p)]^{1/p}$ for $\psi \in C_b(\Omega_T)$. The completion of $C_b(\Omega_T)$ and $L_{ip}(\Omega_T)$ under $\|\cdot\|_{p,G}$ is the same, and we denote them by $L_G^p(\Omega_T)$.

2.2. Forward and Backward Stochastic Differential Equations Driven by G-Brownian Motion. We consider the following stochastic differential equations driven by d dimensional G-Brownian motion (G-SDE):

$$\begin{aligned} X_t = X_0 &+ \int_0^t b(s,X_s)\,ds + \sum_{i,j=1}^d \int_0^t h_{ij}(s,X_s)\,d\left\langle B^i, B^j\right\rangle_s \\ &+ \sum_{j=1}^d \int_0^t \sigma_j(s,X_s)\,dB_s^j, \end{aligned} \tag{14}$$

where $t \in [0,T]$, the initial condition $X_0 \in \mathbb{R}^n$ is a given constant, b, h_{ij}, σ_j are given functions satisfying $b(\cdot,x)$, and $h_{ij}(\cdot,x)$, $\sigma_j(\cdot,x) \in M_G^2(0,T;\mathbb{R}^n)$ for each $x \in \mathbb{R}^n$ and the Lipschitz condition, that is, $|\phi(t,x) - \phi(t,x')| \leq K|x - x'|$, for each $t \in [0,T]$, $x, x' \in \mathbb{R}^n$, $\phi = b, h_{ij}$, and σ_j, respectively. The solution is a process $X \in M_G^2(0,T;\mathbb{R}^n)$ satisfying the G-SDE (14).

Theorem 14 (see [12]). *There exists a unique solution $X \in M_G^2(0,T;\mathbb{R}^n)$ of the stochastic differential equation* (14).

Now, we give the results about BSDEs driven by G-Brownian motion in the G-expectation space

$(\Omega_T, L_G^1(\Omega_T), \widehat{E})$ with $\Omega_T = C_0([0,T],\mathbb{R}^d)$ and $\overline{\sigma}^2 = \widehat{E}[B_1^2] \geq -\widehat{E}[-B_1^2] = \underline{\sigma}^2 > 0$. We consider the following type of G-BSDEs (we always use Einstein convention):

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds + \sum_{i,j=1}^{d} \int_t^T g_{ij}(s, Y_s, Z_s)\, d\left\langle B^i, B^j \right\rangle_s - \int_t^T Z_s dB_s - (K_T - K_t), \tag{15}$$

where $f(t,\omega,y,z), g_{ij}(t,\omega,y,z) : [0,T]\times\Omega_T\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ satisfy the following properties: there exist some $\beta > 1$ such that

(H1) for any y, z, $f(\cdot,\cdot,y,z)$, $g_{ij}(\cdot,\cdot,y,z) \in M_G^{\beta}(0,T)$;

(H2) for some $L > 0$,

$$\begin{aligned} &\left| f(t,\omega,y,z) - f(t,\omega,y',z') \right| \\ &\quad + \sum_{i,j=1}^{d} \left| g_{ij}(t,\omega,y,z) - g_{ij}(t,\omega,y',z') \right| \\ &\leq L\left(\left| y - y' \right| + \left| z - z' \right| \right). \end{aligned} \tag{16}$$

For simplicity, we denote by $\mathfrak{G}(0,T)$ the collection of processes (Y,Z,K) such that $Y \in S_G^{\alpha}(0,T)$, $Z \in H_G^{\alpha}(0,T)$; K is a decreasing G-martingale with $K_0 = 0$ and $K_T \in L_G^{\alpha}(\Omega_T)$. Here, $S_G^{\alpha}(0,T)$ is the completion of $S_G^0(0,T) = \{h(t, B_{t_1\wedge t,\ldots,} B_{t_n\wedge t}) : t_1,\ldots,t_n \in [0,T], h \in C_{b,L_{ip}(\mathbb{R}^{n+1})}\}$ under $\|\cdot\|_{s_G^p} = \{\widehat{E}[\sup_{t\in[0,T]}|\eta_t|^p]\}^{1/p}$, and $H_G^p(0,T)$ is the completion of $M_G^0(0,T)$ under $\|\cdot\|_{H_G^p} = \{\widehat{E}[(\int_0^T |\eta_s|^2 ds)^{p/2}]\}^{1/p}$.

Definition 15. Let $\xi \in L_G^{\beta}(\Omega_T)$ with $\beta > 1$; f and g_{ij} satisfy (H1) and (H2). A triplet of processes (Y,Z,K) is called a solution of (15) if for some $1 < \alpha \leq \beta$ the following properties hold:

(a) $(Y,Z,K) \in \mathfrak{G}_G^{\alpha}(0,T)$;

(b)

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds + \sum_{i,j=1}^{d} \int_t^T g_{ij}(s, Y_s, Z_s)\, d\left\langle B^i, B^j \right\rangle_s - \int_t^T Z_s dB_s - (K_T - K_t). \tag{17}$$

Theorem 16 (see [19]). *Assume that $\xi \in L_G^{\beta}(\Omega_T)$ and f, g satisfy (H1) and (H2) for some $\beta > 1$. Then,* (15) *has a unique solution (Y,Z,K). Moreover, for any $1 < \alpha < \beta$, we have $Y \in S_G^{\alpha}(0,T)$, $Z \in H_G^{\alpha}(0,T;\mathbb{R}^d)$, and $K_T \in L_G^{\alpha}(\Omega_T)$.*

We have the following estimates.

Proposition 17 (see [19]). *Let $\xi \in L_G^{\beta}(\Omega_T)$, and f, g_{ij} satisfy (H1) and (H2) for some $\beta > 1$. For some $1 < \alpha < \beta$, $(Y,Z,K) \in \mathfrak{G}_G^{\alpha}(0,T)$ is solution of equation* (15)*. Then*

(i) *There exists a constant $C_\alpha := C(\alpha,T,G,L) > 0$ such that*

$$\begin{aligned} &\left|Y_t\right|^{\alpha} \leq C_\alpha \widehat{E}_t\left[|\xi|^{\alpha} + \int_t^T \left|h_s^0\right|^{\alpha} ds \right], \\ &\widehat{E}\left[\left(\int_0^T \left|Z_s\right|^2 ds \right)^{\alpha/2} \right] \\ &\leq C_\alpha \left\{ \widehat{E}\left[\sup_{t\in[0,T]} \left|Y_t\right|^{\alpha} \right] + \left(\widehat{E}\left[\sup_{t\in[0,T]} \left|Y_t\right|^{\alpha} \right] \right)^{1/2} \left(\widehat{E}\left[\left(\int_0^T h_s^0 ds \right)^{\alpha} \right] \right)^{1/2} \right\}, \\ &\widehat{E}\left[\left|K_T\right|^{\alpha} \right] \leq C_\alpha \left\{ \widehat{E}\left[\sup_{t\in[0,T]} \left|Y_t\right|^{\alpha} \right] + \widehat{E}\left[\left(\int_0^T h_s^0 ds \right)^{\alpha} \right] \right\}, \end{aligned} \tag{18}$$

where $h_s^0 = |f(s,0,0)| + \sum_{i,j=1}^{d} |g_{ij}(s,0,0)|$.

(ii) *For any given $\alpha < \alpha' < \beta$, there exists a constant $C_{\alpha,\alpha'}$ depending onα, α', T, G, L such that*

$$\begin{aligned} &\widehat{E}\left[\sup_{t\in[0,T]} \left|Y_t\right|^{\alpha} \right] \\ &\leq C_{\alpha,\alpha'} \left\{ \widehat{E}\left[\sup_{t\in[0,T]} \widehat{E}_t\left[|\xi|^{\alpha} \right] \right] + \left(\widehat{E}\left[\sup_{t\in[0,T]} \widehat{E}_t\left[\left(\int_0^T h_s^0 ds \right)^{\alpha'} \right] \right] \right)^{\alpha/\alpha'} + \widehat{E}\left[\sup_{t\in[0,T]} \widehat{E}_t\left[\left(\int_0^T h_s^0 ds \right)^{\alpha'} \right] \right] \right\}. \end{aligned} \tag{19}$$

Proposition 18 (see [20]). *Let $\xi^i \in L_G^{\beta}(\Omega_T)$, $i = 1, 2$, and f^i, g_{ij}^i satisfy (H1) and (H2) for some $\beta > 1$. For some $1 < \alpha < \beta$, $(Y^i, Z^i, K^i) \in \mathfrak{G}_G^{\alpha}(0,T)$ are solutions of* (15) *corresponding to ξ^i, f^i, and g_{ij}^i. Set $\widehat{Y}_t = Y_t^1 - Y_t^2$, $\widehat{Z}_t = Z_t^1 - Z_t^2$, and $\widehat{K}_t = K_t^1 - K_t^2$.*

(i) *There exists a constant $C_\alpha := C(\alpha,T,G,L) > 0$ such that*

$$\left|\widehat{Y}_t\right|^{\alpha} \leq C_\alpha \widehat{E}_t\left[\left|\widehat{\xi}\right|^{\alpha} + \int_t^T \left|\widehat{h}_s\right|^{\alpha} ds \right], \tag{20}$$

where $\widehat{\xi} = \xi^1 - \xi^2$, $\widehat{h}_s = |f^1(s, Y_s^2, Z_s^2) - f^2(s, Y_s^2, Z_s^2)| + \sum_{i,j=1}^{d} |g_{ij}^1(s, Y_s^2, Z_s^2) - g_{ij}^2(s, Y_s^2, Z_s^2)|$.

(ii) *For any given α' with $\alpha < \alpha' < \beta$, there exists a constant $C_{\alpha,\alpha'}$ depending on α, α', T, G, L such that*

$$\begin{aligned}\widehat{E}\left[\sup_{t\in[0,T]}\left|\widehat{Y}_t\right|^{\alpha}\right] &\le C_{\alpha,\alpha'}\left\{\widehat{E}\left[\sup_{t\in[0,T]}\widehat{E}_t\left[\left|\xi\right|^{\alpha}\right]\right]\right.\\ &\quad+\left(\widehat{E}\left[\sup_{t\in[0,T]}\widehat{E}_t\left[\left(\int_0^T\widehat{h}_s ds\right)^{\alpha'}\right]\right]\right)^{\alpha/\alpha'}\\ &\quad\left.+\widehat{E}\left[\sup_{t\in[0,T]}\widehat{E}_t\left[\left(\int_0^T\widehat{h}_s ds\right)^{\alpha'}\right]\right]\right\}.\end{aligned}\tag{21}$$

Theorem 19 (see [20]). *Let $(Y_t^i, Z_t^i, K_t^i)_{t\le T}$, $i = 1, 2$, be the solutions of the following G-BSDEs:*

$$\begin{aligned}Y_t^i &= \xi^i + \int_t^T f_i\left(s, Y_s^i, Z_s^i\right) ds + \int_t^T g_i\left(s, Y_s^i, Z_s^i\right) d\langle B\rangle_s\\ &\quad- \int_t^T Z_s^i dB_s - \left(K_T^i - K_t^i\right),\end{aligned}\tag{22}$$

where $\xi^i \in L_G^{\beta}(\Omega_T)$, f_i, g_i satisfy (H1) and (H2) with $\beta > 1$. If $\xi^1 \ge \xi^2$, $f_1 \ge f_2$, $g_1 \ge g_2$, then $Y_t^1 \ge Y_t^2$.

Theorem 20 (see [20]). *Let $(Y_t^i, Z_t^i, K_t^i)_{t\le T}$, $i = 1, 2$, be the solutions of the following G-BSDEs:*

$$\begin{aligned}Y_t^i &= \xi^i + \int_t^T f_i\left(s, Y_s^i, Z_s^i\right) ds + \int_t^T g_i\left(s, Y_s^i, Z_s^i\right) d\langle B\rangle_s\\ &\quad- \int_t^T Z_s^i dB_s - \left(K_T^i - K_t^i\right) + V_T^i - V_t^i,\end{aligned}\tag{23}$$

where $\xi^i \in L_G^{\beta}(\Omega_T)$, f_i, g_i satisfy (H1) and (H2), $(V_t^i)_{t\le T}$ are RCLL processes such that $\widehat{E}[\sup_{t\in[0,T]}|V_t^i|^{\beta}] < \infty$ with $\beta > 1$. If $\xi^1 \ge \xi^2$, $f_1 \ge f_2$, $g_1 \ge g_2$, and $V_t^1 - V_t^2$ is an increasing process, then $Y_t^1 \ge Y_t^2$.

3. A DPP for Stochastic Optimal Control Problems under G-Expectation

Now we introduce the setting for stochastic optimal control problems under G-expectation. We suppose that the control state space V is a compact metric space. Let the set of admissible control processes $\mathcal{U}$ for the player be a set of V-valued stochastic processes in $M_G^2([t,T];\mathbb{R}^n)$, $t \in [0,T]$. For a given admissible control $v(\cdot) \in \mathcal{U}$, the corresponding orbit, which regards t as the initial time and $\xi \in L_G^2(\Omega_t;\mathbb{R}^n)$ as the initial state, is defined by the solution of the following type of G-SDE:

$$\begin{aligned}dX_s^{t,\xi;v} &= b\left(s, X_s^{t,\xi;v}, v_s\right) ds + \sum_{i,j=1}^{d} h_{ij}\left(s, X_s^{t,\xi;v}, v_s\right) d\left\langle B^i, B^j\right\rangle_s\\ &\quad+ \sum_{j=1}^{d}\sigma_j\left(s, X_s^{t,\xi;v}, v_s\right) dB_s^j, \quad s \in [t,T], \; X_t^{t,\xi;v} = \xi,\end{aligned}\tag{24}$$

where $b, h_{ij}, \sigma_j : [0,T] \times \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$ are deterministic functions and satisfy the following conditions (H3):

(A1) $h_{ij} = h_{ji}$ for $1 \le i, j \le d$;

(A2) For every fixed $(x, v) \in \mathbb{R}^n \times \mathcal{U}$, $b(\cdot, x, v)$, $h_{ij}(\cdot, x, v)$, $\sigma_j(\cdot, x, v)$ are continuous in t;

(A3) There exists a constant $L > 0$, for any $t \in [0,T]$, x, $x' \in \mathbb{R}^n$, v, $v' \in \mathcal{U}$ such that

$$\begin{aligned}&\left|b(t,x,v) - b\left(t,x',v'\right)\right| + \sum_{i,j=1}^{d}\left|h_{ij}(t,x,v) - h_{ij}\left(t,x',v'\right)\right|\\ &\quad+ \sum_{j=1}^{d}\left|\sigma_j(t,x,v) - \sigma_j\left(t,x',v'\right)\right|\\ &\le L\left(\left|x - x'\right| + \left|v - v'\right|\right).\end{aligned}\tag{25}$$

From the assumption (H3), we can get global linear growth conditions for b, h_{ij}, σ_j; that is, there exists $C > 0$ such that, for $t \in [0,T]$, $x \in \mathbb{R}^n$, $v \in \mathcal{U}$, $|b(t,x,v)| + \sum_{i,j=1}^{d}|h_{ij}(t,x,v)| + \sum_{j=1}^{d}|\sigma_j(t,x,v)| \le C(1 + |x| + |v|)$. Obviously, under the above assumptions, for any $v(\cdot) \in \mathcal{U}$, G-SDE (24) has a unique solution. Moreover, we have the following estimates.

Proposition 21. *Let ξ, $\xi' \in L_G^p(\Omega_t;\mathbb{R}^n)$ with $p \ge 2$, $v(\cdot)$, $v'(\cdot) \in \mathcal{U}$, $t \in [0,T]$, and $\delta \in [0, T-t]$; then we have*

$$\begin{aligned}\widehat{E}_t\left[\left|X_{t+\delta}^{t,\xi;v} - X_{t+\delta}^{t,\xi';v'}\right|^p\right] &\le C\left(\left|\xi - \xi'\right|^p + \int_t^{t+\delta}\widehat{E}_t\left|v_r - v_r'\right|^p dr\right),\\ \widehat{E}_t\left[\left|X_{t+\delta}^{t,\xi;v}\right|^p\right] &\le C\left(1 + |\xi|^p\right),\\ \widehat{E}_t\left[\sup_{s\in[t,t+\delta]}\left|X_s^{t,\xi;v} - \xi\right|^p\right] &\le C\left(1 + |\xi|^p\right)\delta^{p/2},\end{aligned}\tag{26}$$

where C depends on L, G, p, n, T.

Proof. The proof is similar to the proof of Proposition 4.1 in [20]. □

Now we give bounded functions $\Phi : \mathbb{R}^n \to \mathbb{R}$, $f : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{U} \to \mathbb{R}$, $g_{ij} : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{U} \to \mathbb{R}$ that satisfy the following conditions: (H4)

(i) $g_{ij} = g_{ji}$ for $1 \le i, j \le d$.

(ii) For every fixed $(x, y, z, v) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{U}$, $f(\cdot, x, y, z, v)$ and $g_{ij}(\cdot, x, y, z, v)$ are continuous in t, $1 \le i, j \le d$.

(iii) There exist a constant $L > 0$, for $t \in [0, T]$, $x, x' \in \mathbb{R}^n$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$, $v, v' \in \mathcal{U}$, such that

$$\begin{gathered} \left|\Phi(x) - \Phi(x')\right| \le L\left(\left|x - x'\right|\right), \\ \left|f(t, x, y, z, v) - f(t, x', y', z', v')\right| \\ + \sum_{i,j=1}^{d} \left|g_{ij}(t, x, y, z, v) - g_{ij}(t, x', y', z', v')\right| \\ \le L\left(\left|x - x'\right| + \left|y - y'\right| + \left|z - z'\right| + \left|v - v'\right|\right). \end{gathered} \tag{27}$$

From (H4), we have that Φ, f, and g_{ij} also satisfy global linear growth condition in x; that is, there exists $C > 0$, such that for all $0 \le t \le T$, $v \in \mathcal{U}$, $x \in \mathbb{R}^n$,

$$\begin{aligned} &|\Phi(x)| + |f(t, x, 0, 0, v)| + \left|g_{ij}(t, x, 0, 0, v)\right| \\ &\quad \le C(1 + |x| + |v|). \end{aligned} \tag{28}$$

For any $v \in \mathcal{U}$ and $\xi \in L_G^2(\Omega_t, \mathbb{R}^n)$, the mappings $f(s, x, y, z, v) := f(s, X_s^{t,\xi;v}, y, z, v_s)$ and $g_{ij}(s, x, y, z, v) = g_{ij}(s, X_s^{t,\xi;v}, y, z, v_s)$, where $(s, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$ satisfy the conditions of Theorem 16 on the interval $[t, T]$. Therefore, there exists a unique solution for the following G-BSDE:

$$\begin{aligned} Y_s^{t,\xi;v} &= \Phi\left(X_T^{t,\xi;v}\right) + \int_s^T f\left(r, X_r^{t,\xi;v}, Y_r^{t,\xi;v}, Z_r^{t,\xi;v}, v_r\right) dr \\ &\quad - \int_s^T Z_r^{t,\xi;v} dB_r - \left(K_T^{t,\xi;v} - K_s^{t,\xi;v}\right) \\ &\quad + \sum_{i,j=1}^{d} \int_s^T g_{ij}\left(r, X_r^{t,\xi;v}, Y_r^{t,\xi;v}, Z_r^{t,\xi;v}, v_r\right) d\left\langle B^i, B^j\right\rangle_r, \end{aligned} \tag{29}$$

where $X^{t,\xi;v}$ is introduced by (24).

Proposition 22. *For each ξ, $\xi' \in L_G^p(\Omega_t; \mathbb{R}^n)$ with $p \ge 2$ and $v(\cdot)$, $v'(\cdot) \in \mathcal{U}$, we have*

$$\begin{gathered} \left|Y_t^{t,\xi;v} - Y_t^{t,\xi';v}\right| \le C\left|\xi - \xi'\right|, \\ \left|Y_t^{t,\xi;v}\right| \le C\left(1 + |\xi|\right), \\ \left|Y_t^{t,\xi;v} - Y_t^{t,\xi;v'}\right| \le C\left(\int_t^T \widehat{E}_t\left|v(r) - v'(r)\right|^2 dr\right)^{1/2}, \end{gathered} \tag{30}$$

where C depends on L, G, n, and T.

Proof. The proof is similar to the Proposition 4.2 in [20]. □

Given a control process $v(\cdot) \in \mathcal{U}$, we introduce an associated cost functional

$$J(t, x; v) = Y_t^{t,x;v}, \quad (t, x) \in [0, T] \times \mathbb{R}^n, \tag{31}$$

where the process $Y_t^{t,\xi;v}$ is defined by G-BSDE (29). Similar to the proof of Theorem 4.4 in [20], we have that for $t \in [0, T]$, $\xi \in L_G^2(\Omega_t, \mathbb{R}^n)$,

$$J(t, \xi; v) := Y_t^{t,\xi;v}. \tag{32}$$

But we are more interested in the case when $\xi = x$.

Now we define the value function as follows:

$$u(t, x) := \sup_{v(\cdot) \in \mathcal{U}} J(t, x; v). \tag{33}$$

Proposition 23. *$u(t, x)$ is a deterministic function of (t, x).*

Proof. For a partition of $[t, s]$: $t = t_0 < t_1 < \cdots < t_N = s$, $p \ge 2$, $t \le s \le T$, we denote $L_{ip}(\Omega_s^t) := \{\varphi(B_{t_1} - B_t, \dots, B_{t_n} - B_t) : n \ge 1, t_1, \dots, t_n \in [t, s], \varphi \in C_{b,L_{ip}}(\mathbb{R}^{d \times n})\}$, $M_G^{p,0,t}(t, s; \mathbb{R}^n)$ by the collection of simple processes $\eta(r) = \sum_{k=0}^{N-1} \xi_k 1_{[t_k, t_{k+1})}(r)$, where $\xi_k \in L_{ip}(\Omega_{t_k}^t; \mathbb{R}^n)$, $k = 0, 1, 2, \dots, N-1$, and $M_G^{p,t}(t, s; \mathbb{R}^n)$ by the completion of $M_G^{p,0,t}(t, s; \mathbb{R}^n)$ under the norm $\|\eta\|_{M_G^p(t,s;\mathbb{R}^n)} := \{\widehat{E}[\int_t^s |\eta(r)|^p dr]\}^{1/p}$. Use the similar method in Lemma 43 of [27]; we can prove that $v \in M_G^2(t, s; \mathbb{R}^n)$ is a V-valued process; there exists $\{u = \sum_{i=1}^N 1_{A_i} u^i\}_{N \in \mathbb{N}}$; $u^i \in M_G^{2,t}(t, s; \mathbb{R}^n)$ is a V-valued process, A_i is a partition of $\mathscr{B}(\Omega_t)$ such that $u \to v$ under probability measure $P \in \mathscr{P}_M$. When $v(s) \in M_G^{2,t}(t, s; \mathbb{R}^n)$, we note that $J(t, x; v)$ is a deterministic function of (t, x) because b, h_{ij}, σ_j, Φ, f, and g_{ij} are deterministic functions, and $\widetilde{B}_s := B_{t+s} - B_t$ is a G-Brownian motion. So we need to construct a sequence of admissible controls $\{\widetilde{v}^i(\cdot)\}$ of the form

$$\widetilde{v}_s^i = \sum_{j=1}^{N_i} v_s^{ij} 1_{A_{ij}} \tag{34}$$

satisfying $\lim_{i \to \infty} J(t, x; \widetilde{v}^i(\cdot)) = u(t, x)$, where $v^{ij}(\cdot) \in M_G^{2,t}(t, s; \mathbb{R}^n)$ is a V-valued processes and $\{A_{ij}\}_{j=1}^{N_i}$ is a partition of $\mathscr{B}(\Omega_t)$. Firstly, there exists $\{v^k\}_{k \ge 1} \subset \mathcal{U}$, such that $u(t, x) = \sup_{k \ge 1} J(t, x; v^k)$. Then we define $v, v' \in \mathcal{U}$,

$$\left(v \vee v'\right)_s = \begin{cases} 0, & s \in [0, t]; \\ v_s, & s \in (t, T], \text{ on } \left\{J(t, x; v) \ge J(t, x; v')\right\}; \\ v'_s, & s \in (t, T], \text{ on } \left\{J(t, x; v) < J(t, x; v')\right\}. \end{cases} \tag{35}$$

Therefore,

$$J\left(t, x; v \vee v'\right) \ge J(t, x; v) \vee J\left(t, x; v'\right). \tag{36}$$

Set $\overline{v}^1 := v^1 \vee v^1$, $\overline{v}^k := \overline{v}^{k-1} \vee v^i$, $i \ge 2$. So $u(t, x) = \lim_{k \to \infty} J(t, x; \overline{v}^k)$. Without loss of generality, suppose $\widehat{E}[(u(t, x) - J(t, x; \overline{v}^k))^2] \le 1/k$, $k \ge 1$. We denote

$$\widetilde{v}_s^k = \sum_{j,k=0}^{N_i - 1} \overline{v}_{j,k}(s) 1_{A_j^k}, \tag{37}$$

where $\bar{v}_{j,k} \in M_G^{2,t}(t,s;\mathbb{R}^n)$ is a V-valued process and $\{A_j^k\}_{0\le j\le N_k-1}$ is a partition of $\mathscr{B}(\Omega_t)$. Then we can suppose for $k \ge 1$, $\widehat{E}[\int_t^T |\bar{v}_s^k - \tilde{v}_s^k|^2 ds] \le 1/Ck$. From Proposition 22, we have

$$\widehat{E}\left[\left|J\left(t,x;\bar{v}^k\right) - J\left(t,x;\tilde{v}^k\right)\right|^2\right] \le C\widehat{E}\left[\int_t^T \left|\bar{v}_s^k - \tilde{v}_s^k\right|^2 ds\right] \le \frac{1}{k}. \tag{38}$$

Therefore, $\widehat{E}[|u(t,x) - J(t,x;\tilde{v}^k)|^2] \le 4/k$. Then, we have

$$J\left(t,x;\tilde{v}^k\right) = \sum_{j=0}^{N_k-1} 1_{A_j^k} J\left(t,x;\bar{v}_{j,k}\right) \le u(t,x). \tag{39}$$

Now, we suppose that

$$J(t,x,\tilde{v}) \le \max_{0\le j\le N_k-1} J\left(t,x;\bar{v}_{j,k}\right) = J\left(t,x;\bar{v}_{j',k}\right). \tag{40}$$

Because $\widehat{E}[|J(t,x;\tilde{v}^k) - u(t,x)|^2] \to 0$, we have

$$u(t,x) = \lim_{k\to\infty} J\left(t,x;\bar{v}_{j',k}\right), \quad \text{q.s.}. \tag{41}$$

Hence, $\widehat{E}[u(t,x)] = u(t,x)$. We have finished the proof. □

Lemma 24. *For any $t \in [0,T]$, $x, x' \in \mathbb{R}^n$, we have*

$$\left|u(t,x) - u\left(t,x'\right)\right| \le C\left|x - x'\right|, \tag{42}$$

$$|u(t,x)| \le C(1 + |x|). \tag{43}$$

Proof. By Proposition 22, we have for $v(\cdot) \in \mathscr{U}$,

$$\begin{gathered} |J(t,x;v(\cdot))| \le C(1+|x|), \\ \left|J(t,x;v(\cdot)) - J\left(t,x';v(\cdot)\right)\right| \le C\left|x - x'\right|. \end{gathered} \tag{44}$$

Then, $\forall\varepsilon > 0$, there exist $v(\cdot), v'(\cdot) \in \mathscr{U}$ such that

$$\begin{gathered} J(t,x;v(\cdot)) \le u(t,x) \le J(t,x;v(\cdot)) + \varepsilon, \\ J\left(t,x';v'(\cdot)\right) \le u\left(t,x'\right) \le J\left(t,x';v'(\cdot)\right) + \varepsilon. \end{gathered} \tag{45}$$

Now, we have

$$\begin{aligned} -C(1+|x|) &\le J(t,x;v(\cdot)) \le u(t,x) \le J(t,x;v(\cdot)) + \varepsilon \\ &\le C(1+|x|) + \varepsilon. \end{aligned} \tag{46}$$

So, we get (43). Similarly, we obtain

$$\begin{aligned} J&\left(t,x;v'(\cdot)\right) - J\left(t,x';v'(\cdot)\right) - \varepsilon \\ &\le u(t,x) - u\left(t,x'\right) \\ &\le J(t,x;v(\cdot)) - J\left(t,x';v(\cdot)\right) + \varepsilon. \end{aligned} \tag{47}$$

Then,

$$-C\left|x - x'\right| - \varepsilon \le \left|u(t,x) - u\left(t,x'\right)\right| \le C\left|x - x'\right| + \varepsilon. \tag{48}$$

Thus, we have proved (42). □

Lemma 25. *For any $t \in [0,T]$, $\zeta \in L_G^2(\Omega_t;\mathbb{R}^n)$, and ζ is $\mathscr{F}_t^B$ measurable; we have $\forall v(\cdot) \in \mathscr{U}$,*

$$u(t,\zeta) \ge Y_t^{t,\zeta;v}. \tag{49}$$

Conversely, $\forall\varepsilon > 0$, there exists a $v(\cdot) \in \mathscr{U}$, such that

$$u(t,\zeta) \le Y_t^{t,\zeta;v} + \varepsilon. \tag{50}$$

Proof. We already know that $u(t,x)$ is continuous with respect to x and $Y_t^{t,\zeta;v}$ is continuous with respect to $(\zeta, v(\cdot))$. We want to prove (49) and only need to discuss the simple random variables ζ of the form

$$\zeta = \sum_{i=1}^N 1_{A_i} x_i, \tag{51}$$

and $v(\cdot)$ of the form

$$v(\cdot) = \sum_{i=1}^N 1_{A_i} v^i(\cdot). \tag{52}$$

Here, $i = 1, 2, \dots, N$, $x_i \in \mathbb{R}^n$, $v^i \in M_G^{2,t}(t,s;\mathbb{R}^n)$, and $\{A_i\}_{i=1}^N$ is a $\mathscr{B}(\Omega_t)$-partition. Then, from the same technique used in the proof of Theorem 4.4 in [20], we have

$$\begin{aligned} Y_t^{t,\zeta;v} &= \sum_{i=1}^N 1_{A_i} Y_t^{t,x^i;v^i} \le \sum_{i=1}^N 1_{A_i} u(t,x_i) = u\left(t, \sum_{i=1}^N 1_{A_i} x_i\right) \\ &= u(t,\zeta). \end{aligned} \tag{53}$$

So we have proved (49). Now we prove (50) in a similar way. We first construct a random variable $\eta \in L_G^2(\Omega_t;\mathbb{R}^n)$,

$$\eta = \sum_{i=1}^N x_s^i 1_{A_i}, \tag{54}$$

where $(A_i)_{i=1}^N$ is a $\mathscr{B}(\Omega_t)$-partition and $x_i \in \mathbb{R}^n$, such that $|\eta - \zeta| \le \varepsilon/3C$. Then, we have

$$\begin{gathered} \left|Y_t^{t,\eta;v} - Y_t^{t,\zeta;v}\right| \le \frac{\varepsilon}{3}, \\ \left|u(t,\zeta) - u(t,\eta)\right| \le \frac{\varepsilon}{3}, \end{gathered} \tag{55}$$

for $v(\cdot) \in \mathscr{U}$. Now, we choose a control $v^i(\cdot) \in M_G^{2,t}(t,s;\mathbb{R}^n)$, such that $u(t,x_i) \le Y_t^{t,x^i;v^i} + \varepsilon/3$. Set $v(\cdot) := \sum_{i=1}^N v^i(\cdot) 1_{A_i}$. Finally, we get

$$\begin{aligned} Y_t^{t\zeta;v} &\ge -\left|Y_t^{t,\eta;v} - Y_t^{t,\zeta;v}\right| + Y_t^{t,\eta;v} \\ &\ge -\frac{\varepsilon}{3} + \sum_{i=1}^N Y_t^{t,x^i;v^i} 1_{A_i} \\ &\ge -\frac{\varepsilon}{3} + \sum_{i=1}^N \left(u(t,x_i) - \frac{\varepsilon}{3}\right) 1_{A_i} \\ &= -\frac{2\varepsilon}{3} + \sum_{i=1}^N u(t,x_i) 1_{A_i} \\ &= -\frac{2\varepsilon}{3} + u(t,\eta) \ge -\varepsilon + u(t,\zeta). \end{aligned} \tag{56}$$

So, we have (50). □

Now, we give a type of DPP for our stochastic optimal control problems. Firstly, we define a family of backward semigroups associated with the G-BSDE (29). Given the initial data (t, x), a positive number $\delta \leq T - t$, and a random variable $\eta \in L_G^p(\Omega; \mathbb{R})$ with $p > 1$, we set

$$G_{t,t+\delta}^{t,x;v}[\eta] := Y_s^{t,x;v}, \tag{57}$$

where $(Y_s^{t,x;v})_{t\leq s\leq t+\delta}$ is the solution of the following G-BSDE with the time horizon $t + \delta$:

$$\begin{aligned} Y_s^{t,x;v} = \eta &+ \int_s^{t+\delta} f\left(r, X_r^{t,x;v}, Y_r^{t,x;v}, Z_r^{t,x;v}, v_r\right) dr \\ &- \int_s^{t+\delta} Z_r^{t,x;v} dB_r - \left(K_T^{t,x;v} - K_t^{t,x;v}\right) \\ &+ \sum_{i,j=1}^{d} \int_s^{t+\delta} g_{ij}\left(r, X_r^{t,x;v}, Y_r^{t,x;v}, Z_r^{t,x;v}, v_r\right) d\left\langle B^i, B^j\right\rangle_r. \end{aligned} \tag{58}$$

Obviously, for the solution $Y^{t,x;v}$ of G-BSDE (29), we have

$$G_{t,T}^{t,x;v}\left[\Phi\left(X_T^{t,x;v}\right)\right] = G_{t,t+\delta}^{t,x;v}\left[Y_{t+\delta}^{t,x;v}\right]. \tag{59}$$

Then, we can obtain the DPP for our stochastic optimal control problems as follows.

Theorem 26. *The value function $u(t, x)$ has the following proposition: for every $0 \leq \delta \leq T - t$, we have*

$$u(t,x) = \sup_{v(\cdot)\in\mathcal{U}} G_{t,t+\delta}^{t,x;v}\left[u\left(t+\delta, X_{t+\delta}^{t,x;v}\right)\right]. \tag{60}$$

Proof. We have

$$u(t,x) = \sup_{v(\cdot)\in\mathcal{U}} G_{t,T}^{t,x;v}\left[\Phi\left(X_T^{t,x;v}\right)\right] = \sup_{v(\cdot)\in\mathcal{U}} G_{t,t+\delta}^{t,x;v}\left[Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t,x;v};v}\right]. \tag{61}$$

Obviously, $X_{t+\delta}^{t,x;v}$ is $\mathcal{F}_{t+\delta}^B$ measurable. So, by Lemma 25 and Theorem 19, we have

$$u(t,x) \leq \sup_{v(\cdot)\in\mathcal{U}} G_{t,t+\delta}^{t,x;v}\left[u\left(t+\delta, X_{t+\delta}^{t,x;v}\right)\right]. \tag{62}$$

Besides, for $\varepsilon > 0$, there exists an admissible control $\overline{v}(\cdot) \in \mathcal{U}$ such that

$$u\left(t+\delta, X_{t+\delta}^{t,x;v}\right) \leq Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t,x;v};\overline{v}} + \varepsilon. \tag{63}$$

Then,

$$\begin{aligned} u(t,x) &\geq \sup_{v(\cdot)\in\mathcal{U}} G_{t,t+\delta}^{t,x;v}\left[u\left(t+\delta, X_{t+\delta}^{t,x;v}\right) - \varepsilon\right] \\ &\geq \sup_{v(\cdot)\in\mathcal{U}} G_{t,t+\delta}^{t,x;v}\left[u\left(t+\delta, X_{t+\delta}^{t,x;v}\right)\right] - C\varepsilon. \end{aligned} \tag{64}$$

Because ε can be arbitrarily small, we get (60). □

Proposition 27. *$u(t, x)$ is 1/2-Hölder continuous in t.*

Proof. For any given $(t, x) \in [0, T] \times \mathbb{R}^n$ and $\delta > 0$ $(t + \delta \leq T)$, from Theorem 26, we know that for $\varepsilon > 0$, there exists a $v(\cdot) \in \mathcal{U}$ such that

$$\begin{aligned} G_{t,t+\delta}^{t,x;v}\left[u\left(t+\delta, X_{t+\delta}^{t,x;v}\right)\right] + \varepsilon &\geq u(t,x) \\ &\geq G_{t,t+\delta}^{t,x;v}\left[u\left(t+\delta, X_{t+\delta}^{t,x;v}\right)\right]. \end{aligned} \tag{65}$$

Then, we need to prove

$$u(t,x) - u(t+\delta,x) \leq C\delta^{1/2} \quad \left(\text{resp.}, \geq -C\delta^{1/2}\right). \tag{66}$$

We only check the first inequality in (66). The second can be proved similarly. We have $\forall \varepsilon > 0$,

$$u(t,x) - u(t+\delta,x) \leq I_\delta^1 + I_\delta^2 + \varepsilon, \tag{67}$$

where

$$\begin{aligned} I_\delta^1 &= G_{t,t+\delta}^{t,x;v}\left[u\left(t+\delta, X_{t+\delta}^{t,x;v}\right)\right] - G_{t,t+\delta}^{t,x;v}\left[u(t+\delta,x)\right], \\ I_\delta^2 &= G_{t,t+\delta}^{t,x;v}\left[u(t+\delta,x)\right] - u(t+\delta,x). \end{aligned} \tag{68}$$

From Proposition 21, we have

$$\widehat{E}_t\left[\left|X_{t+\delta}^{t,x;v} - x\right|^2\right] \leq C\left(1+|x|^2\right)\delta. \tag{69}$$

By Proposition 22 and Lemma 24, we deduce that

$$\begin{aligned} \left|I_\delta^1\right| &\leq \left[C\widehat{E}_t\left[\left|u\left(t+\delta, X_{t+\delta}^{t,x;v}\right) - u(t+\delta,x)\right|^2\right]\right]^{1/2} \\ &\leq \left[C\widehat{E}_t\left[\left|X_{t+\delta}^{t,x;v} - x\right|^2\right]\right]^{1/2} \leq C'\delta^{1/2}. \end{aligned} \tag{70}$$

Based on the definition of $G_{t,t+\delta}^{t,x;v}$, we get

$$\begin{aligned} I_\delta^2 &= \widehat{E}_t\Bigg[u(t+\delta,x) + \int_t^{t+\delta} f\left(s, X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v}, v_s\right) ds \\ &\qquad + \sum_{i,j=1}^{d} \int_t^{t+\delta} g_{ij}\left(s, X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v}, v_s\right) d\left\langle B^i, B^j\right\rangle_s \\ &\qquad - \int_t^{t+\delta} Z_s^{t,x;v} dB_s - \left(K_T^{t,x;v} - K_t^{t,x;v}\right)\Bigg] - u(t+\delta,x) \\ &= \widehat{E}_t\Bigg[\int_t^{t+\delta} f\left(s, X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v}, v_s\right) ds \\ &\qquad + \sum_{i,j=1}^{d} \int_t^{t+\delta} g_{ij}\left(s, X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v}, v_s\right) d\left\langle B^i, B^j\right\rangle_s\Bigg] \\ &\leq C'\delta^{1/2} \\ &\quad \times \left(1 + \widehat{E}_t\left[\int_t^{t+\delta} \left|X_s^{t,x;v}\right|^2 + \left|Y_s^{t,x;v}\right|^2 + \left|Z_s^{t,x;v}\right|^2 ds\right]^{1/2}\right). \end{aligned} \tag{71}$$

By Proposition 22, we can prove the following inequality easily by the similar method in Proposition 3.5 of [11]

$$\widehat{E}_t\left[\int_t^{t+\delta}\left|Z_s^{t,x;v}\right|^2 ds\right]^{1/2} \leq C(1+|x|). \tag{72}$$

So we have $I_\delta^2 \leq C'\delta^{1/2}$. Hence, by (67) we have

$$u(t,x) - u(t+\delta,x) \leq C'\delta^{1/2} + \varepsilon. \tag{73}$$

Let $\varepsilon \to 0$; we obtain the first inequality of (66). The proof is completed. □

4. Value Function and Viscosity Solution of Fully Nonlinear Second-Order Partial Differential Equation

In this section, we consider the following fully nonlinear second-order partial differential equation

$$\begin{gathered}\partial_t u + F\left(D_x^2 u, D_x u, u, x, t\right) = 0, \quad (t,x) \in [0,T]\times\mathbb{R}^n, \\ u(T,x) = \Phi(x),\end{gathered} \tag{74}$$

where

$$\begin{aligned}
&F\left(D_x^2 u, D_x u, u, x, t\right) \\
&= \sup_{v\in V}\left\{G\left(H\left(D_x^2 u, D_x u, u, x, t, v\right)\right) + \langle b(t,x,v), D_x u\rangle\right. \\
&\qquad + f\left(t,x,u,\langle\sigma_1(t,x), D_x u\rangle,\right. \\
&\qquad\qquad \left.\left.\ldots,\langle\sigma_d(t,x), D_x u\rangle, v\right)\right\}, \\
&H_{ij}\left(D_x^2 u, D_x u, u, x, t, v\right) \\
&\quad = \left\langle D_x^2 u \cdot \sigma_i(t,x,v), \sigma_j(t,x,v)\right\rangle \\
&\qquad + 2\left\langle D_x u, h_{ij}(t,x,v)\right\rangle \\
&\qquad + 2g_{ij}\left(t,x,u,\langle\sigma_1(t,x,v), D_x u\rangle,\right. \\
&\qquad\qquad \left.\ldots,\langle\sigma_d(t,x,v), D_x u\rangle, v\right).
\end{aligned} \tag{75}$$

Remark 28. The definition and uniqueness of the viscosity solution of above second-order partial differential equation can be found in Appendix C in Peng [12]. So, we only need to prove that $u(t,x)$ is a viscosity solution of (74). Besides, from the result of Section 3, we can have that $u(t,x)$ is continuous in $[0,T]\times\mathbb{R}^n$.

Definition 29. A real-valued continuous function $u(t,x) \in C([0,T]\times\mathbb{R}^n)$, $u(T,x) \leq \Phi(x)$, for any $x \in \mathbb{R}^n$, is called a viscosity subsolution (super-solution) of (74); if for all functions $\varphi \in C^{2,3}([0,T]\times\mathbb{R}^n)$ satisfy $\varphi \geq u$ and $\varphi(t,x) = u(t,x)$ at fixed $(t,x) \in [0,T)\times\mathbb{R}^n$, we have

$$\begin{aligned}&\partial_t\varphi(t,x) + F\left(D_x^2\varphi(t,x), D_x\varphi(t,x), \varphi(t,x), x, t\right) \\ &\quad \geq 0\ (\leq 0).\end{aligned} \tag{76}$$

Theorem 30. *Under the assumptions (H3) and (H4), the value function $u(t,x)$ defined by* (33) *is a viscosity solution of* (74).

In order to prove the Theorem, we need three lemmas. Firstly, we set

$$\begin{aligned}
&F_1(r,x,y,z,v) \\
&\quad = \langle b(r,x,v), D_x\varphi(r,x)\rangle + \partial_t\varphi(t,x) \\
&\qquad + f\left(r,x,y+\varphi(r,x),\right. \\
&\qquad\qquad z + \left(\langle\sigma_1(t,x,v), D_x\varphi(r,x)\rangle,\right. \\
&\qquad\qquad\qquad \left.\left.\ldots,\langle\sigma_d(t,x,v), D_x\varphi(r,x)\rangle\right), v\right), \\
&F_2^{ij}(r,x,y,z,v) \\
&\quad = \left\langle D_x\varphi(r,x), h_{ij}(r,x,v)\right\rangle \\
&\qquad + \frac{1}{2}\left\langle D_x^2\varphi(r,x)\sigma_i(r,x,v), \sigma_j(r,x,v)\right\rangle \\
&\qquad + g_{ij}\left(r,x,y+\varphi(r,x),\right. \\
&\qquad\qquad z + \left(\langle\sigma_1(t,x,v), D_x\varphi(r,x)\rangle,\right. \\
&\qquad\qquad\qquad \left.\left.\ldots,\langle\sigma_d(r,x,v), D_x\varphi(r,x)\rangle\right), v\right).
\end{aligned} \tag{77}$$

Then, we consider a G-BSDE defined on the interval $[t,t+\delta]$ $(0<\delta\leq T-t)$:

$$\begin{aligned}
Y_s^{1,v} &= \int_s^{t+\delta} F_1\left(r, X_r^{t,x;v}, Y_r^{1,v}, Z_r^{1,v}, v_r\right) dr \\
&\quad + \int_s^{t+\delta} Z_r^{1,v} dB_r - \left(K_{t+\delta}^1 - K_s^1\right) \\
&\quad - \sum_{i,j=1}^d \int_s^{t+\delta} F_2^{ij}\left(r, X_r^{t,x;v}, Y_r^{1,v}, Z_r^{1,v}, v_r\right) d\left\langle B^i, B^j\right\rangle_r,
\end{aligned} \tag{78}$$

where $v(\cdot) \in \mathcal{U}$ and $X_s^{t,x;v}$ is defined by (24).

Lemma 31. *For $s \in [t,t+\delta]$, we have*

$$G_{s,t+\delta}^{t,x;v}\left[\varphi\left(X_{t+\delta}^{t,x;v}, t+\delta\right)\right] - \varphi\left(X_s^{t,x;v}, s\right) \tag{79}$$

which is the solution of (78).

Proof. From the definition of $G_{s,t+\delta}^{t,x;v}$, we know that $G_{s,t+\delta}^{t,x;v}[\varphi(X_{t+\delta}^{t,x;v}, t+\delta)]$ is the solution of G-BSDE (29) on $[t,t+\delta]$ with terminal condition $\varphi(X_{t+\delta}^{t,x;v}, t+\delta)$. Applying Itô's formula to $\varphi(X_s^{t,x;v}, s)$, we can obtain the result. □

Now we construct a simple G-BSDE by replacing the driving process $X_s^{t,x;v}$ by its deterministic initial value x as follows:

$$\begin{aligned} Y_s^{2,v} = & \int_s^{t+\delta} F_1\left(r, x, Y_r^{2,v}, Z_r^{2,v}, v_r\right) dr \\ & + \sum_{i,j=1}^{d} \int_s^{t+\delta} F_2^{ij}\left(r, x, Y_r^{2,v}, Z_r^{2,v}, v_r\right) d\left\langle B^i, B^j\right\rangle_r \\ & - \int_s^{t+\delta} Z_r^{2,v} dB_r - \left(K_{t+\delta}^2 - K_s^2\right). \end{aligned} \tag{80}$$

Lemma 32. *We have the following estimate, for $v(\cdot) \in \mathcal{U}$,*

$$\left|Y_t^{1,v} - Y_t^{2,v}\right| \le C\delta^{3/2}, \tag{81}$$

where C is independent of the control processes $v(\cdot)$.

Proof. By Proposition 21, we have the estimate for $p \ge 2$

$$\widehat{E}_t\left[\sup_{s\in[t,t+\delta]} \left|X_s^{t,x;v} - x\right|^p\right] \le C\left(1 + |x|^p\right)\delta^{p/2}. \tag{82}$$

By Proposition 18, we get for fixed $p > 2$ and $2 < p < \beta$,

$$\begin{aligned} \left|Y_t^{1,v} - Y_t^{2,v}\right|^2 & \le \widehat{E}\left[\sup_{s\in[t,t+\delta]} \left|Y_t^{1,v} - Y_t^{2,v}\right|^2\right] \\ & \le C\left\{\widehat{E}\left[\sup_{s\in[t,t+\delta]} \widehat{E}_s\left[\left(\int_t^{t+\delta} \widehat{F}_r dr\right)^p\right]\right]^{2/p}\right. \\ & \qquad \left. + \widehat{E}\left[\sup_{s\in[t,t+\delta]} \widehat{E}_s\left[\left(\int_t^{t+\delta} \widehat{F}_r dr\right)^p\right]\right]\right\}, \end{aligned} \tag{83}$$

where

$$\begin{aligned} \widehat{F}_r = & \left|F_1\left(r, X_r^{t,x;v}, Y_r^{2,v}, Z_r^{2,v}, v_r\right) - F_1\left(r, x, Y_r^{2,v}, Z_r^{2,v}, v_r\right)\right| \\ & + \sum_{i,j=1}^{d} \left|F_1^{i,j}\left(r, X_r^{t,x;v}, Y_r^{2,v}, Z_r^{2,v}, v_r\right)\right. \\ & \qquad \left. - F_2^{i,j}\left(r, x, Y_r^{2,v}, Z_r^{2,v}, v_r\right)\right|. \end{aligned} \tag{84}$$

It is easy to prove that

$$\widehat{F}_r \le C\left|X_r^{t,x;v} - x\right|. \tag{85}$$

Then, we can deduce that $|Y_t^{1,v} - Y_t^{2,v}| \le C\delta^{3/2}$. □

Lemma 33. *We have*

$$\sup_{v(\cdot)\in\mathcal{U}} Y_t^{2,v} = Y^0(t), \tag{86}$$

where $Y_0(\cdot)$ is the solution of the following ODE:

$$\begin{aligned} -dY_s^0 &= F^0\left(s, x, Y_r^0, 0\right) ds, \quad s \in [t, t+\delta], \\ Y_{t+\delta}^0 &= 0, \end{aligned} \tag{87}$$

where $F^0(r, x, y, z) = \sup_{v\in V}\{F_1(r, x, y, z, v) + 2G[(F_2^{ij}(r, x, y, z, v))_{i,j=1}^d]\}$.

Proof. By Theorem 16, we know that the G-BSDE (80) have a unique solution (Y, Z, K). Hence, there exists a process

$$\begin{aligned} V_s^{2,v} = & \sum_{i,j=1}^{d} \int_t^s F_2^{ij}\left(r, x, Y_r^{2,v}, Z_r^{2,v}, v\right) d\left\langle B^i, B^j\right\rangle_r \\ & - \int_t^s 2G\left(\left(F_2^{ij}\left(r, x, Y_r^{2,v}, Z_r^{2,v}, v\right)\right)_{i,j=1}^d\right) dr. \end{aligned} \tag{88}$$

Here, $V_s^{2,v}$, $s \in [t, t+\delta]$ is a decreasing and continuous process by [28]. Besides, it satisfies $\widehat{E}[\sup_{s\in[t,t+\delta]} |V_s^{2,v}|^\beta] < \infty$ obviously. So, $Y_s^{2,v}$ is the solution of the following G-BSDE:

$$\begin{aligned} Y_s^{2,v} = & \int_s^{t+\delta} \left[F_1\left(r, x, Y_r^{2,v}, Z_r^{2,v}, v_r\right)\right. \\ & \qquad \left. + 2G\left[\left(F_2^{ij}\left(r, x, Y_r^{2,v}, Z_r^{2,v}, v_r\right)\right)_{i,j=1}^d\right]\right] dr \\ & - \int_s^{t+\delta} Z_r^{2,v} dB_r - \left(K_{t+\delta}^2 - K_s^2\right) + V_{t+\delta}^{2,v} - V_s^{2,v}, \end{aligned} \tag{89}$$

where $v(\cdot) \in \mathcal{U}$. In addition, we have

$$\begin{aligned} Y_t^0 = & \int_s^{t+\delta} F^0\left(r, x, Y_r^0, Z_r^0\right) dr \\ & - \int_s^{t+\delta} Z_r^0 dB_r - \left(K_{t+\delta}^0 - K_s^0\right) + \left(V_{t+\delta}^0 - V_s^0\right), \end{aligned} \tag{90}$$

where $(Z, K, V) = 0$. By the comparison of Theorem 20 and the definition of F^0, we have for $v(\cdot) \in \mathcal{U}$,

$$Y_s^{2,v} \le Y_s^0, \quad s \in [t, t+\delta]. \tag{91}$$

On the other hand, there exists a measurable function $v'(r, x, y, z) : [t, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to V$ such that

$$\begin{aligned} F^0(r, x, y, z) = & F_1\left(r, x, y, z, v'\right) \\ & + 2G\left[\left(F_2^{ij}\left(r, x, y, 0, v'\right)\right)_{i,j=1}^d\right]. \end{aligned} \tag{92}$$

Then, we have $v'(r, x, Y_r^0, Z_r^0) \in \mathcal{U}$, and Y_t^0 is the solution of the following G-BSDE:

$$\begin{aligned} Y_s^0 = & \int_s^{t+\delta} F_1\left(r, x, Y_r^0, Z_r^0, v_r'\right) dr \\ & + \sum_{i,j=1}^{d} \int_s^{t+\delta} F_2^{ij}\left(r, x, Y_r^0, Z_r^0, v_r'\right) d\left\langle B^i, B^j\right\rangle_r \\ & - \int_s^{t+\delta} Z_r^0 dB_r - \left(K_{t+\delta}^0 - K_s^0\right), \end{aligned} \tag{93}$$

where $Z_{r,v}^0 = 0$,

$$\begin{aligned} K_s^0 = & \sum_{i,j=1}^{d} \int_t^s F_2^{ij}\left(r, x, Y_r^0, 0, v'\right) d\left\langle B^i, B^j\right\rangle_r \\ & - \int_t^s 2G\left(\left(F_2^{ij}\left(r, x, Y_r^0, 0, v'\right)\right)_{i,j=1}^d\right) dr. \end{aligned} \tag{94}$$

So, $Y_t^0 \le \sup_{v(\cdot)\in\mathcal{U}} Y_t^{2,v}$. Now, we have proved the lemma. □

Then, we give the proof of Theorem 30:

Proof. We set $\varphi \in C^{2,3}([0,T] \times \mathbb{R}^n)$ and $\varphi(t,x) = u(t,x)$ for fixed $(t,x) \in [0,T] \times \mathbb{R}^n$. From Theorem 26, we know

$$\varphi(t,x) = u(t,x) = \sup_{v(\cdot)\in\mathcal{U}} G^{t,x;v}_{t,t+\delta}\left[u\left(X^{t,x}_{t+\delta}, t+\delta\right)\right]. \tag{95}$$

By $\varphi \geq u$ $(\varphi \leq u)$ and the definition of G,

$$\sup_{v(\cdot)\in\mathcal{U}} \left\{G^{t,x;v}_{t,t+\delta}\left[u\left(X^{t,x}_{t+\delta}, t+\delta\right)\right] - \varphi(t,x)\right\} \geq 0\,(\leq 0). \tag{96}$$

Then, form Lemma 31,

$$\sup_{v(\cdot)\in\mathcal{U}} Y^{1,v}_t \geq 0\,(\leq 0). \tag{97}$$

Besides, from Lemma 32,

$$\sup_{v(\cdot)\in\mathcal{U}} Y^{2,v}_t \geq C\delta^{3/2}\left(\leq C\delta^{3/2}\right). \tag{98}$$

Finally, Lemma 33 implies

$$Y^0(t) \geq C\delta^{3/2}\left(\leq C\delta^{3/2}\right). \tag{99}$$

So, $F^0(r,x,0,0) \geq 0$ (≤ 0) and from the definition of viscosity solution of (74), we know $u(t,x)$ is a viscosity solution of (74). □

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