

Research Article

Hopf Bifurcation Analysis for a Computer Virus Model with Two Delays

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Received 18 August 2013; Accepted 26 August 2013

Academic Editor: Luca Guerrini

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This paper is concerned with a computer virus model with two delays. Its dynamics are studied in terms of local stability and Hopf bifurcation. Sufficient conditions for local stability of the positive equilibrium and existence of the local Hopf bifurcation are obtained by regarding the possible combinations of the two delays as a bifurcation parameter. Furthermore, explicit formulae for determining direction of the Hopf bifurcation and stability of the bifurcating periodic solutions are obtained by using the normal form method and center manifold theory. Finally, some numerical simulations are presented to support the theoretical results.

1. Introduction

Since the pioneering work of Kephart and White [1, 2], many classical epidemic models such as SIR [3–5], SIRS [6–8], SEIR [9, 10], and SEIRS [11, 12], SEIQV [13] have been used to describe the spread of a computer virus in computer network due to the high similarity between computer viruses and biological viruses. In [9], Yuan and Chen proposed the following SEIR model:

$$\begin{aligned}\frac{dS(t)}{dt} &= \mu N - \nu(t)S(t) - \rho_{SR}S(t) - \mu S(t), \\ \frac{dE(t)}{dt} &= \nu(t)S(t) - m_E \alpha E(t), \\ \frac{dI(t)}{dt} &= \alpha E(t) - (\gamma + \mu)I(t), \\ \frac{dR(t)}{dt} &= \rho_{SR}S(t) + \rho_{ER}E(t) + \gamma I(t) - \mu R(t),\end{aligned}\tag{1}$$

where $S(t)$, $E(t)$, $I(t)$, and $R(t)$ denote the numbers of nodes at time t in states susceptible, exposed, infectious, and recovered, respectively. Yuan and Chen [9] studied the behaviors of virus propagation with the presence of antivirus

countermeasures by analyzing the equilibrium stability of system (1).

It is well known that time delays of one or other reasons can cause a stable equilibrium to become unstable and make a system bifurcate periodic solutions and dynamical systems with delay have been studied by many scholars [14–23]. Starting from this point and considering that the antivirus software may use a period to clean the viruses in a computer, Dong et al. [10] proposed the following model with delay:

$$\begin{aligned}\frac{dS(t)}{dt} &= \mu N - \beta I(t)S(t) - (\rho_{SR} + \mu)S(t), \\ \frac{dE(t)}{dt} &= \beta I(t)S(t) - (\alpha + \mu)E(t) - \rho_{ER}E(t - \tau), \\ \frac{dI(t)}{dt} &= \alpha E(t) - (\gamma + \mu)I(t), \\ \frac{dR(t)}{dt} &= \rho_{SR}S(t) + \rho_{ER}E(t - \tau) + \gamma I(t - \tau) - \mu R(t),\end{aligned}\tag{2}$$

where N is the total number of computers in a network. μ describes the impact of quarantine or replacement. ρ_{SR} describes the impact of implementing real-time immunization. ρ_{ER} describes the impact of cleaning the virus and

immunizing the computers. β is the transmission coefficient. α and γ are the state transition rates. And τ is the period which a computer uses antivirus software to clean viruses. Dong et al. [10] discussed the local stability and existence of local Hopf bifurcation of system (2). Properties of the Hopf bifurcation were also investigated in [10].

However, Yuan and Chen [9] Dong et al. [10] supposed that the recovered computers have a permanent immunization period and can no longer be infected. This is not consistent with real situation. In order to overcome limitation and considering that the recovered computers may be infected again after a temporary immunity period, we investigate the following system with two delays in this paper:

$$\begin{aligned}\frac{dS(t)}{dt} &= \mu N - \beta I(t)S(t) - (\rho_{SR} + \mu)S(t) + \varepsilon R(t - \tau_2), \\ \frac{dE(t)}{dt} &= \beta I(t)S(t) - (\alpha + \mu)E(t) - \rho_{ER}E(t - \tau_1), \\ \frac{dI(t)}{dt} &= \alpha E(t) - (\gamma + \mu)I(t), \\ \frac{dR(t)}{dt} &= \rho_{SR}S(t) + \rho_{ER}E(t - \tau_1) + \gamma I(t - \tau_1) \\ &\quad - \mu R(t) - \varepsilon R(t - \tau_2),\end{aligned}\quad (3)$$

where τ_1 is the period that a computer uses antivirus software to clean viruses and τ_2 is the temporary immunity period after which a recovered computer may be infected again. ε is the transition rate from R to S .

This paper is organized as follows. In Section 2, local stability of the positive equilibrium and the existence of local Hopf bifurcation are discussed. In Section 3, the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions are determined by using the normal form theory and center manifold theorem. In order to illustrate the validity of the theoretical analysis, some numerical simulations are presented in Section 4. Some main conclusions are drawn in Section 5.

2. Stability and Existence of Local Hopf Bifurcation

In this section, we will study the stability of positive equilibrium and the existence of Hopf bifurcation. It is not difficult to verify if

$$R_0 = \frac{\alpha\beta\mu N(\mu + \varepsilon) + \varepsilon\rho_{SR}(\mu + \gamma)(\alpha + \mu + \rho_{ER})}{(\mu + \varepsilon)(\mu + \gamma)(\mu + \rho_{SR})(\alpha + \mu + \rho_{ER})} > 1 \quad (4)$$

system (3) has a unique positive equilibrium $D_*(S_*, E_*, I_*, R_*)$, where

$$\begin{aligned}S_* &= \frac{(\mu + \gamma)(\alpha + \mu + \rho_{ER})}{\alpha\beta}, \\ I_* &= (\alpha\beta\mu N(\mu + \varepsilon) + \varepsilon\rho_{SR}(\mu + \gamma)(\alpha + \mu + \rho_{ER}) \\ &\quad - (\mu + \varepsilon)(\mu + \gamma)(\mu + \rho_{SR})(\alpha + \mu + \rho_{ER})) \\ &\quad \times (\alpha\beta\mu(\mu + \varepsilon + \gamma) + \beta\mu(\mu + \gamma)(\mu + \varepsilon + \rho_{ER}))^{-1},\end{aligned}$$

$$E_* = \frac{\mu + \gamma}{\alpha} I_*,$$

$$R_* = \frac{\rho_{SR}(\mu + \gamma)(\alpha + \mu + \rho_{ER}) + [\alpha\beta\gamma + \beta\rho_{ER}(\mu + \gamma)]I_*}{\alpha\beta(\mu + \varepsilon)}. \quad (5)$$

R_0 is called the basic reproduction number.

Linearizing system (3) at the positive equilibrium D_* yields the following linear system:

$$\begin{aligned}\frac{dS(t)}{dt} &= a_{11}S(t) + a_{13}I(t) + c_{14}R(t - \tau_2), \\ \frac{dE(t)}{dt} &= a_{21}S(t) + a_{22}E(t) + a_{23}I(t) + b_{22}E(t - \tau_1), \\ \frac{dI(t)}{dt} &= a_{32}E(t) + a_{33}I(t), \\ \frac{dR(t)}{dt} &= a_{41}S(t) + a_{44}R(t) + b_{42}E(t - \tau_1) + b_{43}I(t - \tau_1) \\ &\quad + c_{44}R(t - \tau_2),\end{aligned}\quad (6)$$

where

$$\begin{aligned}a_{11} &= -(\mu + \rho_{SR} + \beta I_*), & a_{13} &= -\beta S_*, \\ a_{21} &= \beta I_*, & a_{22} &= -(\alpha + \mu), & a_{23} &= \beta S_*, \\ a_{32} &= \alpha, & a_{33} &= -(\mu + \gamma), \\ a_{41} &= \rho_{SR}, & a_{44} &= -\mu, \\ b_{22} &= -\rho_{ER}, & b_{42} &= \rho_{ER}, & b_{43} &= \gamma, \\ c_{14} &= \varepsilon, & c_{44} &= -\varepsilon.\end{aligned}\quad (7)$$

Thus, the characteristic equation of system (3) at D_* is

$$\begin{aligned}\lambda^4 &+ A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 \\ &+ (B_3\lambda^3 + B_2\lambda^2 + B_1\lambda + B_0)e^{-\lambda\tau_1} \\ &+ (C_3\lambda^3 + C_2\lambda^2 + C_1\lambda + C_0)e^{-\lambda\tau_2} \\ &+ (D_2\lambda^2 + D_1\lambda + D_0)e^{-\lambda(\tau_1+\tau_2)} = 0,\end{aligned}\quad (8)$$

where

$$\begin{aligned}A_0 &= a_{11}a_{44}(a_{22}a_{33} - a_{23}a_{32}) + a_{13}a_{21}a_{32}a_{44}, \\ A_1 &= a_{23}a_{32}(a_{11} + a_{44}) - a_{11}a_{22}(a_{33} + a_{44}) - a_{13}a_{21}a_{32} \\ &\quad - a_{33}a_{44}(a_{11} + a_{22}), \\ A_2 &= (a_{11} + a_{22})(a_{33} + a_{44}) + a_{11}a_{22} + a_{33}a_{44} - a_{23}a_{32}, \\ A_3 &= -(a_{11} + a_{22} + a_{33} + a_{44}), & B_0 &= a_{11}a_{33}a_{44}b_{22}, \\ B_1 &= -a_{11}a_{33}b_{22} - a_{44}b_{22}(a_{11} + a_{33}),\end{aligned}$$

$$\begin{aligned}
 B_2 &= b_{22}(a_{11} + a_{33} + a_{44}), & B_3 &= -b_{22}, \\
 C_0 &= (a_{22}a_{33} - a_{23}a_{32})(a_{11}c_{44} - a_{41}c_{14}) + a_{13}a_{21}a_{32}c_{44}, \\
 C_1 &= a_{41}c_{14}(a_{22} + a_{33}) + c_{44}(a_{23}a_{32} - a_{11}a_{22}) \\
 &\quad - a_{33}a_{44}(a_{11} + a_{22}), \\
 C_2 &= c_{44}(a_{11} + a_{22} + a_{33}) - a_{41}c_{14}, \\
 C_3 &= -c_{44}, & D_2 &= b_{22}c_{44}, \\
 D_1 &= c_{14}(a_{41}b_{22} - a_{21}b_{42}) - b_{22}c_{44}(a_{11} + a_{33}), \\
 D_0 &= a_{33}(a_{11}b_{22}c_{44} + a_{21}b_{42}c_{14}) \\
 &\quad - c_{14}(a_{21}a_{32}b_{43} + a_{33}a_{41}b_{22}).
 \end{aligned} \tag{9}$$

Case 1 ($\tau_1 = \tau_2 = 0$). Equation (8) reduces to

$$\lambda^4 + A_{13}\lambda^3 + A_{12}\lambda^2 + A_{11}\lambda + A_{10} = 0, \tag{10}$$

where

$$\begin{aligned}
 A_{10} &= A_0 + B_0 + C_0 + D_0, & A_{11} &= A_1 + B_1 + C_1 + D_1, \\
 A_{12} &= A_2 + B_2 + C_2 + D_2, & A_{13} &= A_3 + B_3 + C_3.
 \end{aligned} \tag{11}$$

By the Routh-Hurwitz criterion, sufficient conditions for all roots of (10) to have a negative real part are given in the following form:

$$\begin{aligned}
 \text{Det}_1 &= A_{13} > 0, \\
 \text{Det}_2 &= \begin{vmatrix} A_{13} & 1 \\ A_{11} & A_{12} \end{vmatrix} > 0, \\
 \text{Det}_3 &= \begin{vmatrix} A_{13} & 1 & 0 \\ A_{11} & A_{12} & A_{13} \\ 0 & A_{10} & A_{11} \end{vmatrix} > 0, \\
 \text{Det}_4 &= \begin{vmatrix} A_{13} & 1 & 0 & 0 \\ A_{11} & A_{12} & A_{13} & 1 \\ 0 & A_{10} & A_{11} & A_{12} \\ 0 & 0 & 0 & A_{10} \end{vmatrix} > 0.
 \end{aligned} \tag{12}$$

Thus, if condition (H_1) (12) holds, then the positive equilibrium D_* is locally asymptotically stable in the absence of delay.

Case 2 ($\tau_1 > 0, \tau_2 = 0$). When $\tau_1 > 0, \tau_2 = 0$, (8) becomes the following form:

$$\begin{aligned}
 \lambda^4 + A_{23}\lambda^3 + A_{22}\lambda^2 + A_{21}\lambda + A_{20} \\
 + (B_{23}\lambda^3 + B_{22}\lambda^2 + B_{21}\lambda + B_{20})e^{-\lambda\tau_1} = 0,
 \end{aligned} \tag{13}$$

where

$$\begin{aligned}
 A_{23} &= A_3 + C_3, & A_{22} &= A_2 + C_2, \\
 A_{21} &= A_1 + C_1, & A_{20} &= A_0 + C_0, \\
 B_{23} &= B_3, & B_{22} &= B_2 + D_2, \\
 B_{21} &= B_1 + D_1, & B_{20} &= B_0 + D_0.
 \end{aligned} \tag{14}$$

Let $\lambda = i\omega_1$ ($\omega_1 > 0$) be the root of (13). Then, we can get

$$\begin{aligned}
 (B_{21}\omega_1 - B_{23}\omega_1^3)\sin\tau_1\omega_1 + (B_{20} - B_{22}\omega_1^2)\cos\tau_1\omega_1 \\
 = A_{22}\omega_1^2 - \omega_1^4 - A_{20}, \\
 (B_{21}\omega_1 - B_{23}\omega_1^3)\cos\tau_1\omega_1 - (B_{20} - B_{22}\omega_1^2)\sin\tau_1\omega_1 \\
 = A_{23}\omega_1^3 - A_{21}\omega_1,
 \end{aligned} \tag{15}$$

which follows that

$$\omega_1^8 + e_{23}\omega_1^6 + e_{22}\omega_1^4 + e_{21}\omega_1^2 + e_{20} = 0, \tag{16}$$

where

$$\begin{aligned}
 e_{20} &= A_{20}^2 - B_{20}^2, \\
 e_{21} &= A_{21}^2 - B_{21}^2 - 2A_{20}A_{22} + 2B_{20}B_{22}, \\
 e_{22} &= A_{22}^2 - B_{22}^2 - 2A_{21}A_{23} + 2A_{20} + 2B_{21}B_{23}, \\
 e_{23} &= A_{23}^2 - B_{23}^2 - 2A_{22}.
 \end{aligned} \tag{17}$$

Let $\omega_1^2 = \nu_1$; then, (16) becomes

$$\nu_1^4 + e_{23}\nu_1^3 + e_{22}\nu_1^2 + e_{21}\nu_1 + e_{20} = 0. \tag{18}$$

Discussion about the roots of (18) is similar to that in [15]. Therefore, we have the following lemma.

Lemma 1. Let

$$\begin{aligned}
 f_1(\nu_1) &= \nu_1^4 + e_{23}\nu_1^3 + e_{22}\nu_1^2 + e_{21}\nu_1 + e_{20}, \\
 p &= \frac{1}{2}e_{22} - \frac{3}{16}e_{23}^2, & q &= \frac{1}{32}e_{23}^3 - \frac{1}{8}e_{22}e_{23} + e_{21}, \\
 \alpha_1 &= \left(\frac{q}{2}\right)^3 + \left(\frac{p}{3}\right)^3, & \beta_1 &= -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \\
 \gamma_1 &= \sqrt[3]{-\frac{q}{2} + \sqrt{\alpha_1}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\alpha_1}}, \\
 \gamma_2 &= \beta_1\sqrt[3]{-\frac{q}{2} + \sqrt{\alpha_1}} + \beta_1^2\sqrt[3]{-\frac{q}{2} - \sqrt{\alpha_1}}, \\
 \gamma_3 &= \beta_1^2\sqrt[3]{-\frac{q}{2} + \sqrt{\alpha_1}} + \beta_1\sqrt[3]{-\frac{q}{2} - \sqrt{\alpha_1}}, \\
 \nu_{1i} &= \gamma_i - \frac{3e_{23}}{4}, \quad i = 1, 2, 3.
 \end{aligned} \tag{19}$$

Then, for (18), one has the following:

- (i) if $e_{20} < 0$, (18) has at least one positive root;
- (ii) if $e_{20} \geq 0$ and $\alpha_1 \geq 0$, (18) has positive roots if and only if $v_{11} > 0$ and $f_1(v_{11}) < 0$;
- (iii) if $e_{20} \geq 0$ and $\alpha_1 < 0$, (18) has positive roots if and only if there exists at least one $v_{1*} \in \{v_{11}, v_{12}, v_{13}\}$ such that $v_{1*} > 0$ and $f_1(v_{1*}) \leq 0$.

In what follows, we assume that (H_{21}) : the coefficients in $f_1(v_1)$ satisfy one of the following conditions in (a)–(c):

- (a) $e_{20} < 0$;
- (b) $e_{20} \geq 0$, $\alpha_1 \geq 0$, $v_{11} > 0$, and $f_1(v_{11}) < 0$;
- (c) $e_{20} \geq 0$, $\alpha_1 < 0$, and there exists at least one $v_{1*} \in \{v_{11}, v_{12}, v_{13}\}$ such that $v_{1*} > 0$ and $f_1(v_{1*}) \leq 0$.

If (H_{21}) holds, we know that (16) has at least a positive root ω_{10} such that (13) has a pair of purely imaginary root $\pm i\omega_{10}$. The corresponding critical value of the delay is

$$\tau_{1k} = \frac{1}{\omega_{10}} \arccos \frac{p_6 \omega_{10}^6 + p_4 \omega_{10}^4 + p_2 \omega_{10}^2 + p_0}{q_6 \omega_{10}^6 + q_4 \omega_{10}^4 + q_2 \omega_{10}^2 + q_0} + \frac{2k\pi}{\omega_{10}}, \quad k = 0, 1, 2, \dots, \quad (20)$$

where

$$\begin{aligned} p_0 &= -A_{20}B_{20}, & p_2 &= A_{20}B_{22} - A_{21}B_{21} + A_{22}B_{20}, \\ p_4 &= A_{21}B_{23} - A_{22}B_{22} + A_{23}B_{21} - B_{20}, \\ p_6 &= B_{22} - A_{23}B_{23}, & q_0 &= B_{20}^2, \\ q_2 &= B_{21}^2 - 2B_{20}B_{22}, \\ q_4 &= B_{22}^2 - 2B_{21}B_{23}, & q_6 &= B_{23}^2. \end{aligned} \quad (21)$$

Differentiating both sides of (13) regarding τ_1 , we can obtain

$$\begin{aligned} \left[\frac{d\lambda}{d\tau_1} \right]^{-1} &= -\frac{4\lambda^3 + 3A_{23}\lambda^2 + 2A_{22}\lambda + A_{21}}{\lambda^5 + A_{23}\lambda^4 + A_{22}\lambda^3 + A_{21}\lambda^2 + A_{20}\lambda} \\ &\quad + \frac{3B_{23}\lambda^2 + 2B_{22}\lambda + B_{21}}{B_{23}\lambda^4 + B_{22}\lambda^3 + B_{21}\lambda^2 + B_{20}\lambda} - \frac{\tau_1}{\lambda}. \end{aligned} \quad (22)$$

Thus,

$$\begin{aligned} \operatorname{Re} \left[\frac{d\lambda}{d\tau} \right]_{\tau_1=\tau_{10}}^{-1} &= (4\omega_{10}^6 + 3(A_{23}^2 - 2A_{22})\omega_{10}^4 \\ &\quad + 2(A_{22}^2 + 2A_{20} - 2A_{21}A_{23})\omega_{10}^2 + A_{21}^2 - 2A_{20}A_{22}) \end{aligned}$$

$$\begin{aligned} &\times (\omega_{10}^8 + (A_{23}^2 - 2A_{22})\omega_{10}^6 \\ &\quad + (A_{22}^2 + 2A_{20} - 2A_{21}A_{23})\omega_{10}^4 \\ &\quad + (A_{21}^2 - 2A_{20}A_{22})\omega_{10}^2 + A_{20}^2)^{-1} \\ &\quad - (3B_{23}^2\omega_{10}^4 + 2(B_{22}^2 - 2B_{21}B_{23})\omega_{10}^2 \\ &\quad + B_{21}^2 - 2B_{20}B_{22}) \\ &\quad \times (B_{23}^2\omega_{10}^6 + (B_{22}^2 - 2B_{21}B_{23})\omega_{10}^4 \\ &\quad + (B_{21}^2 - 2B_{20}B_{22})\omega_{10}^2 + B_{20}^2)^{-1}. \end{aligned} \quad (23)$$

From (16), we can get

$$\begin{aligned} \operatorname{Re} \left[\frac{d\lambda}{d\tau} \right]_{\tau_1=\tau_{10}}^{-1} &= f_1'(v_1^*) \times (B_{23}^2\omega_{10}^6 + (B_{22}^2 - 2B_{21}B_{23})\omega_{10}^4 \\ &\quad + (B_{21}^2 - 2B_{20}B_{22})\omega_{10}^2 + B_{20}^2)^{-1}, \end{aligned} \quad (24)$$

where $v_1^* = \omega_{10}^2$.

Thus, if condition (H_{22}) $f_1'(v_1^*) \neq 0$ holds, then the transversality condition is satisfied. According to the Hopf bifurcation theorem in [24], we have the following results.

Theorem 2. Suppose that the conditions (H_{21}) – (H_{22}) hold. Consider

- (i) the positive equilibrium $D_*(S_*, E_*, I_*, R_*)$ of system (3) is asymptotically stable for $\tau_1 \in [0, \tau_{10})$;
- (ii) system (3) undergoes a Hopf bifurcation at the positive equilibrium $D_*(S_*, E_*, I_*, R_*)$ when $\tau_1 = \tau_{10}$ and a family of periodic solutions bifurcate from the positive equilibrium $D_*(S_*, E_*, I_*, R_*)$ near $\tau_1 = \tau_{10}$.

Case 3 ($\tau_1 = 0$, $\tau_2 > 0$). When $\tau_1 = 0$, $\tau_2 > 0$, (8) becomes the following form:

$$\begin{aligned} &\lambda^4 + A_{33}\lambda^3 + A_{32}\lambda^2 + A_{31}\lambda + A_{30} \\ &\quad + (C_{33}\lambda^3 + C_{32}\lambda^2 + C_{31}\lambda + C_{30})e^{-\lambda\tau_2} = 0, \end{aligned} \quad (25)$$

where

$$\begin{aligned} A_{33} &= A_3 + B_3, & A_{32} &= A_2 + B_2, \\ A_{31} &= A_1 + B_1, & A_{30} &= A_0 + B_0, \\ C_{33} &= C_{33}, & C_{32} &= C_2 + D_2, \\ C_{31} &= C_1 + D_1, & C_{30} &= C_0 + D_0. \end{aligned} \quad (26)$$

Let $\lambda = i\omega_2$ ($\omega_2 > 0$) be the root of (25), and we have

$$\begin{aligned} & (C_{31}\omega_2 - C_{33}\omega_2^3) \sin \tau_2\omega_2 + (C_{30} - C_{32}\omega_2^2) \cos \tau_2\omega_2 \\ & = A_{32}\omega_2^2 - \omega_2^4 - A_{30}, \\ & (C_{31}\omega_2 - C_{33}\omega_2^3) \cos \tau_2\omega_2 - (C_{30} - C_{32}\omega_2^2) \sin \tau_2\omega_2 \\ & = A_{33}\omega_2^3 - A_{31}\omega_2, \end{aligned} \quad (27)$$

from which, we can obtain

$$\omega_2^8 + e_{33}\omega_2^6 + e_{32}\omega_2^4 + e_{31}\omega_2^2 + e_{30} = 0, \quad (28)$$

where

$$\begin{aligned} e_{30} &= A_{30}^2 - C_{30}^2, \\ e_{31} &= A_{31}^2 - C_{31}^2 - 2A_{30}A_{32} + 2C_{20}C_{32}, \\ e_{32} &= A_{32}^2 - B_{32}^2 - 2A_{31}A_{33} + 2A_{30} + 2C_{31}C_{33}, \\ e_{33} &= A_{33}^2 - C_{33}^2 - 2A_{32}. \end{aligned} \quad (29)$$

Let $\omega_2^2 = v_2$, then (28) becomes

$$v_2^4 + e_{33}v_2^3 + e_{32}v_2^2 + e_{31}v_2 + e_{30} = 0. \quad (30)$$

Define

$$\begin{aligned} f_2(v_2) &= v_2^4 + e_{33}v_2^3 + e_{32}v_2^2 + e_{31}v_2 + e_{30}, \\ m &= \frac{1}{2}e_{32} - \frac{3}{16}e_{33}^2, \quad n = \frac{1}{32}e_{33}^3 - \frac{1}{8}e_{32}e_{33} + e_{31}, \\ \alpha_2 &= \left(\frac{n}{2}\right)^3 + \left(\frac{m}{3}\right)^3, \quad \beta_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \\ z_1 &= \sqrt[3]{-\frac{n}{2} + \sqrt{\alpha_2}} + \sqrt[3]{-\frac{n}{2} - \sqrt{\alpha_2}}, \\ z_2 &= \beta_2 \sqrt[3]{-\frac{n}{2} + \sqrt{\alpha_2}} + \beta_2^2 \sqrt[3]{-\frac{n}{2} - \sqrt{\alpha_2}}, \\ z_3 &= \beta_2^2 \sqrt[3]{-\frac{n}{2} + \sqrt{\alpha_2}} + \beta_2 \sqrt[3]{-\frac{n}{2} - \sqrt{\alpha_2}}, \\ v_{2i} &= z_i - \frac{3e_{33}}{4}, \quad i = 1, 2, 3. \end{aligned} \quad (31)$$

Next, we assume that (H_{31}) : the coefficients in $f_2(v_2)$ satisfy one of the following conditions in (a')-(c'):

- (a') $e_{30} < 0$;
- (b') $e_{30} \geq 0$, $\alpha_2 \geq 0$, $v_{21} > 0$, and $f_2(v_{21}) < 0$;
- (c') $e_{30} \geq 0$, $\alpha_2 < 0$, and there exists at least one $v_{2*} \in \{v_{21}, v_{22}, v_{23}\}$ such that $v_{2*} > 0$ and $f_2(v_{2*}) \leq 0$.

According to Lemma 1, we know that if the condition (H_{31}) holds, then (28) has at least a positive root ω_{20} such that (25)

has a pair of purely imaginary roots $\pm i\omega_{20}$. The corresponding critical value of the delay is

$$\begin{aligned} \tau_{2k} &= \frac{1}{\omega_{20}} \arccos \frac{m_6\omega_{20}^6 + m_4\omega_{20}^4 + m_2\omega_{20}^2 + m_0}{n_6\omega_{20}^6 + n_4\omega_{20}^4 + n_2\omega_{20}^2 + n_0} \\ &+ \frac{2k\pi}{\omega_{20}}, \quad k = 0, 1, 2, \dots, \end{aligned} \quad (32)$$

where

$$\begin{aligned} m_0 &= -A_{30}C_{30}, \quad m_2 = A_{30}C_{32} - A_{31}C_{31} + A_{32}C_{30}, \\ m_4 &= A_{31}C_{33} - A_{32}C_{32} + A_{33}C_{31} - C_{30}, \\ m_6 &= C_{32} - A_{33}C_{33}, \quad n_0 = C_{30}^2, \quad n_2 = C_{31}^2 - 2C_{30}C_{32}, \\ n_4 &= C_{32}^2 - 2C_{31}C_{33}, \quad n_6 = C_{33}^2. \end{aligned} \quad (33)$$

As in Case 2, we know that if $f_2'(v_2^*) \neq 0$ where $v_2^* = \omega_{20}^2$, then $\text{Re}[d\lambda/d\tau_2]_{\tau_2=\tau_{20}}^{-1} \neq 0$. That is, the transversality condition is satisfied if the condition (H_{32}) $f_2'(v_2^*) \neq 0$ holds. Thus, according to the Hopf bifurcation theorem in [24], we have the following results.

Theorem 3. Suppose the conditions (H_{31}) -(H_{32}) hold.

- (i) The positive equilibrium $D_*(S_*, E_*, I_*, R_*)$ of system (3) is asymptotically stable for $\tau_2 \in [0, \tau_{20})$.
- (ii) System (3) undergoes a Hopf bifurcation at the positive equilibrium $D_*(S_*, E_*, I_*, R_*)$ when $\tau_2 = \tau_{20}$ and a family of periodic solutions bifurcate from the positive equilibrium $D_*(S_*, E_*, I_*, R_*)$ near $\tau_2 = \tau_{20}$.

Case 4 ($\tau_1 = \tau_2 = \tau > 0$). Substituting $\tau_1 = \tau_2 = \tau > 0$ into (8), then (8) becomes

$$\begin{aligned} & \lambda^4 + A_{43}\lambda^3 + A_{42}\lambda^2 + A_{41}\lambda + A_{40} \\ & + (B_{43}\lambda^3 + B_{42}\lambda^2 + B_{41}\lambda + B_{40})e^{-\lambda\tau} \\ & + (D_{42}\lambda^2 + D_{41}\lambda + D_{40})e^{-2\lambda\tau} = 0, \end{aligned} \quad (34)$$

where

$$\begin{aligned} A_{43} &= A_3, \quad A_{42} = A_2, \quad A_{41} = A_1, \quad A_{40} = A_0, \\ B_{43} &= B_3 + C_3, \quad B_{42} = B_2 + C_2, \quad B_{41} = B_1 + C_1, \\ B_{40} &= B_0 + C_0, \quad D_{42} = D_2, \quad D_{41} = D_1, \\ D_{40} &= D_0. \end{aligned} \quad (35)$$

Multiplying by $e^{\lambda\tau}$, (34) becomes

$$\begin{aligned} & B_{43}\lambda^3 + B_{42}\lambda^2 + B_{41}\lambda + B_{40} \\ & \times (\lambda^4 + A_{43}\lambda^3 + A_{42}\lambda^2 + A_{41}\lambda + A_{40})e^{\lambda\tau} \\ & + (D_{42}\lambda^2 + D_{41}\lambda + D_{40})e^{-\lambda\tau} = 0. \end{aligned} \quad (36)$$

Let $\lambda = i\omega$ ($\omega > 0$) be the root of (36); consider

$$\begin{aligned} & (\omega^4 - (A_{42} + D_{42})\omega^2 + A_{40} + D_{40}) \cos \tau\omega \\ & + (A_{43}\omega^3 - (A_{41} - D_{41})\omega) \sin \tau\omega = B_{32}\omega^2 - B_{40}, \\ & (\omega^4 - (A_{42} - D_{42})\omega^2 + A_{40} - D_{40}) \sin \tau\omega \\ & - (A_{43}\omega^3 - (A_{41} + D_{41})\omega) \cos \tau\omega = B_{43}\omega^3 - B_{41}\omega, \end{aligned} \quad (37)$$

which follows that

$$\begin{aligned} \sin \tau\omega &= \frac{g_7\omega^7 + g_5\omega^5 + g_3\omega^3 + g_1\omega}{\omega^8 + h_6\omega^6 + h_4\omega^4 + h_2\omega^2 + h_0}, \\ \cos \tau\omega &= \frac{g_6\omega^6 + g_4\omega^4 + g_2\omega^2 + g_0}{\omega^8 + h_6\omega^6 + h_4\omega^4 + h_2\omega^2 + h_0}, \end{aligned} \quad (38)$$

where

$$\begin{aligned} g_0 &= B_{40}(D_{40} - A_{40}), \\ g_1 &= B_{40}(A_{41} + D_{41}) - B_{41}(A_{40} + D_{40}), \\ g_2 &= B_{40}(A_{42} - D_{42}) - B_{41}(A_{41} - D_{41}) + B_{42}(A_{40} - D_{40}), \\ g_3 &= B_{41}(A_{42} + D_{42}) - B_{42}(A_{41} + D_{41}) \\ &+ B_{43}(A_{40} + D_{40}) - A_{43}B_{40}, \\ g_4 &= B_{42}(D_{42} - A_{42}) + B_{43}(A_{41} - D_{41}) + A_{43}B_{41} - B_{40}, \\ g_5 &= A_{42}B_{43} - B_{41} - B_{43}(A_{42} + D_{42}), \\ g_6 &= B_{42} - A_{43}B_{43}, \quad g_7 = B_{43}, \\ h_0 &= A_{40}^2 - D_{40}^2, \\ h_2 &= A_{41}^2 - D_{41}^2 - 2A_{40}A_{42} + 2D_{40}D_{42}, \\ h_4 &= A_{42}^2 - D_{42}^2 - 2A_{41}A_{43} + 2A_{40}, \\ h_6 &= A_{43}^2 - 2A_{42}. \end{aligned} \quad (39)$$

As is known, $\sin^2 \tau\omega + \cos^2 \tau\omega = 1$. Thus, we have

$$\begin{aligned} & \omega^{16} + e_{47}\omega^{14} + e_{46}\omega^{12} + e_{45}\omega^{10} + e_{44}\omega^8 \\ & + e_{43}\omega^6 + e_{42}\omega^4 + e_{41}\omega^2 + e_{40} = 0, \end{aligned} \quad (40)$$

where

$$\begin{aligned} e_{40} &= h_0^2 - g_0^2, \quad e_{41} = 2h_0h_2 - 2g_0g_2 - g_1^2, \\ e_{42} &= h_2^2 - g_2^2 + 2h_0h_4 - 2g_0g_4 - 2g_1g_3, \\ e_{43} &= 2h_0h_6 + 2h_2h_4 - 2g_0g_6 - 2g_1g_5 - 2g_2g_4 - g_3^3, \\ e_{44} &= h_4^2 - g_4^2 + 2h_0 + 2h_2h_6 - 2g_1g_7 - 2g_2g_6 - 2g_3g_5, \\ e_{45} &= 2h_2 + 2h_4h_6 - 2g_3g_7 - 2g_4g_6 - g_5^2, \end{aligned}$$

$$\begin{aligned} e_{46} &= h_6^2 - g_6^2 + 2h_4 - 2g_5g_7, \\ e_{47} &= 2h_6 - g_7^2. \end{aligned} \quad (41)$$

Let $\omega^2 = v$, then (40) becomes

$$\begin{aligned} & v^8 + e_{47}v^7 + e_{46}v^6 + e_{45}v^5 + e_{44}v^4 + e_{43}v^3 \\ & + e_{42}v^2 + e_{41}v + e_{40} = 0. \end{aligned} \quad (42)$$

In order to give the main results in this paper, we make the following assumption.

(H_{41}) (42) has at least one positive real root.

Suppose that condition (H_{41}) holds. Without loss of generality, we assume that (42) has eight positive real roots, which are denoted as v_1, v_2, \dots, v_8 , respectively. Then (40) has eight positive real roots $\omega_k = \sqrt{v_k}$. For every fixed ω_k , the corresponding critical value of time delay is

$$\begin{aligned} \tau_k^{(j)} &= \frac{1}{\omega_k} \arccos \frac{g_6\omega_k^6 + g_4\omega_k^4 + g_2\omega_k^2 + g_0}{\omega_k^8 + h_6\omega_k^6 + h_4\omega_k^4 + h_2\omega_k^2 + h_0} + \frac{2j\pi}{\omega_k}, \\ & k = 1, 2, \dots; \quad j = 0, 1, 2, \dots \end{aligned} \quad (43)$$

Define

$$\tau_0 = \min \{ \tau_k^{(0)} \mid k = 1, 2, \dots, 8 \}, \quad \omega_0 = \omega_k|_{\tau=\tau_0}. \quad (44)$$

Taking the derivative of with respect to (36), it is easy to obtain

$$\begin{aligned} & \left[\frac{d\lambda}{d\tau} \right]^{-1} \\ &= - \left((3B_{43}\lambda^2 + 2B_{42}\lambda + B_{41} + (2D_{42}\lambda + D_{41}))e^{-\lambda\tau} \right. \\ & \quad + (4\lambda^3 + 3A_{43}\lambda^2 + 2A_{42}\lambda + A_{41})e^{\lambda\tau} \\ & \quad \times ((\lambda^5 + A_{43}\lambda^4 + A_{42}\lambda^3 + A_{41}\lambda^2 + A_{40}\lambda)e^{\lambda\tau} \\ & \quad \left. - (D_{42}\lambda^3 + D_{41}\lambda^2 + D_{40}\lambda)e^{-\lambda\tau})^{-1} \right) - \frac{\tau}{\lambda}. \end{aligned} \quad (45)$$

Thus, we have

$$\operatorname{Re} \left[\frac{d\lambda}{d\tau} \right]_{\tau=\tau_0}^{-1} = - \frac{P_R Q_R + P_I Q_I}{Q_R^2 + Q_I^2}, \quad (46)$$

where

$$\begin{aligned} P_R &= (A_{41} + D_{41} - 3A_{43}\omega_0^2) \cos \tau_0\omega_0 \\ &+ 2(2\omega_0^3 - A_{42}\omega_0 + D_{42}\omega_0) \sin \tau_0\omega_0 - 3B_{43}\omega_0^2 + B_{41}, \\ P_I &= (A_{41} - D_{41} - 3A_{43}\omega_0^2) \sin \tau_0\omega_0 \\ &- 2(2\omega_0^3 - A_{42}\omega_0 - D_{42}\omega_0) \cos \tau_0\omega_0 + 2B_{42}\omega_0, \\ Q_R &= (\omega_0^5 - (A_{42} + D_{42})\omega_0^3 + (A_{40} + D_{40})\omega_0) \sin \tau_0\omega_0 \\ &+ (A_{43}\omega_0^4 - (A_{41} + D_{41})\omega_0^2) \cos \tau_0\omega_0, \end{aligned}$$

$$Q_I = (\omega_0^5 - (A_{42} + D_{42})\omega_0^3 + (A_{40} + D_{40})\omega_0) \cos \tau_0 \omega_0 - (A_{43}\omega_0^4 - (A_{41} + D_{41})\omega_0^2) \sin \tau_0 \omega_0. \quad (47)$$

Obviously, if condition $P_R Q_R + P_I Q_I \neq 0$ holds, then $\text{Re}[d\lambda/d\tau]_{\tau=\tau_0}^{-1} \neq 0$. Thus, by the Hopf bifurcation theorem in [24], we have the following results.

Theorem 4. Suppose that conditions (H_{41}) – (H_{42}) hold.

- (i) The positive equilibrium $D_*(S_*, E_*, I_*, R_*)$ of system (3) is asymptotically stable for $\tau \in [0, \tau_0)$.
- (ii) System (3) undergoes a Hopf bifurcation at the positive equilibrium $D_*(S_*, E_*, I_*, R_*)$ when $\tau = \tau_0$ and a family of periodic solutions bifurcate from the positive equilibrium $D_*(S_*, E_*, I_*, R_*)$ near $\tau = \tau_0$.

Case 5 ($\tau_1 > 0, \tau_2 \in (0, \tau_{20})$). We consider (8) with τ_2 in its stable interval, regarding τ_1 as a parameter.

Let $\lambda = i\omega_{1*}$ ($\omega_{1*} > 0$) be the root of (8); we obtain

$$c_0(\omega_{1*}) + 2c_1(\omega_{1*}) \cos \tau_2 \omega_{1*} + 2c_2(\omega_{1*}) \sin \tau_2 \omega_{1*} = 0, \quad (48)$$

where

$$\begin{aligned} c_0(\omega_{1*}) &= \omega_{1*}^8 + (A_3^2 - B_3^2 + C_3^2 - 2A_2)\omega_{1*}^6 \\ &\quad + (A_2^2 - B_2^2 + C_2^2 - D_2^2 + 2A_0 - 2A_1A_3 + 2B_1B_3 \\ &\quad - 2C_1C_3)\omega_{1*}^4 \\ &\quad + (A_1^2 - B_1^2 + C_1^2 - D_1^2 - 2A - 0A_2 \\ &\quad + 2B_0B_2 - 2C_0C_2 + 2D_0D_2)\omega_{1*}^2 \\ &\quad + A_0^2 - B_0^2 + C_0^2 - D_0^2, \\ c_1(\omega_{1*}) &= (A_3C_3 - C_2)\omega_{1*}^6 \\ &\quad + (A_2C_2 + B_3D_1 - B_2D_2 + C_0 - A_1C_3 - A_3C_1) \\ &\quad \times \omega_{1*}^4 + (A_1C_1 - A_0C_2 - A_2C_0 + B_0D_2 \\ &\quad + B_2D_0 - B_1D_1)\omega_{1*}^2 + A_0C_0 + B_0D_0, \\ c_2(\omega_{1*}) &= -C_3\omega_{1*}^7 + (A_2C_3 - A_3C_2 + B_3D_2 + C_1)\omega_{1*}^5 \\ &\quad + (A_1C_2 + A_3C_0 - A_0C_3 - A_2C_1 - B_1D_2 \\ &\quad + B_2D_1 - B_3D_0)\omega_{1*}^3 \\ &\quad \times (A_0C_1 - A_1C_0 - B_0D_1 + B_1D_0)\omega_{1*}. \end{aligned} \quad (49)$$

We assume that (H_{51}) (48) has at least finite positive roots. And we denote the positive roots of (48) as $\omega_{11*}, \omega_{12*}$, and ω_{1k*} . For every fixed ω_{1i*} , the corresponding critical value of time delay is

$$\tau_{1i*}^{(j)} = \frac{1}{\omega_{1i*}} \arccos \frac{\Delta_1 \Delta_4 + \Delta_2 \Delta_3}{\Delta_1^2 + \Delta_2^2} + \frac{2j\pi}{\omega_{1i*}}, \quad (50)$$

$$i = 1, 2, k; \quad j = 0, 1, 2, \dots,$$

where

$$\begin{aligned} \Delta_1 &= D_1 \omega_{1i*} \cos \tau_2 \omega_{1i*} + (D_2 \omega_{1i*}^2 - D_0) \sin \tau_2 \omega_{1i*} \\ &\quad - B_3 \omega_{1i*}^3 + B_1 \omega_{1i*}, \\ \Delta_2 &= D_1 \omega_{1i*} \sin \tau_2 \omega_{1i*} - (D_2 \omega_{1i*}^2 - D_0) \cos \tau_2 \omega_{1i*} \\ &\quad - B_2 \omega_{1i*}^2 + B_0, \\ \Delta_3 &= (C_3 \omega_{1i*}^3 - C_1 \omega_{1i*}) \sin \tau_2 \omega_{1i*} \\ &\quad + (C_2 \omega_{1i*}^2 - C_0) \cos \tau_2 \omega_{1i*} + A_2 \omega_{1i*}^2 - \omega_{1i*}^4 - A_0, \\ \Delta_4 &= (C_3 \omega_{1i*}^3 - C_1 \omega_{1i*}) \cos \tau_2 \omega_{1i*} \\ &\quad + (C_2 \omega_{1i*}^2 - C_0) \sin \tau_2 \omega_{1i*} + A_3 \omega_{1i*}^3 - A_1 \omega_{1i*}. \end{aligned} \quad (51)$$

Let

$$\tau_1^* = \min \{ \tau_{1i*}^{(0)} \mid i = 1, 2, \dots, k \}, \quad \omega_1^* = \omega_{1i*} |_{\tau_1 = \tau_1^*}. \quad (52)$$

Next, we make the following assumption: (H_{51}) $\text{Re}[d\lambda/d\tau]_{\tau_1 = \tau_1^*}^{-1} \neq 0$. Thus, by the Hopf bifurcation theorem in [24], we have the following results.

Theorem 5. Suppose that conditions (H_{51}) – (H_{52}) hold and $\tau_2 \in (0, \tau_{20})$.

- (i) The positive equilibrium $D_*(S_*, E_*, I_*, R_*)$ of system (3) is asymptotically stable for $\tau_1 \in [0, \tau_1^*)$.
- (ii) System (3) undergoes a Hopf bifurcation at the positive equilibrium $D_*(S_*, E_*, I_*, R_*)$ when $\tau_1 = \tau_1^*$ and a family of periodic solutions bifurcate from the positive equilibrium $D_*(S_*, E_*, I_*, R_*)$ near $\tau_1 = \tau_1^*$.

3. Direction and Stability of the Hopf Bifurcation

In this section, we will investigate the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions w.r.t. τ_1 for $\tau_2 \in (0, \tau_{20})$ by using the normal form theory and the center manifold argument in [24]. We assume that $\tau_2^* < \tau_1^*$ where $\tau_2^* \in (0, \tau_{20})$.

Let $u_1(t) = S(t) - S_*$, $u_2(t) = E(t) - E_*$, $u_3(t) = I(t) - I_*$, $u_4(t) = R(t) - R_*$, $\tau_1 = \tau_1^* + \mu$, $\mu \in \mathbb{R}$, and normalize $t \rightarrow (t/\tau_1)$. Then system (3) can be transformed into the following form:

$$\dot{u}(t) = L_\mu u_t + F(\mu, u_t), \quad (53)$$

where $u_t = (u_1(t), u_2(t), u_3(t), u_4(t))^T \in C = C([-1, 0], \mathbb{R}^4)$ and

$$\begin{aligned} L_\mu \phi &= (\tau_1^* + \mu) \left(A' \phi(0) + C' \phi \left(-\frac{\tau_2^*}{\tau_1^*} \right) + B' \phi(-1) \right), \\ F(\mu, \phi) &= (\tau_1^* + \mu) \begin{pmatrix} -\beta \phi_1(0) \phi_3(0) \\ \beta \phi_1(0) \phi_3(0) \\ 0 \\ 0 \end{pmatrix}, \end{aligned} \quad (54)$$

where

$$\begin{aligned} A' &= \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & 0 \\ a_{41} & 0 & 0 & a_{44} \end{pmatrix}, \\ B' &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & b_{42} & b_{43} & 0 \end{pmatrix}, \\ C' &= \begin{pmatrix} 0 & 0 & 0 & c_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} \end{pmatrix}. \end{aligned} \quad (55)$$

By the Riesz representation theorem, there exists a 4×4 matrix function $\eta(\theta, \mu) : [-1, 0] \rightarrow R^4$ whose elements are of bounded variation such that

$$L_\mu \phi = \int_{-1}^0 d\eta(\theta, \mu) \phi(\theta), \quad \phi \in C. \quad (56)$$

In fact, we choose

$$\eta(\theta, \mu) = \begin{cases} (\tau_1^* + \mu)(A' + B' + C'), & \theta = 0, \\ (\tau_1^* + \mu)(B' + C'), & \theta \in \left[-\frac{\tau_2^*}{\tau_1^*}, 0\right), \\ (\tau_1^* + \mu)B', & \theta \in \left(-1, -\frac{\tau_2^*}{\tau_1^*}\right), \\ 0, & \theta = 1. \end{cases} \quad (57)$$

For $\phi \in C([-1, 0], R^4)$, we define

$$\begin{aligned} A(\mu)\phi &= \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \leq \theta < 0, \\ \int_{-1}^0 d\eta(\theta, \mu) \phi(\theta), & \theta = 0, \end{cases} \\ R(\mu)\phi &= \begin{cases} 0, & -1 \leq \theta < 0, \\ F(\mu, \phi), & \theta = 0. \end{cases} \end{aligned} \quad (58)$$

Then system (53) can be transformed into the following operator equation:

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t. \quad (59)$$

The adjoint operator A^* of A is defined by

$$A^*(\varphi) = \begin{cases} -\frac{d\varphi(s)}{ds}, & 0 < s \leq 1, \\ \int_{-1}^0 d\eta^T(s, 0) \varphi(-s), & s = 0, \end{cases} \quad (60)$$

associated with a bilinear form

$$\begin{aligned} \langle \varphi(s), \phi(\theta) \rangle &= \bar{\varphi}(0) \phi(0) \\ &\quad - \int_{\theta=-1}^0 \int_{\xi=0}^\theta \bar{\varphi}(\xi - \theta) d\eta(\theta) \phi(\xi) d\xi, \end{aligned} \quad (61)$$

where $\eta(\theta) = \eta(\theta, 0)$.

Let $q(\theta) = (1, q_2, q_3, q_4)^T e^{i\tau_1^* \omega_1^* \theta}$ be the eigenvector of $A(0)$ corresponding to $+i\tau_1^* \omega_1^*$, and let $q^*(s) = D(1, q_2^*, q_3^*, q_4^*) e^{i\tau_1^* \omega_1^* s}$ be the eigenvector of $A^*(0)$ corresponding to $-i\tau_1^* \omega_1^*$. From the definition of $A(0)$ and $A^*(0)$, we can get

$$\begin{aligned} q_2 &= \frac{a_{21}(i\omega_1^* - a_{33})}{(i\omega_1^* - a_{33})(i\omega_1^* - a_{22} - b_{22}e^{-\tau_1^* \omega_1^*}) - a_{23}a_{32}}, \\ q_3 &= \frac{a_{21}a_{32}}{(i\omega_1^* - a_{33})(i\omega_1^* - a_{22} - b_{22}e^{-\tau_1^* \omega_1^*}) - a_{23}a_{32}}, \\ q_4 &= \frac{i\omega_1^* - a_{11}}{c_{14}e^{-i\tau_2^* \omega_1^*}} \\ &\quad - \frac{a_{13}a_{21}a_{32}e^{i\tau_2^* \omega_1^*}}{c_{14}(i\omega_1^* - a_{33})(i\omega_1^* - a_{22} - b_{22}e^{-\tau_1^* \omega_1^*}) - a_{23}a_{32}c_{14}}, \\ q_2^* &= \frac{a_{41}c_{14}e^{i\tau_2^* \omega_1^*}}{a_{21}(i\omega_1^* + a_{44} + c_{44}e^{i\tau_2^* \omega_1^*})} - \frac{i\omega_1^* + a_{11}}{a_{21}}, \\ q_3^* &= -\frac{(i\omega_1^* + a_{22} + b_{22}e^{i\tau_1^* \omega_1^*})q_2^* + b_{42}e^{i\tau_1^* \omega_1^*}q_4^*}{a_{32}}, \\ q_4^* &= -\frac{c_{14}e^{i\tau_2^* \omega_1^*}}{i\omega_1^* + a_{44} + c_{44}e^{i\tau_2^* \omega_1^*}}. \end{aligned} \quad (62)$$

From (61), we can get

$$\begin{aligned} \langle q^*, q \rangle &= \bar{D} \left[1 + q_2 \bar{q}_2^* + q_3 \bar{q}_3^* + q_4 \bar{q}_4^* \right. \\ &\quad \left. + \tau_2^* e^{-i\tau_2^* \omega_1^*} q_4 (c_{14} + c_{44} \bar{q}_4^*) \right. \\ &\quad \left. + \tau_1^* e^{-i\tau_1^* \omega_1^*} (q_2 (b_{22} \bar{q}_2^* + b_{42} \bar{q}_4^*) + b_{43} q_3 \bar{q}_4^*) \right]. \end{aligned} \quad (63)$$

Then, we choose

$$\begin{aligned} \bar{D} &= \left[1 + q_2 \bar{q}_2^* + q_3 \bar{q}_3^* + q_4 \bar{q}_4^* + \tau_2^* e^{-i\tau_2^* \omega_1^*} q_4 (c_{14} + c_{44} \bar{q}_4^*) \right. \\ &\quad \left. + \tau_1^* e^{-i\tau_1^* \omega_1^*} (q_2 (b_{22} \bar{q}_2^* + b_{42} \bar{q}_4^*) + b_{43} q_3 \bar{q}_4^*) \right]^{-1}, \end{aligned} \quad (64)$$

such that $\langle q^*, q \rangle = 1$, $\langle q^*, \bar{q} \rangle = 0$.

Next, we can get the coefficients which can be used to determine the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions by following the algorithms introduced in [24]:

$$\begin{aligned} g_{20} &= 2\beta \tau_1^* \bar{D} q_3 (\bar{q}_2^* - 1), \\ g_{02} &= 2\beta \tau_1^* \bar{D} \bar{q}_3 (\bar{q}_2^* - 1), \\ g_{11} &= b \tau_1^* \bar{D} (q_3 + \bar{q}_3) (\bar{q}_2^* - 1), \end{aligned} \quad (65)$$

$$\begin{aligned} g_{21} &= 2\beta \tau_1^* \bar{D} (\bar{q}_2^* - 1) \left(W_{11}^{(1)}(0) q_3 + \frac{1}{2} W_{20}^{(1)}(0) \bar{q}_3 \right. \\ &\quad \left. + W_{11}^{(3)}(0) + \frac{1}{2} W_{20}^{(3)}(0) \right), \end{aligned}$$

with

$$W_{20}(\theta) = \frac{ig_{20}q(0)}{\tau_1^* \omega_1^*} e^{i\tau_1^* \omega_1^* \theta} + \frac{i\bar{g}_{02}\bar{q}(0)}{3\tau_1^* \omega_1^*} e^{-i\tau_1^* \omega_1^* \theta} + E_1 e^{2i\tau_1^* \omega_1^* \theta},$$

$$W_{11}(\theta) = -\frac{ig_{11}q(0)}{\tau_1^* \omega_1^*} e^{i\tau_1^* \omega_1^* \theta} + \frac{i\bar{g}_{11}\bar{q}(0)}{\tau_1^* \omega_1^*} e^{-i\tau_1^* \omega_1^* \theta} + E_2, \quad (66)$$

where E_1 and E_2 can be determined by the following equations respectively

$$E_1 = 2 \begin{pmatrix} 2i\omega_1^* - a_{11} & 0 & -a_{13} & -c_{14}e^{-2i\tau_2^* \omega_1^*} \\ -a_{21} & 2i\omega_1^* - a_{22} - b_{22}e^{-2i\tau_1^* \omega_1^*} & -a_{23} & 0 \\ 0 & -a_{32} & 2i\omega_1^* - a_{33} & 0 \\ -a_{41} & -b_{42}e^{-2i\tau_1^* \omega_1^*} & -b_{43}e^{-2i\tau_1^* \omega_1^*} & 2i\omega_1^* - a_{44} - c_{44}e^{-2i\tau_2^* \omega_1^*} \end{pmatrix}^{-1} \begin{pmatrix} E_1^{(1)} \\ E_1^{(2)} \\ 0 \\ 0 \end{pmatrix}, \quad (67)$$

$$E_2 = - \begin{pmatrix} a_{11} & 0 & a_{13} & c_{14} \\ a_{21} & a_{22} + b_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & 0 \\ a_{41} & b_{42} & b_{43} & a_{44} + c_{44} \end{pmatrix}^{-1} \begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \\ 0 \\ 0 \end{pmatrix},$$

with

$$\begin{aligned} E_1^{(1)} &= -\beta q_3, & E_1^{(2)} &= \beta q_3, \\ E_2^{(1)} &= -\beta(q_3 + \bar{q}_3), & E_2^{(2)} &= \beta(q_3 + \bar{q}_3). \end{aligned} \quad (68)$$

Therefore, we can calculate the following values:

$$\begin{aligned} C_1(0) &= \frac{i}{2\tau_1^* \omega_1^*} \left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3} \right) + \frac{g_{21}}{2}, \\ \bar{\mu} &= -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_1^*)\}}, \\ \bar{\beta} &= 2\operatorname{Re}\{C_1(0)\}, \\ \bar{T} &= -\frac{\operatorname{Im}\{C_1(0)\} + \bar{\mu}\operatorname{Im}\{\lambda'(\tau_1^*)\}}{\tau_1^* \omega_1^*}. \end{aligned} \quad (69)$$

Based on the discussion above, we can obtain the following results.

Theorem 6. For system (3), if $\bar{\mu} > 0$ ($\bar{\mu} < 0$), the Hopf bifurcation is supercritical (subcritical). If $\bar{\beta} < 0$ ($\bar{\beta} > 0$) the bifurcating periodic solutions are stable (unstable). If $\bar{T} > 0$ ($\bar{T} < 0$), the period of the bifurcating periodic solutions increases (decreases).

4. Numerical Simulation

In this section, we present some numerical simulations to verify the theoretical analysis in Sections 2 and 3. Let $N = 1000$, $\alpha = 0.1$, $\beta = 0.01$, $\gamma = 0.08$, $\varepsilon = 0.15$, $\mu = 0.01$,

$\rho_{SR} = 0.2$, $\rho_{ER} = 0.2$. Then, we can get a particular case of system (3):

$$\begin{aligned} \frac{dS(t)}{dt} &= 10 - 0.01I(t)S(t) - 0.21S(t) + 0.15R(t - \tau_2), \\ \frac{dE(t)}{dt} &= 0.01I(t)S(t) - 0.11E(t) - 0.2E(t - \tau_1), \\ \frac{dI(t)}{dt} &= 0.1E(t) - 0.09I(t), \\ \frac{dR(t)}{dt} &= 0.2S(t) + 0.2E(t - \tau_1) + 0.08I(t - \tau_1) \\ &\quad - 0.01R(t) - 0.15R(t - \tau_2). \end{aligned} \quad (70)$$

Then, we can get $R_0 = 2.5996 > 1$ and the unique positive equilibrium $D_*(27.9, 239.2915, 265.8794, 466.9290)$ of system (70). Further, we have $\operatorname{Det}_1 = 3.4288 > 0$, $\operatorname{Det}_2 = 5.4964 > 0$, $\operatorname{Det}_3 = 0.6898 > 0$, $\operatorname{Det}_4 = 9.6572e - 004 > 0$. That is, condition (H_1) holds.

For $\tau_1 > 0$, $\tau_2 = 0$, by some complicated computation, we obtain $\omega_{10} = 0.6102$, $\tau_{10} = 17.2530$. By Theorem 2, we know that when $\tau_1 \in [0, 17.2530]$ the positive equilibrium D_* is asymptotically stable which can be illustrated by Figures 1 and 2. However, if we let $\tau_1 = 20.05 > 17.2530 = \tau_{10}$, the positive equilibrium D_* becomes unstable and a Hopf bifurcation occurs and a branch of periodic solutions bifurcates from the positive equilibrium D_* . This property can be shown in Figures 3 and 4. Similarly, we have $\omega_{20} = 0.1181$, $\tau_{20} = 52.4927$ for $\tau_1 = 0$, $\tau_2 > 0$. The corresponding wave forms and plots are shown in Figures 5, 6, 7, and 8.

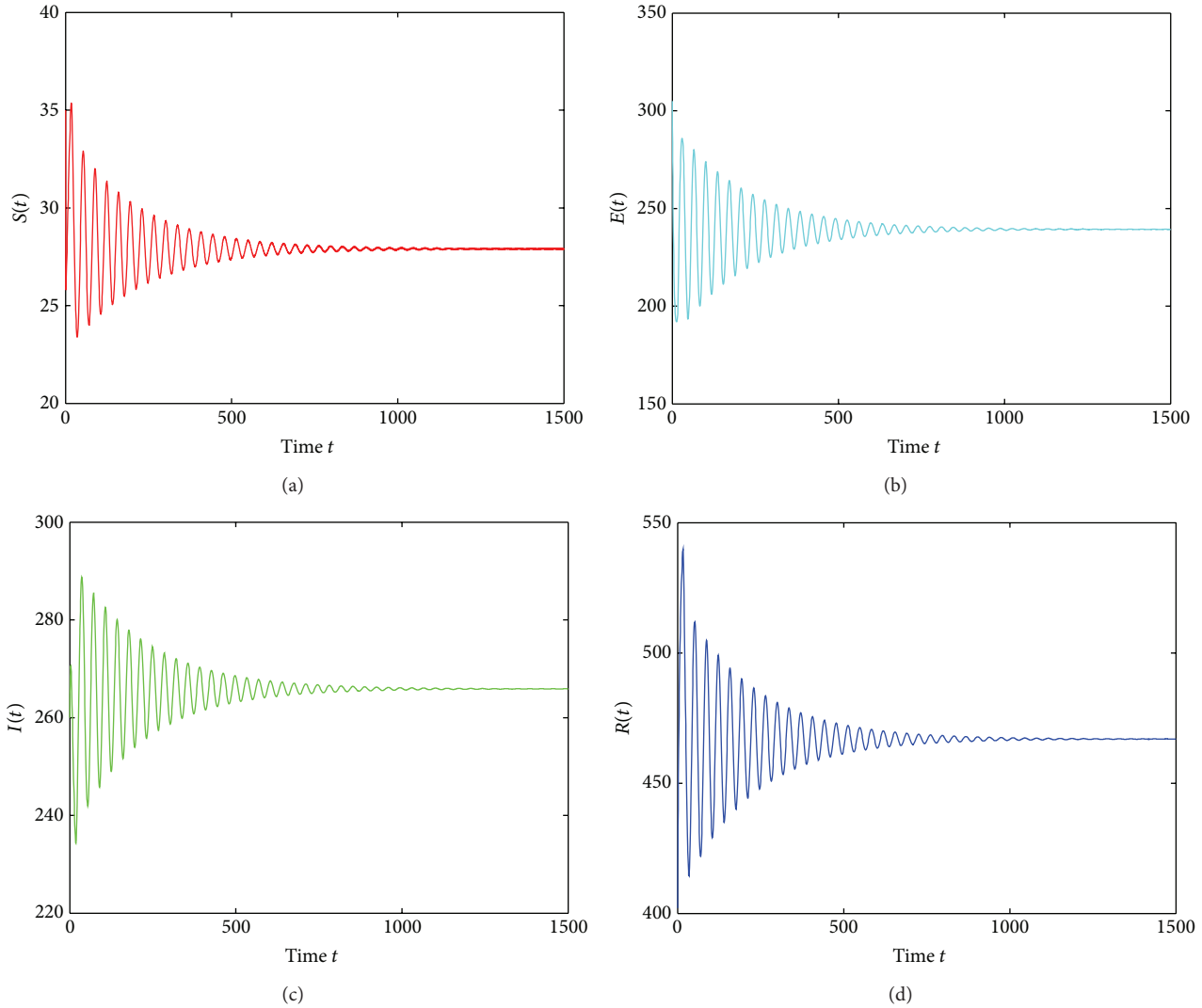


FIGURE 1: The track of the states S , E , I , and R for $\tau_1 = 15.5000 < 17.2530 = \tau_{10}$.

For $\tau_1 = \tau_2 = \tau > 0$, we get $\omega_0 = 2.0670$, $\tau_0 = 5.1094$. According to Theorem 4, we can conclude that when $\tau \in [0, 5.1094)$, the positive equilibrium D_* is asymptotically stable, which can be seen from Figures 9 and 10. If $\tau = 5.23 > 5.1094 = \tau_0$; then, the positive equilibrium D_* becomes unstable, and a branch of bifurcating periodic solutions occurs, which can be shown in Figures 11 and 12.

Lastly, we have $\omega_1^* = 1.6115$, $\tau_1^* = 7.8533$ for $\tau_1 > 0$, $\tau_2 = 3 \in (0, \tau_{20})$. The corresponding wave forms and phase plots are shown in Figures 13, 14, 15, and 16. In addition, we have $\lambda'(\tau_1^*) = 0.3938 + 0.1113i$, $C_1(0) = -0.2079 - 0.6067i$, $\bar{\mu} = 0.5279$, $\bar{\beta} = -0.4158 < 0$, $\bar{T} = 0.0433 > 0$. Thus, by Theorem 6, we can conclude that the Hopf bifurcation is supercritical, the bifurcating periodic solutions are stable, and the period of the periodic solutions increases.

5. Conclusions

Based on the SEIR model considered in the literature [10], we propose an SEIRS model with two delays for the propagation

of computer viruses in networks. The effects of the two delays on the dynamics of the model are investigated. It is found that the two delays may lead to local Hopf bifurcation and make the model unstable under some certain conditions. When the corresponding delay is suitable and small, the positive equilibrium is asymptotically stable. In this case, the propagation of computer viruses can be predicted and controlled. However, a local Hopf bifurcation occurs and a branch of periodic solutions bifurcates from the positive equilibrium when the corresponding delay passes through a critical value. In such conditions, propagation of the computer viruses is out of control. This phenomenon is not welcome in networks. Therefore, In order to control and even eliminate the propagation of computer viruses, the two delays τ_1 and τ_2 in the model should remain less than the corresponding critical value. Furthermore, by using the normal form theory and center manifold theorem, the properties of the Hopf bifurcation such as direction and stability are determined. Some numerical simulations are also included to testify the theoretical analysis.

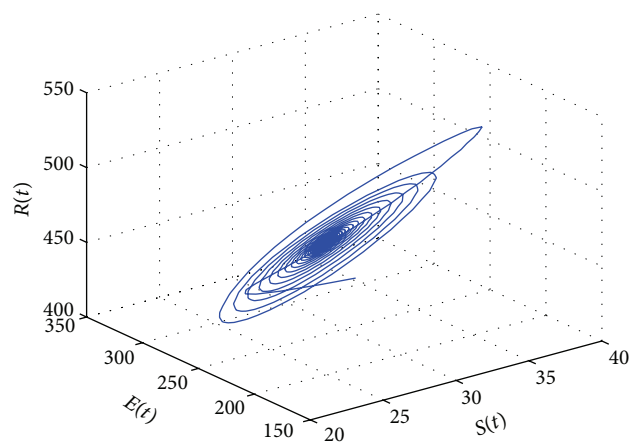


FIGURE 2: The phase plot of the states S , E , and R for $\tau_1 = 15.5000 < 17.2530 = \tau_{10}$.

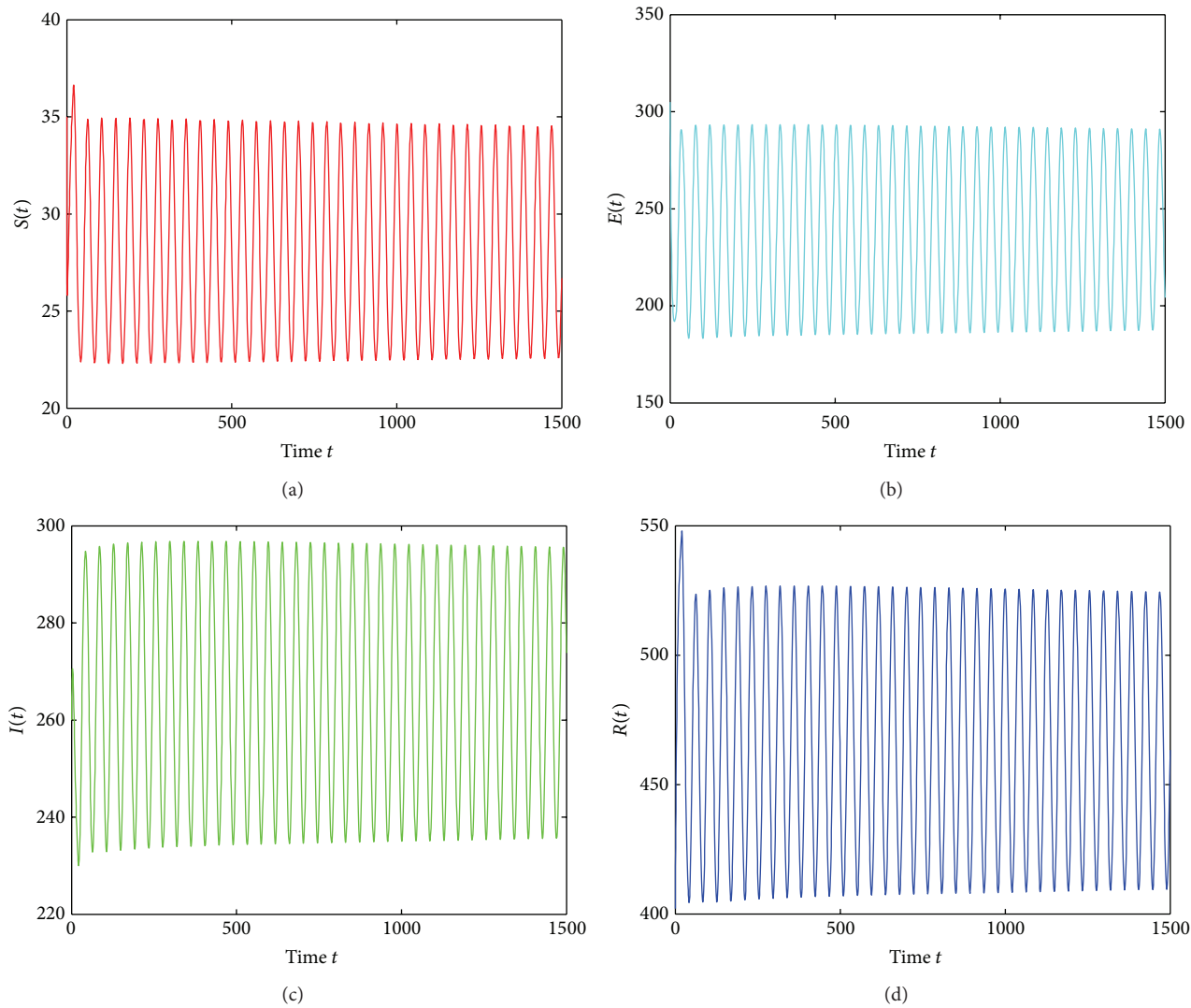


FIGURE 3: The track of the states S , E , I , and R for $\tau_1 = 20.0500 > 17.2530 = \tau_{10}$.

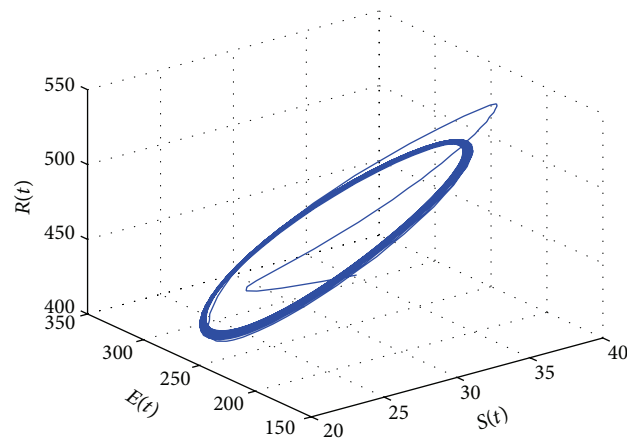
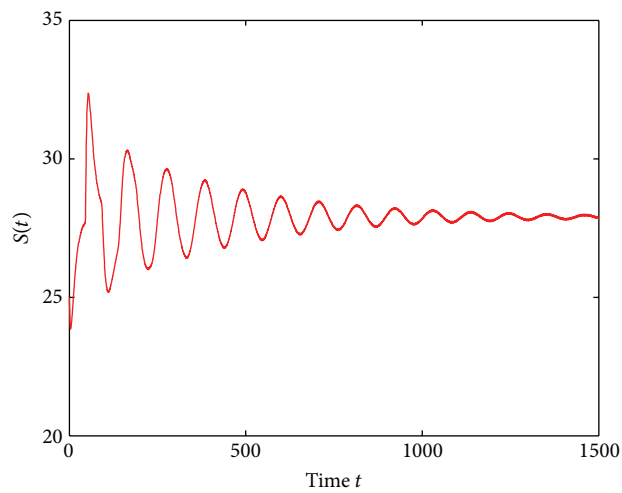
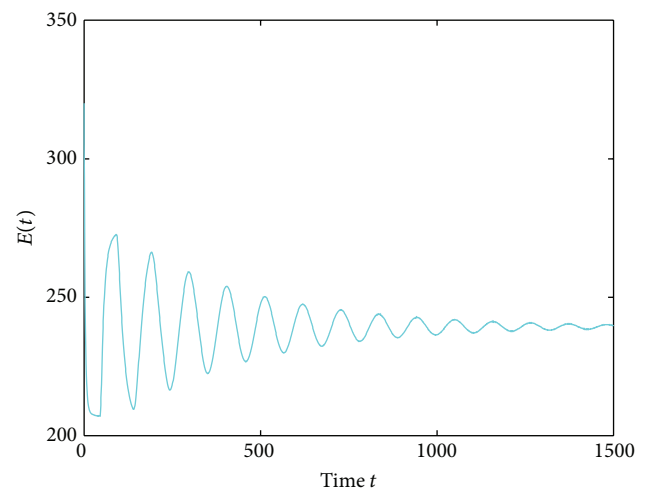


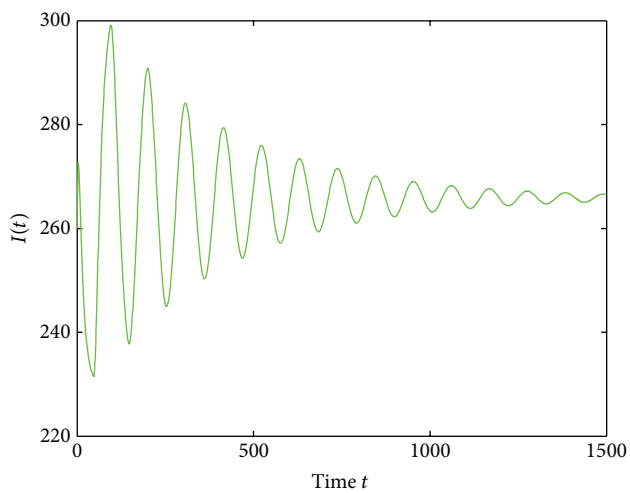
FIGURE 4: The phase plot of the states S , E , and R for $\tau_1 = 20.0500 > 17.2530 = \tau_{10}$.



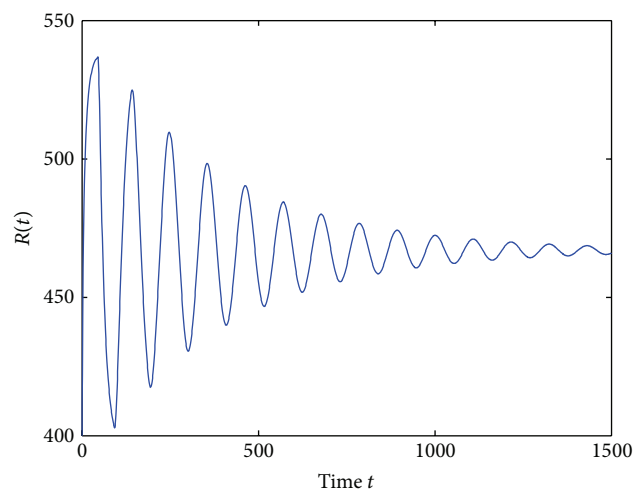
(a)



(b)



(c)



(d)

FIGURE 5: The track of the states S , E , I , and R for $\tau_2 = 45.5000 < 52.4927 = \tau_{20}$.

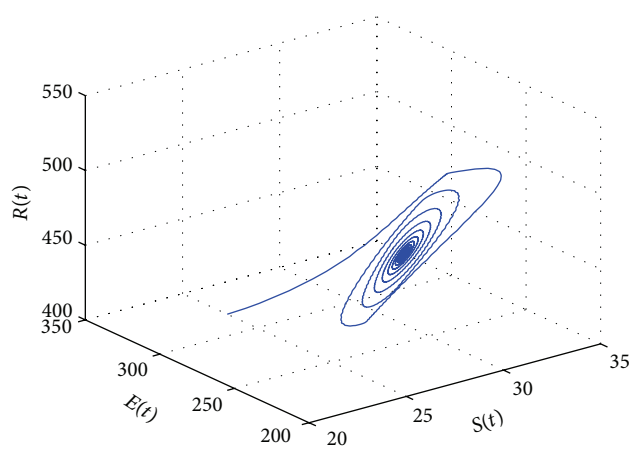


FIGURE 6: The phase plot of the states S , E , and R for $\tau_2 = 45.5000 < 52.4927 = \tau_{20}$.

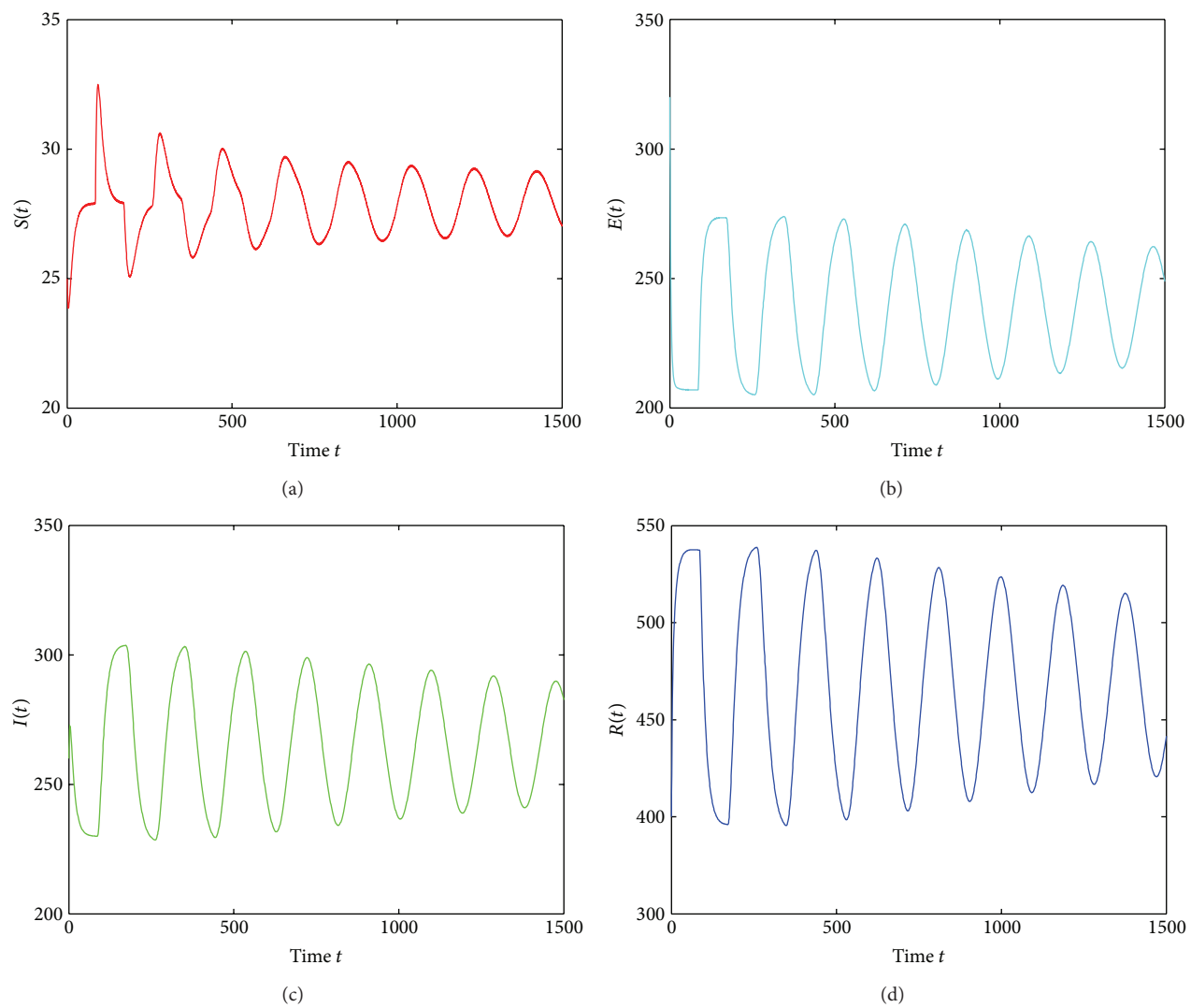


FIGURE 7: The track of the states S , E , I , and R for $\tau_2 = 67.5000 > 52.4927 = \tau_{20}$.

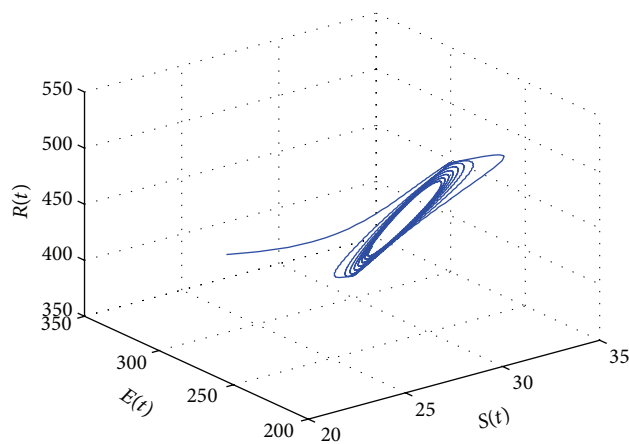


FIGURE 8: The phase plot of the states S , E , and R for $\tau_2 = 67.5000 > 52.4927 = \tau_{20}$.

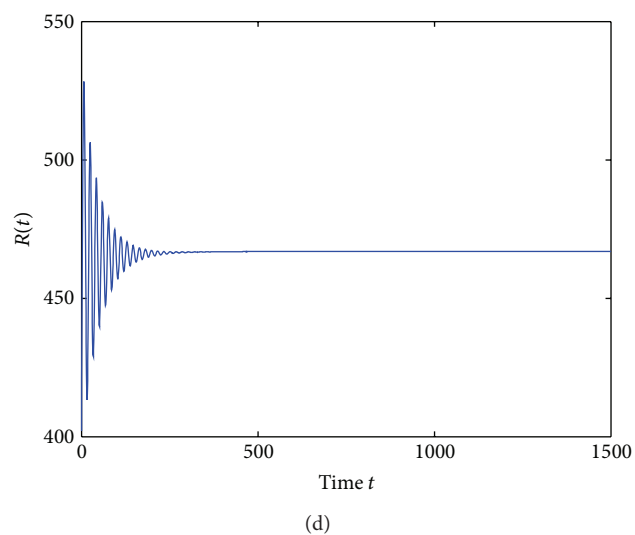
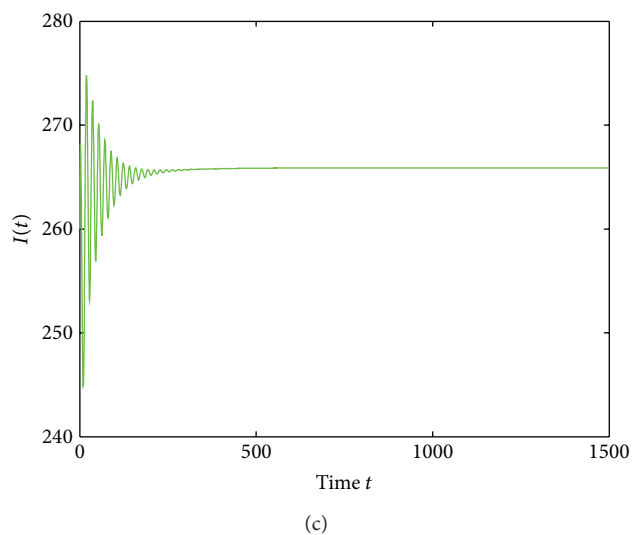
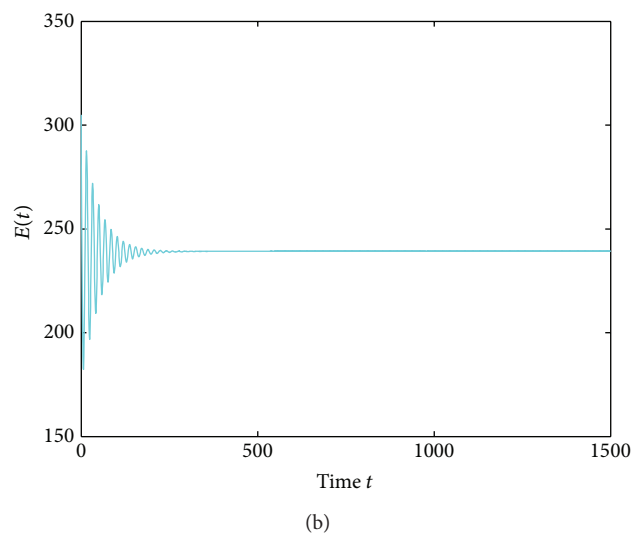
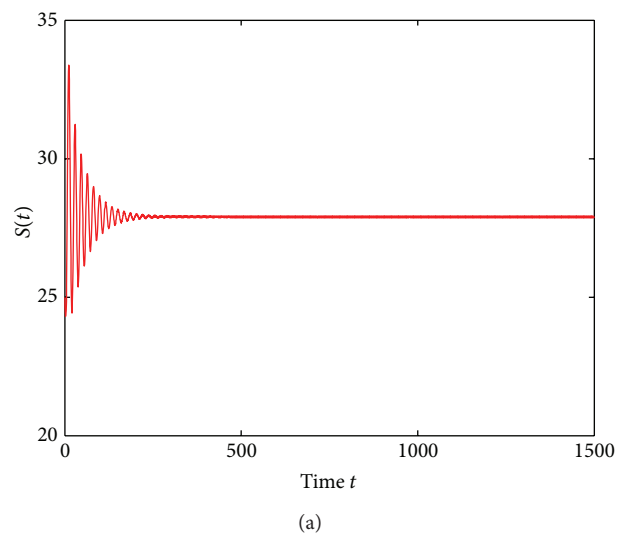


FIGURE 9: The track of the states S , E , I , and R for $\tau = 4.7500 < 5.1094 = \tau_0$.

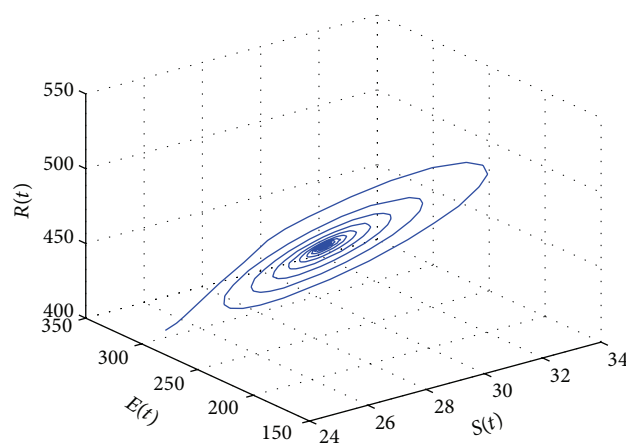


FIGURE 10: The phase plot of the states S , E , and R for $\tau = 4.7500 < 5.1094 = \tau_0$.

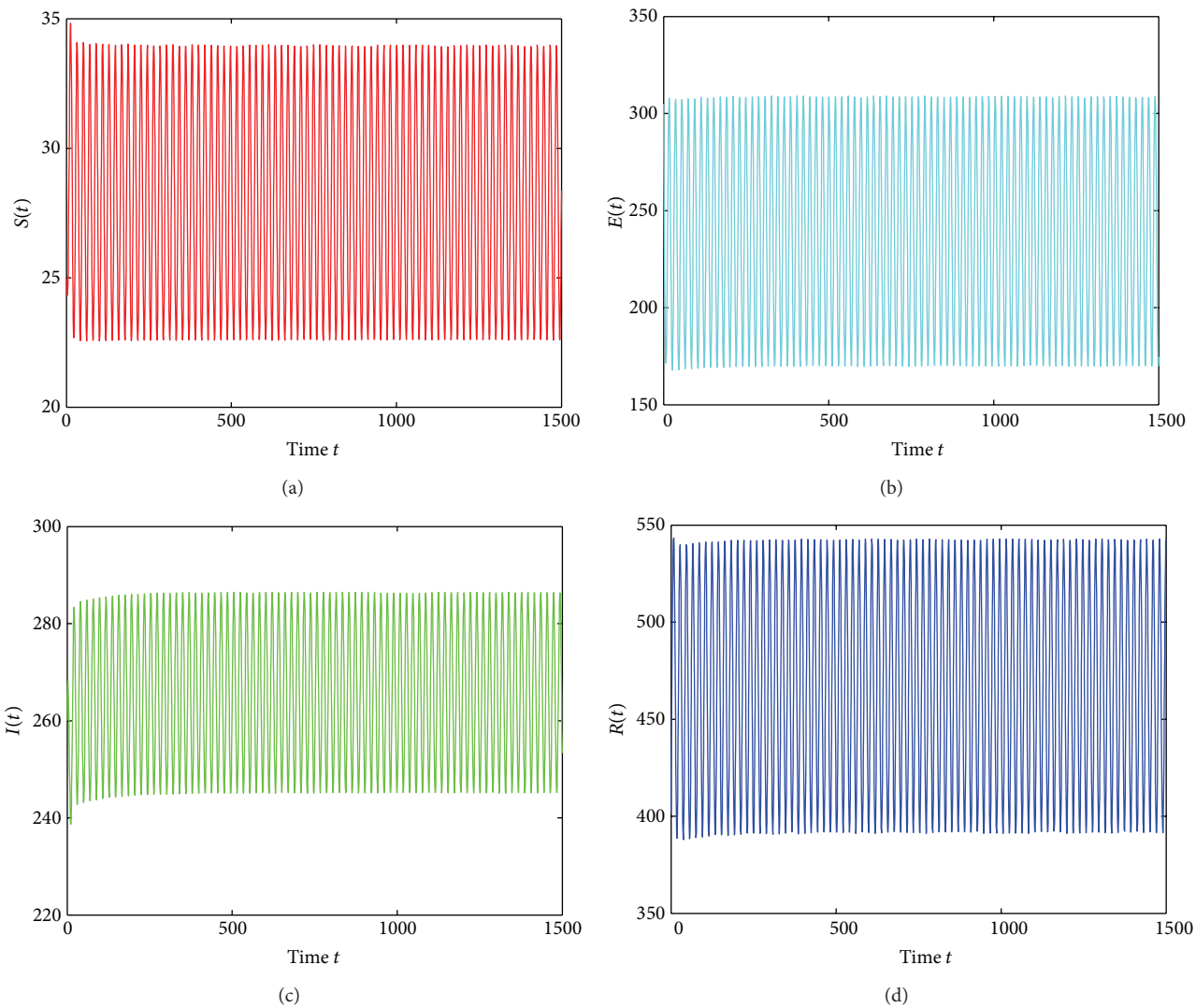


FIGURE 11: The track of the states S , E , I , and R for $\tau = 5.2300 > 5.1094 = \tau_0$.

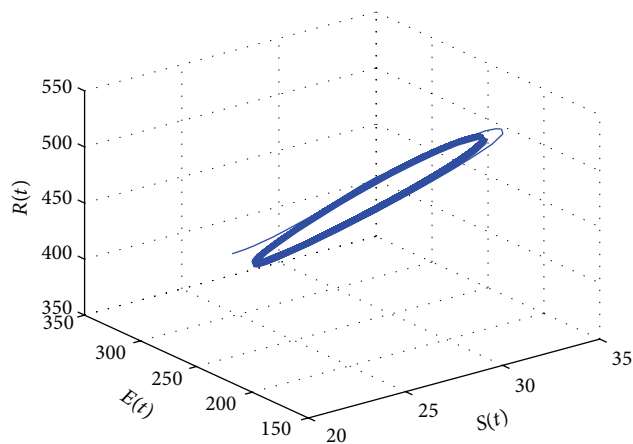
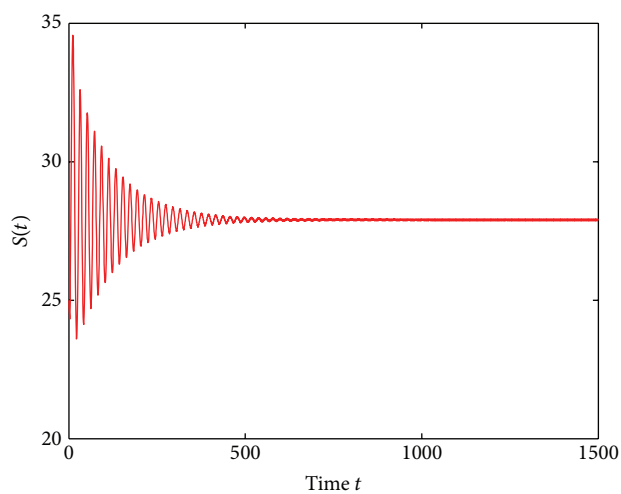
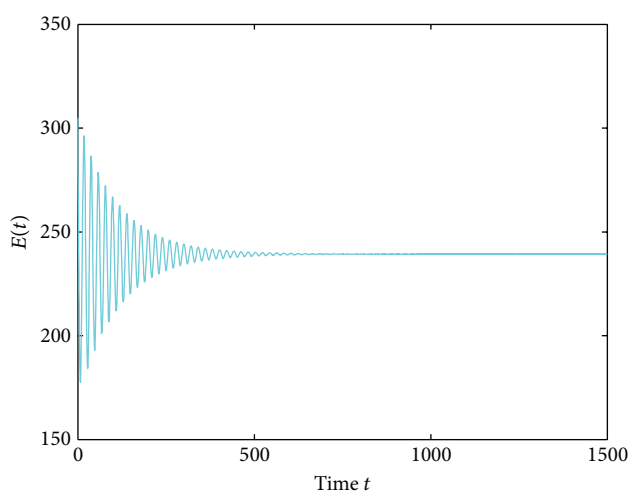


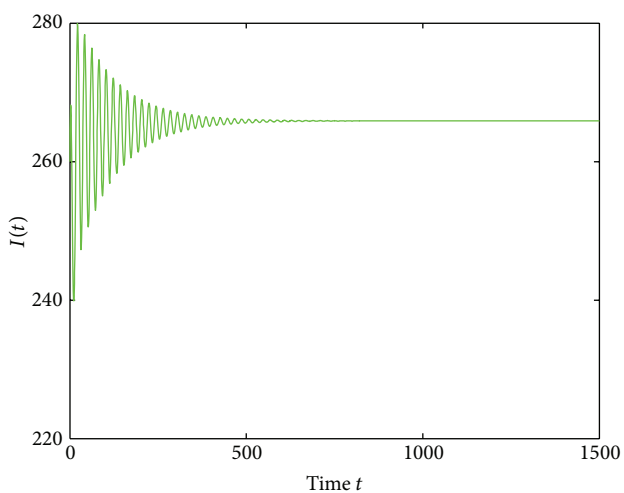
FIGURE 12: The phase plot of the states S , E , and R for $\tau = 5.2300 > 5.1094 = \tau_0$.



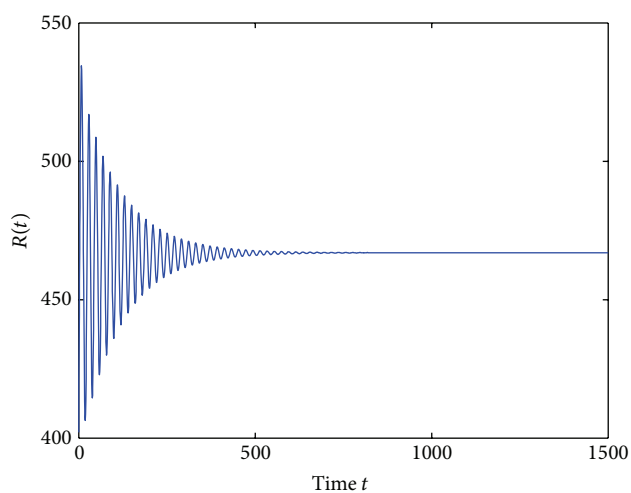
(a)



(b)



(c)



(d)

FIGURE 13: The track of the states S , E , I , and R for $\tau_1 = 7.2000 < 7.8533 = \tau_1^*$ and $\tau_2^* = 3$.

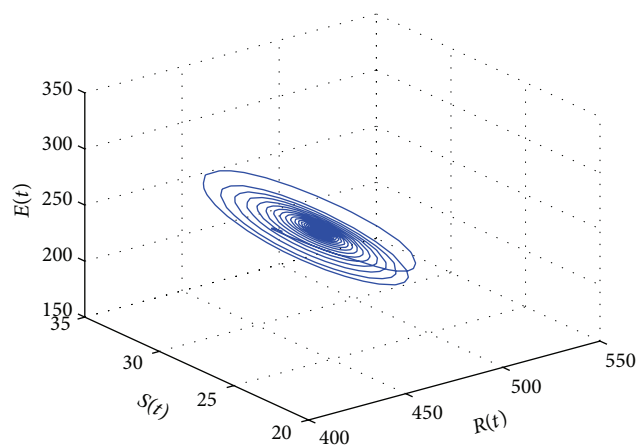


FIGURE 14: The phase plot of the states R , S , and E for $\tau_1 = 7.2000 < 7.8533 = \tau_1^*$ and $\tau_2^* = 3$.

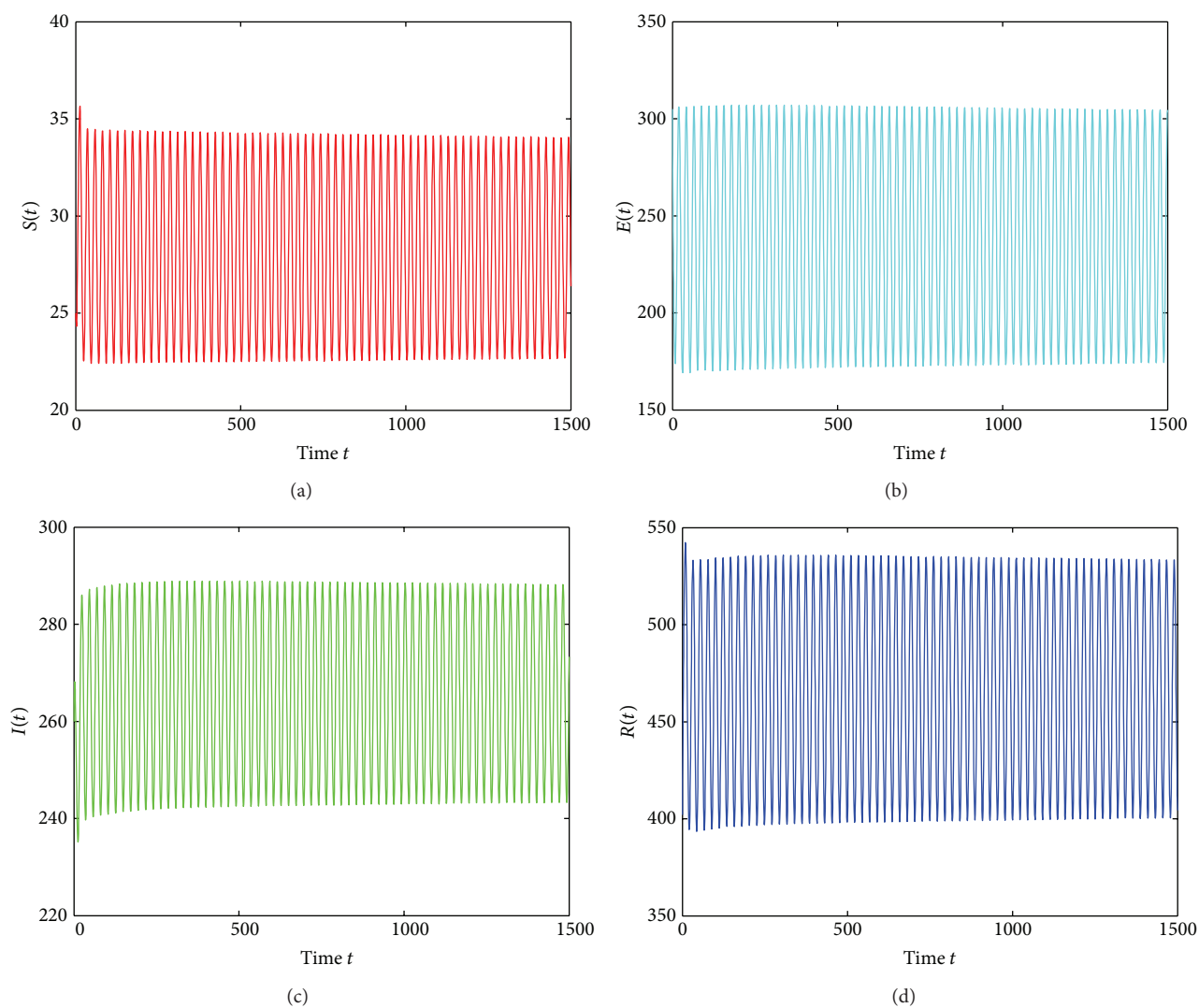


FIGURE 15: The track of the states S , E , I , and R for $\tau_1 = 8.4500 > 7.8533 = \tau_1^*$ and $\tau_2^* = 3$.

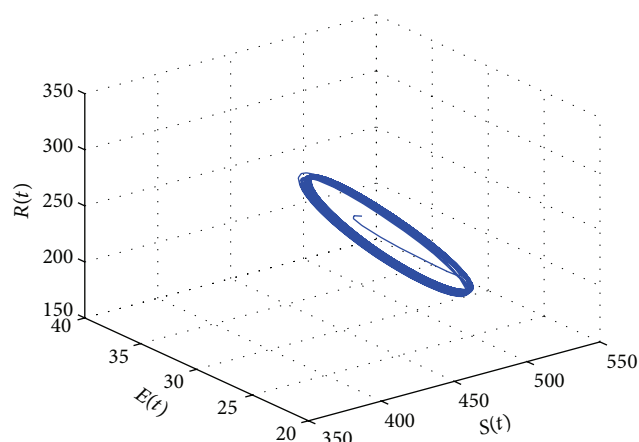


FIGURE 16: The phase plot of the states R , S , and E for $\tau_1 = 8.4500 > 7.8533 = \tau_1^*$ and $\tau_2^* = 3$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

Acknowledgment

This work was supported by the National Natural Science Foundation of China (61273070), a project funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions.

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