

Research Article

A New Method Based on the RKHSM for Solving Systems of Nonlinear IDDEs with Proportional Delays

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An efficient computational method is given in order to solve the systems of nonlinear infinite-delay-differential equations (IDDEs) with proportional delays. Representation of the solution and an iterative method are established in the reproducing kernel space. Some examples are displayed to demonstrate the computation efficiency of the method.

1. Introduction

In functional-differential equations (FDEs), there is a class of infinite-delay-differential equations (IDDEs) with proportional delays such systems are often encountered in many scientific fields such as electric mechanics, quantum mechanics, and optics. In view of this, developing the research for this class of IDDEs possesses great significance on theory and practice, for this attracts constant interest of researchers. Ones have found that there exist very different mathematical challenges between FDEs with proportional delays and those with constant delays. Some researches on the numerical solutions and the corresponding analysis for the linear FDEs with proportional delays have been presented by several authors. In the last few years, there has been a growing interest in studying the existence of solutions of functional differential equations with state dependent delay [1–7]. Initial-value problem for neutral functional-differential equations with proportional time delays had been studied in [8–11]; in the literature [11] authors had discussed the existence and uniqueness of analytic solution of linear proportional delays equations.

Ishiwata et al. used the rational approximation method and the collocation method to compute numerical solutions of delay-differential equations with proportional delays in [12, 13]. At [14–17], Hu et al. gave the numerical method to compute numerical solutions of neutral delay differential equations. For neutral delay differential equations

with proportional delays, Chen and Wang proposed the variational iteration method [18] and the homotopy perturbation method [19]. Recently, time-delay systems become interested in applications like population growth models, transportation, communications, and agricultural models so those systems were widely studied both in a theoretical aspect and in that of related applications [20–22].

We consider the following nonlinear infinite delay-differential equation (IDDE) with proportional delay [23]:

$$\begin{aligned}u'(x) &= g(x, u(x), u(px)), \\u(0) &= \eta,\end{aligned}\tag{1}$$

where $p \in (0, 1)$, η is a given initial value, $u(x) \in W_2[0, +\infty)$, and for $x \in [0, +\infty)$, $y, z \in (-\infty, +\infty)$, $g(x, y, z)$ is a continuous function; $g(x, y, z) \in W_1[0, +\infty)$ as $y = y(x)$, $z = z(x) \in W_1[0, +\infty)$.

Next, the following system of nonlinear infinite-delay-differential equations (IDDEs) with proportional delays will be studied:

$$\begin{aligned}u'(x) &= F(x, u(x), v(px)), \\v'(x) &= G(x, v(x), u(qx)), \\u(0) &= 0, \quad v(0) = 0,\end{aligned}\tag{2}$$

where $p, q \in (0, 1)$, for $x \in [0, +\infty)$, $y, z \in (-\infty, +\infty)$, $f(x, y, z)$, $g(x, y, z)$ are continuous bounded function, and

$f(x, y, z) \in W_1[0, +\infty)$, $g(x, y, z) \in W_1[0, +\infty)$ as $y = y(x)$, $z = z(x)$. $W_2[0, +\infty)$ and $W_1[0, \infty)$ are reproducing kernel spaces. Equations (2) are obtained through homogenization of initial condition for model in [24]. In this study, existence and a new iterative algorithm are established for the nonlinear infinite-delay-differential equations (IDDEs) with proportional delays in the reproducing kernel spaces.

The paper is organized as follows. In Section 2, some definitions of the reproducing kernel spaces are introduced. In Section 3, main results and the structure of the solution for operator equation are discussed. Existence of the solution to (2) and an iterative method are developed for the kind of problems in the reproducing kernel space. We verify that the approximate solution converges to the exact solution uniformly. In Section 4, some experiments are given to demonstrate the computation efficiency of the algorithm. The conclusion is given in Section 5.

2. Preliminaries

Definition 1 (see [25] (reproducing kernel)). Let M be a nonempty abstract set. A function $K : M \times M \rightarrow C$ is a reproducing kernel of the Hilbert space H if and only if

- (a) $\forall y \in M, K(\cdot, x) \in H,$
- (b) $\forall y \in M, \forall u \in H, \langle u(\cdot), K(\cdot, y) \rangle = u(y).$

The condition (b) is called “the reproducing property”; a Hilbert space which possesses a reproducing kernel is called a Reproducing Kernel Hilbert Space (RKHS).

Next, two reproducing kernel spaces are given.

Definition 2. $W_2[0, +\infty) = \{u(x) \mid u' \text{ is absolutely continuous real value functions, } u'' \in L^2[0, +\infty), u(0) = 0\}.$

$W_2[0, +\infty)$ is a Hilbert space, for $u(x), v(x) \in W_2[0, +\infty)$; the inner product and norm in $W_2[0, +\infty)$ are given by

$$\langle u(x), v(x) \rangle_{W_2} = \int_0^{+\infty} (4uv + 5u'v' + u''v'') dx, \tag{3}$$

$$\|u\|_{W_2} = \sqrt{\langle u, u \rangle_{W_2}},$$

respectively.

Theorem 3. *The space $W_2[0, +\infty)$ is a reproducing kernel space that is, for any $u(y) \in W_2[0, +\infty)$ and each fixed $x \in [0, +\infty)$, there exists $R_x(y) \in W_2[0, +\infty)$, $y \in [0, +\infty)$, such that $u(x) = \langle R_x(y), u(y) \rangle$. And the corresponding reproducing kernel can be represented as follows [26]:*

$$R_x(y) = \begin{cases} -\frac{1}{12}e^{-2(x+y)}(-1+e^{2y})(1+e^{2y}-2e^{x+y}), & y \leq x, \\ -\frac{1}{12}e^{-2(x+y)}(-1+e^{2x})(1+e^{2x}-2e^{x+y}), & y > x. \end{cases} \tag{4}$$

Definition 4. $W_1[0, +\infty) = \{u(x) \mid u \text{ is absolutely continuous real-valued function, } u' \in L^2[0, +\infty)\}.$

The inner product and norm in $W_1[0, +\infty)$ can be defined by

$$\langle u(x), v(x) \rangle_{W_1} = \int_0^{+\infty} (uv + u'v') dx, \tag{5}$$

$$\|u\|_{W_1} = \sqrt{\langle u, u \rangle_{W_1}},$$

respectively, where $u(x), v(x) \in W_1[0, +\infty)$. It has been proved that $W_1[0, +\infty)$ is a complete reproducing kernel space and its reproducing kernel is as follows [27]:

$$Q_x(y) = \begin{cases} \frac{1}{2}e^{-x-y}(1+e^{2x}), & x < y, \\ \frac{1}{2}e^{-x-y}(1+e^{2y}), & x \geq y. \end{cases} \tag{6}$$

3. Statements of the Main Results

In this section, the implementation method of obtaining the solution of (2) is proposed in the reproducing kernel space $W_2[0, +\infty)$.

Put the differential operator $\mathcal{L} = d/dx$; then we can convert (2) into the following form:

$$\begin{aligned} \mathcal{L}u(x) &= F(x, u(x), v(px)), \\ \mathcal{L}v(x) &= G(x, v(x), u(qx)), \\ u(0) &= v(0) = 0, \end{aligned} \tag{7}$$

where $x \in [0, +\infty)$, and

$$\begin{aligned} F(x, u(x), v(px)) &\stackrel{\text{def}}{=} N(u(x), v(px)) + f(x), \\ G(x, v(x), u(qx)) &\stackrel{\text{def}}{=} D(v(x), u(qx)) + g(x). \end{aligned} \tag{8}$$

It is clear that $\mathcal{L} : W_2[0, +\infty) \rightarrow W_1[0, +\infty)$ is a bounded linear operator. Let $\varphi_i(x) = Q_{x_i}(x)$, where $\{x_i\}_{i=1}^{\infty}$ is dense in the interval $[0, +\infty)$, and $\psi_i(x) = \mathcal{L}^* \varphi_i(x)$, where \mathcal{L}^* is the conjugate operator of \mathcal{L} . Define the normal orthogonal system $\bar{\psi}_i(x)$ in $W_2[0, +\infty)$, which derives from Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$,

$$\bar{\psi}_i(x) = \sum_{k=1}^i \beta_{ik} \psi_k(x), \quad \beta_{ii} > 0, \quad i = 1, 2, \dots \tag{10}$$

Lemma 5. *Assume that $\{x_i\}_{i=1}^{\infty}$ are dense in $[0, +\infty)$; then $\{\psi_i(x)\}_{i=1}^{\infty}$ is a complete system in $W_2[0, +\infty)$ and $\{\psi_i(x)\} = (d/dy)R_x(y)|_{y=x_i}$.*

Proof. Ones have

$$\begin{aligned} \psi_i(x) &= \langle \psi_i(y), R_x(y) \rangle_{W_2} = \langle \mathcal{L}^* \varphi_i(y), R_x(y) \rangle_{W_2} \\ &= \langle \varphi_i(y), \mathcal{L}R_x(y) \rangle_{W_1} = \mathcal{L}R_x(x_i) \\ &= \left. \frac{d}{dy}R_x(y) \right|_{y=x_i}. \end{aligned} \tag{11}$$

Clearly, $\psi_i(x) \in W_2[0, +\infty)$.

For any $u(x) \in W_2[0, +\infty)$, let $\langle u(x), \psi_i(x) \rangle = 0, i = 1, 2, \dots$, which means that

$$\langle u(x), (\mathcal{L}^* \varphi_i(x)) \rangle = \langle \mathcal{L}u(\cdot), \varphi_i(\cdot) \rangle = (\mathcal{L}u)(x_i) = 0. \tag{12}$$

Note that $\{x_i\}_{i=1}^\infty$ is dense in $[0, +\infty)$; therefore, $\mathcal{L}u(x) = 0$. It follows that $u(x) \equiv 0$ from the existence of \mathcal{L}^{-1} . \square

3.1. Construction the of Iterative Sequence $\tilde{u}_n(x)$ and $\tilde{v}_n(x)$. Next we construct the iterative sequence $\tilde{u}_n(x)$ and $\tilde{v}_n(x)$, putting

$$\begin{aligned} \mathcal{L}M_n(x) &= F(x, \tilde{u}_{n-1}(x), \tilde{v}_{n-1}(px)), \\ \tilde{u}_n(x) &= P_n M_n(x), \\ \mathcal{L}N_n(x) &= G(x, \tilde{v}_{n-1}(x), \tilde{u}_{n-1}(qx)), \\ \tilde{v}_n(x) &= P_n N_n(x), \end{aligned} \tag{13}$$

where $M_n(x), N_n(x) \in W_2[0, +\infty)$ and $P_n : W_2 \rightarrow \text{Span}\{\bar{\psi}_1, \bar{\psi}_2, \dots, \bar{\psi}_n\}$ is a orthogonal projection operator. Then by (13) it followed that:

$$\begin{aligned} M_n(x) &= \int_0^x F(\mu, \tilde{u}_{n-1}(\mu), \tilde{v}_{n-1}(p\mu)) d\mu, \\ M'_n(x) &= F(x, \tilde{u}_{n-1}(x), \tilde{v}_{n-1}(px)), \\ M''_n(x) &= \frac{d}{dx} F(x, \tilde{u}_{n-1}(x), \tilde{v}_{n-1}(px)), \\ N_n(x) &= \int_0^x G(\mu, \tilde{v}_{n-1}(\mu), \tilde{u}_{n-1}(q\mu)) d\mu, \\ N'_n(x) &= G(x, \tilde{v}_{n-1}(x), \tilde{u}_{n-1}(qx)), \\ N''_n(x) &= \frac{d}{dx} G(x, \tilde{v}_{n-1}(x), \tilde{u}_{n-1}(qx)). \end{aligned} \tag{15}$$

Lemma 6. Let $\{x_i\}_{i=1}^\infty$ be dense on $[0, +\infty)$; if the solution of (2) is unique, then the solution satisfies the form

$$\begin{aligned} M_n(x) &= \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} F(x_k, \tilde{u}_{n-1}(x_k), \tilde{v}_{n-1}(px_k)) \bar{\psi}_i(x), \\ N_n(x) &= \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} G(x_k, \tilde{v}_{n-1}(x_k), \tilde{u}_{n-1}(qx_k)) \bar{\psi}_i(x). \end{aligned} \tag{16}$$

Proof. Note that $\langle u(x), \varphi_i(x) \rangle = u(x_i)$ and $\{\bar{\psi}_i(x)\}_{i=1}^\infty$ is an orthonormal basis of $W_2[0, +\infty)$; hence according to Lemma 5 we have

$$\begin{aligned} M_n(x) &= \sum_{i=1}^\infty \langle M_n(x), \bar{\psi}_i(x) \rangle \bar{\psi}_i(x) \\ &= \sum_{i=1}^\infty \left\langle M_n(x), \sum_{k=1}^i \beta_{ik} \psi_k(x) \right\rangle \bar{\psi}_i(x) \\ &= \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} \langle M_n(x), \psi_k(x) \rangle \bar{\psi}_i(x) \\ &= \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} \langle \mathcal{L}M_n(x), \varphi_k(x) \rangle \bar{\psi}_i(x) \\ &= \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} F(x_k, u_{n-1}(x_k), v_{n-1}(px_k)) \bar{\psi}_i(x). \end{aligned} \tag{17}$$

In the same manner

$$N_n(x) = \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} G(x_k, v_{n-1}(x_k), u_{n-1}(qx_k)) \bar{\psi}_i(x). \tag{18}$$

\square

Take $u_0(x) = v_0(x) = 0$; define the iterative sequence

$$\begin{aligned} u_n(x) &= P_n M_n(x) \\ &= \sum_{i=1}^n \sum_{k=1}^i \beta_{ik} F(x_k, u_{n-1}(x_k), v_{n-1}(px_k)) \bar{\psi}_i(x), \\ v_n(x) &= P_n N_n(x) \\ &= \sum_{i=1}^n \sum_{k=1}^i \beta_{ik} G(x_k, v_{n-1}(x_k), u_{n-1}(qx_k)) \bar{\psi}_i(x). \end{aligned} \tag{19}$$

3.2. The Boundedness of Sequence $u_n(x)$ and $v_n(x)$. From

$$\partial_y^3 R_x(x+0) - \partial_y^3 R_x(x-0) = 1, \tag{20}$$

we have

$$\begin{aligned} \partial_y^4 R_x(y) &= g_x(y) + (\partial_y^3 R_x(x+0) - \partial_y^3 R_x(x-0)) \\ &\quad \times \delta(x-y) \\ &= g_x(y) + \delta(x-y), \end{aligned} \tag{21}$$

where

$$g_x(y) = \begin{cases} g_x^1, & y < x, \\ g_x^2, & y > x, \\ 0, & y = x, \end{cases} \tag{22}$$

$g_x^1(y) = \partial_y^4(a_1 e^y + a_2 e^{-y} + a_3 e^{2y} + a_4 e^{-2y})$; $g_x^2(y) = \partial_y^4(b_1 e^y + b_2 e^{-y} + b_3 e^{2y} + b_4 e^{-2y})$; and $g_x^1(y), g_x^2(y), \partial_y g_x^1(y), \partial_y g_x^2(y)$ are bounded functions with respect to x, y , respectively.

Lemma 7. For $x \in [0, +\infty)$, $y, z \in (-\infty, +\infty)$, and $F(x, y, z)$, $G(x, y, z)$ are continuous bounded functions on $[0, +\infty)$, we have that $M_n^{(i)}(x)$ and $N_n^{(i)}(x)$ ($i = 0, 1, 2$) are bounded.

Proof. By the expression of $M_n(x)$, $M_n'(x)$ and the assumptions, we know that $M_n(x)$ and $M_n'(x)$ are bounded. In the following, we will discuss the boundedness of $M_n''(x)$.

Since the function which is in $C_{[0,+\infty)}^2$ is dense in $W_2[0, +\infty)$, without loss of generality, we assume that $M_n''(x)$ is continuous.

Note that

$$4R_x(y) - 5\partial_y^2 R_x(y) + \partial_y^4 R_x(y) = \delta(x - y). \tag{23}$$

One gets

$$\begin{aligned} M_n''(x) &= \int_0^\infty M_n''(y) \delta(x - y) dy \\ &= \int_0^\infty M_n''(y) [4R_x(y) - 5\partial_y^2 R_x(y) + \partial_y^4 R_x(y)] dy \\ &= \int_0^\infty M_n''(y) [4R_x(y) - 5\partial_y^2 R_x(y)] dy \\ &\quad + \int_0^\infty M_n''(y) \partial_y^4 R_x(y) dy \\ &= M_n'(y) [4R_x(y) - 5\partial_y^2 R_x(y)] \Big|_0^\infty \\ &\quad - \int_0^{+\infty} M_n'(y) [4\partial_y R_x(y) - 5\partial_y^3 R_x(y)] dy \\ &\quad + \int_0^\infty M_n''(y) [g_x(y) + \delta(x - y)] dy \\ &= m(x) + \int_0^\infty M_n''(y) [g_x(y) + \delta(x - y)] dy \\ &= m(x) + \int_0^\infty M_n''(y) g_x(y) dy + M_n''(x), \end{aligned} \tag{24}$$

where $m(x) = M_n'(y)[4R_x(y) - 5\partial_y^2 R_x(y)] \Big|_0^\infty - \int_0^\infty M_n'(y)[4\partial_y R_x(y) - 5\partial_y^3 R_x(y)] dy$. Thus, we have

$$\begin{aligned} 0 &= m(x) - \int_0^\infty M_n''(y) g_x(y) dy \\ &= m(x) - \int_0^x M_n''(y) g_x^1(y) dy - \int_x^\infty M_n''(y) g_x^2(y) dy \\ &= m(x) - M_n'(y) g_x^1(y) \Big|_0^x + \int_0^x M_n'(y) \partial_y g_x^1(y) dy \\ &\quad - M_n'(y) g_x^2(y) \Big|_x^M + \int_x^\infty M_n'(y) \partial_y g_x^2(y) dy \end{aligned}$$

$$\begin{aligned} &= m(x) - M_n'(x) (g_x^1(x) - g_x^2(x)) \\ &\quad + (M_n'(0) g_x^1(0) - M_n'(M) g_x^2(M)) \\ &\quad + \int_0^x M_n'(y) \partial_y g_x^1(y) dy \\ &\quad + \int_x^\infty M_n'(y) \partial_y g_x^2(y) dy. \end{aligned} \tag{25}$$

So

$$\begin{aligned} M_n'(x) (g_x^1(x) - g_x^2(x)) &= m(x) + (M_n'(0) g_x^1(0) - M_n'(M) g_x^2(M)) \\ &\quad + \int_0^x M_n'(y) \partial_y g_x^1(y) dy \\ &\quad + \int_x^\infty M_n'(y) \partial_y g_x^2(y) dy. \end{aligned} \tag{26}$$

Furthermore, we see that

$$\begin{aligned} M_n''(x) &= \frac{1}{g_x^1(x) - g_x^2(x)} \\ &\quad \times \left\{ \partial_x \left[m(x) + (M_n'(0) g_x^1(0) - M_n'(M) g_x^2(M)) \right. \right. \\ &\quad \left. \left. + \int_0^x M_n'(y) \partial_y g_x^1(y) \right. \right. \\ &\quad \left. \left. + \int_x^\infty M_n'(y) \partial_y g_x^2(y) dy \right] \right. \\ &\quad \left. - M_n'(x) \partial_x (g_x^1(x) - g_x^2(x)) \right\}. \end{aligned} \tag{27}$$

In view of the expression of $R_x(y)$, we know that $\partial_y^m R_x(y)$ are bounded for $m = 1, 2, 3$. It follows that $M_n''(x)$ is bounded from the boundedness of $g_x^1(y)$, $g_x^2(y)$, $\partial_y g_x^1(y)$, $\partial_y g_x^2(y)$, and $M_n'(x)$. In the same way, $N_n(x)$, $N_n'(x)$, and $N_n''(x)$ are bounded. \square

Lemma 8. Assume that $F(x, y, z)$, $G(x, y, z)$ are continuous bounded functions for $x \in [0, +\infty)$, $y, z \in (-\infty, +\infty)$, and $F(x, y, z), G(x, y, z) \in W_1[0, +\infty)$ as $y = y(x) \in W_2, z = z(x) \in W_2$, then

$$\|u_n\|_{W_2} \leq C, \quad \|v_n\|_{W_2} \leq D, \quad (C, D \text{ are constants}). \tag{28}$$

Proof.

$$\|M_n\|_{W_2}^2 = \langle M_n, M_n \rangle = \int_0^\infty (4M_n^2 + 5M_n'^2 + M_n''^2) dx. \tag{29}$$

Note that (14) and the assumptions, by Lemma 7, $|M_n^{(i)}(x)| \leq C^i$, $i = 0, 1, 2$, thus $\|M_n\| \leq C$, where $C = \max\{C^0, C^1, C^2\}$, from (19), we have $\|u_n\|_{W_2} \leq \|M_n\|_{W_2} \leq C$. In the same way, we obtain $\|v_n\|_{W_2} \leq D$. \square

3.3. Construction the of Another Iterative Sequence
 $u_n(x)$ and $v_n(x)$

Theorem 9. *Let $\{x_i\}_{i=1}^\infty$ be dense on $[0, +\infty)$; if the solution of (2) exists and unique, then the solution satisfies the form*

$$\begin{aligned} u(x) &= \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} F(x_k, u(x_k), v(px_k)) \bar{\psi}_i(x), \\ v(x) &= \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} G(x_k, v(x_k), u(qx_k)) \bar{\psi}_i(x). \end{aligned} \tag{30}$$

Proof. Note that $\langle u(x), \varphi_i(x) \rangle = u(x_i)$ and $\{\bar{\psi}_i(x)\}_{i=1}^\infty$ is an orthonormal basis of $W_2[0, +\infty)$; hence we have

$$\begin{aligned} u(x) &= \sum_{i=1}^\infty \langle u(x), \bar{\psi}_i(x) \rangle \bar{\psi}_i(x) \\ &= \sum_{i=1}^\infty \left\langle u(x), \sum_{k=1}^i \beta_{ik} \psi_k(x) \right\rangle \bar{\psi}_i(x) \\ &= \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} \langle u(x), \psi_k(x) \rangle \bar{\psi}_i(x) \\ &= \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} \langle \mathcal{L}u(x), \varphi_k(x) \rangle \bar{\psi}_i(x) \\ &= \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} F(x_k, u(x_k), v(px_k)) \bar{\psi}_i(x). \end{aligned} \tag{31}$$

In the same manner

$$v(x) = \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} G(x_k, v(x_k), u(qx_k)) \bar{\psi}_i(x). \tag{32}$$

\square

In the following, a new method of solving (2) is presented. Equations (31) and (32) can be denoted by

$$u(x) = \sum_{i=1}^\infty A_i \bar{\psi}_i(x), \tag{33}$$

$$v(x) = \sum_{i=1}^\infty B_i \bar{\psi}_i(x), \tag{34}$$

where $A_i = \sum_{k=1}^i \beta_{ik} F(x_k, u(x_k), v(px_k))$ and $B_i = \sum_{k=1}^i \beta_{ik} G(x_k, v(x_k), u(qx_k))$. In fact, A_i and B_i are unknown; we will approximate A_i and B_i by using the known \bar{A}_i and \bar{B}_i .

We take $u_1(x) = 0, v_1(x) = 0$ and define the following iterative sequence

$$\begin{aligned} u_n(x) &= \sum_{i=1}^{n-1} \bar{A}_i \bar{\psi}_i(x), \\ v_n(x) &= \sum_{i=1}^{n-1} \bar{B}_i \bar{\psi}_i(x), \end{aligned} \tag{35}$$

where

$$\begin{aligned} \bar{A}_1 &= \beta_{11} F(x_1, u_1(x_1), v_1(px_1)) \\ \bar{A}_2 &= \sum_{k=1}^2 \beta_{2k} F(x_k, u_k(x_k), v_k(px_k)) \\ &\vdots \\ \bar{A}_{n-1} &= \sum_{k=1}^{n-1} \beta_{nk} F(x_k, u_k(x_k), v_k(px_k)), \\ \bar{B}_1 &= \beta_{11} G(x_1, v_1(x_1), u_1(qx_1)) \\ \bar{B}_2 &= \sum_{k=1}^2 \beta_{2k} G(x_k, v_k(x_k), u_k(qx_k)) \\ &\vdots \\ \bar{B}_{n-1} &= \sum_{k=1}^{n-1} \beta_{nk} G(x_k, v_k(x_k), u_k(qx_k)). \end{aligned} \tag{36}$$

Next, lemmas are given.

Lemma 10. *The following iterative sequences*

$$\begin{aligned} u_n(x) &= \sum_{i=1}^{n-1} \beta_{ik} F(x_k, u_k(x_k), v_k(px_k)) \bar{\psi}_i(x), \\ v_n(x) &= \sum_{i=1}^{n-1} \beta_{ik} G(x_k, v_k(x_k), u_k(qx_k)) \bar{\psi}_i(x) \end{aligned} \tag{38}$$

satisfy

$$\begin{aligned} \mathcal{L}u_n(x_j) &= F(x_j, u_j(x_j), v_j(px_j)), \\ \mathcal{L}v_n(x_j) &= G(x_j, v_j(x_j), u_j(qx_j)), \quad j \leq n-1, \end{aligned} \tag{39}$$

respectively.

Proof. If $j = 1$,

$$\begin{aligned} \mathcal{L}u_n(x_1) &= \sum_{i=1}^{n-1} \beta_{ik} F(x_k, u_k(x_k), v_k(px_k)) \mathcal{L}\bar{\psi}_i(x_1) \\ &= \sum_{i=1}^{n-1} \beta_{ik} F(x_k, u_k(x_k), v_k(px_k)) \\ &\quad \times \langle \mathcal{L}\bar{\psi}_i(x), \varphi_1(x) \rangle \\ &= \sum_{i=1}^{n-1} \beta_{ik} F(x_k, u_k(x_k), v_k(px_k)) \\ &\quad \times \langle \bar{\psi}_i(x), \psi_1(x) \rangle, \\ \beta_{11} \mathcal{L}u_n(x_1) &= \sum_{i=1}^{n-1} \beta_{ik} F(x_k, u_k(x_k), v_k(px_k)) \\ &\quad \times \langle \bar{\psi}_i(x), \bar{\psi}_1(x) \rangle \\ &= \beta_{11} F(x_1, u_1(x_1), v_1(px_1)), \end{aligned} \quad (40)$$

so,

$$\mathcal{L}u_n(x_1) = F(x_1, u_1(x_1), v_1(px_1)). \quad (41)$$

If $j = 2$,

$$\begin{aligned} \mathcal{L}u_n(x_2) &= \sum_{i=1}^{n-1} \beta_{ik} F(x_k, u_k(x_k), v_k(px_k)) \mathcal{L}\bar{\psi}_i(x_2) \\ &= \sum_{i=1}^{n-1} \beta_{ik} F(x_k, u_k(x_k), v_k(px_k)) \langle \bar{\psi}_i(x), \psi_2(x) \rangle, \end{aligned} \quad (42)$$

$\beta_{21} \times (40) + \beta_{22} \times (42)$, we have

$$\begin{aligned} &\beta_{21} \mathcal{L}u_n(x_1) + \beta_{22} \mathcal{L}u_n(x_2) \\ &= \sum_{i=1}^{n-1} \beta_{ik} F(x_k, u_k(x_k), v_k(px_k)) \langle \bar{\psi}_i(x), \bar{\psi}_2(x) \rangle \\ &= \beta_{21} F(x_1, u_1(x_1), v_1(px_1)) \\ &\quad + \beta_{22} F(x_2, u_2(x_2), v_2(px_2)), \end{aligned} \quad (43)$$

by (41), $\mathcal{L}u_n(x_2) = F(x_2, u_2(x_2), v_2(px_2))$. In the same way, we have

$$\mathcal{L}u_n(x_j) = F(x_j, u_j(x_j), v_j(px_j)), \quad j \leq n-1. \quad (44)$$

Similarly,

$$\mathcal{L}v_n(x_j) = F(x_j, v_j(x_j), u_j(qx_j)), \quad j \leq n-1. \quad (45)$$

□

Theorem 11. *The iterative form*

$$\begin{aligned} \tilde{u}_n(x) &= \sum_{i=1}^n \sum_{k=1}^i \beta_{ik} F(x_k, \tilde{u}_{n-1}(x_k), \tilde{v}_{n-1}(px_k)) \bar{\psi}_i(x), \\ \tilde{v}_n(x) &= \sum_{i=1}^n \sum_{k=1}^i \beta_{ik} G(x_k, \tilde{v}_{n-1}(x_k), \tilde{u}_{n-1}(qx_k)) \bar{\psi}_i(x), \\ \tilde{u}_1(0) &= \tilde{v}_1(0) = 0 \end{aligned} \quad (46)$$

and the iterative form

$$\begin{aligned} u_n(x) &= \sum_{i=1}^{n-1} \sum_{k=1}^i \beta_{ik} F(x_k, u_k(x_k), v_k(px_k)) \bar{\psi}_i(x), \\ v_n(x) &= \sum_{i=1}^{n-1} \sum_{k=1}^i \beta_{ik} G(x_k, v_k(x_k), u_k(qx_k)) \bar{\psi}_i(x), \\ u_1(0) &= v_1(0) = 0 \end{aligned} \quad (47)$$

are the same.

Proof. In Lemma 10, let $n-1 = k$; then

$$\mathcal{L}u_{k+1}(x_k) = F(x_k, u_k(x_k), v_k(px_k)), \quad (48)$$

but

$$\mathcal{L}u_n(x_k) = F(x_k, u_k(x_k), v_k(px_k)), \quad (49)$$

thus

$$\begin{aligned} u_n(x) &= \sum_{i=1}^{n-1} \sum_{k=1}^i \beta_{ik} \mathcal{L}u_{k+1}(x_k) \bar{\psi}_i(x) \\ &= \sum_{i=1}^{n-1} \sum_{k=1}^i \beta_{ik} \mathcal{L}u_n(x_k) \bar{\psi}_i(x) \\ &= \sum_{i=1}^{n-1} \sum_{k=1}^i \beta_{ik} \langle \mathcal{L}u_n(x), \varphi_k(x) \rangle \bar{\psi}_i(x) \\ &= \sum_{i=1}^{n-1} \sum_{k=1}^i \beta_{ik} \langle u_n(x), \psi_k(x) \rangle \bar{\psi}_i(x) \\ &= \sum_{i=1}^{n-1} \langle u_n(x), \bar{\psi}_i(x) \rangle \bar{\psi}_i(x). \end{aligned} \quad (50)$$

Equation (46) can be written as

$$\begin{aligned} \mathcal{L}M_n(x) &= F(x, \tilde{u}_{n-1}(x), \tilde{v}_{n-1}(px)), \\ \tilde{u}_n(x) &= P_n M_n(x). \end{aligned} \quad (51)$$

In fact, $M_n(x) = \sum_{i=1}^{\infty} \sum_{k=1}^i \beta_{ik} F(x_k, \tilde{u}_{n-1}(x_k), \tilde{v}_{n-1}(px_k)) \bar{\psi}_i(x)$; then

$$\begin{aligned} \mathcal{L}\tilde{u}_n(x_j) &= \mathcal{L}P_n M_n(x_j) \\ &= \langle \mathcal{L}P_n M_n(x), \varphi_j(x) \rangle \\ &= \langle P_n M_n(x), \psi_j(x) \rangle \\ &= \langle M_n(x), \psi_j(x) \rangle \\ &= \langle \mathcal{L}M_n(x), \varphi_j(x) \rangle \\ &= \mathcal{L}M_n(x_j), \end{aligned} \tag{52}$$

by (51) and (52), we have

$$\begin{aligned} \tilde{u}_n(x) &= \sum_{i=1}^{n-1} \sum_{k=1}^i \beta_{ik} F(x_k, \tilde{u}_{n-1}(x_k), \tilde{v}_{n-1}(px_k)) \bar{\psi}_i(x) \\ &= \sum_{i=1}^{n-1} \sum_{k=1}^i \beta_{ik} \mathcal{L}M_n(x_k) \bar{\psi}_i(x) \\ &= \sum_{i=1}^{n-1} \sum_{k=1}^i \beta_{ik} \mathcal{L}\tilde{u}_n(x_k) \bar{\psi}_i(x) \\ &= \sum_{i=1}^{n-1} \sum_{k=1}^i \beta_{ik} \langle \mathcal{L}\tilde{u}_n(x), \varphi_k(x) \rangle \bar{\psi}_i(x) \\ &= \sum_{i=1}^{n-1} \sum_{k=1}^i \beta_{ik} \langle \tilde{u}_n(x), \psi_k(x) \rangle \bar{\psi}_i(x) \\ &= \sum_{i=1}^{n-1} \langle \tilde{u}_n(x), \bar{\psi}_i(x) \rangle \bar{\psi}_i(x). \end{aligned} \tag{53}$$

Equation (53) is the same as (50). We may prove for $\tilde{v}_n(x)$ similarly. \square

So, by Theorem 11 and Lemma 8, we have the following Theorem.

Theorem 12. Under the conditions of Lemma 8,

$$u_n(x) = \sum_{i=1}^{n-1} \beta_{ik} F(x_k, u_k(x_k), v_k(px_k)) \bar{\psi}_i(x), \tag{54}$$

$$v_n(x) = \sum_{i=1}^{n-1} \beta_{ik} G(x_k, v_k(x_k), u_k(qx_k)) \bar{\psi}_i(x) \tag{55}$$

satisfy $\|u_n(x)\| \leq C, \|v_n(x)\| \leq D$.

Lemma 13. If $u(x)$ and $v(x) \in W_2[0, +\infty)$, then there exists $M_1, M_2 > 0$, such that $|u(x)| \leq M_1 \|u(x)\|_{W_2}$ and $|v(x)| \leq M_2 \|v(x)\|_{W_2}$.

Proof. It is easy to obtain from the properties in the reproducing kernel space. \square

By Lemma 13 and Theorem 12, it is easy to obtain the following Lemma 14.

Lemma 14. If $u_n(x) \xrightarrow{\|\cdot\|_{W_2}} \bar{u}(x) (n \rightarrow \infty)$, $v_n(x) \xrightarrow{\|\cdot\|_{W_2}} \bar{v}(x) (n \rightarrow \infty)$, $x_n \rightarrow y (n \rightarrow \infty)$, and $F(x, y, z)$ and $G(x, y, z)$ satisfy the conditions of Lemma 8, then

$$\begin{aligned} &F(x_n, u_{n-1}(x_n), v_{n-1}(px_n)) \\ &\rightarrow F(y, \bar{u}(y), \bar{v}(py)) \quad \text{as } n \rightarrow \infty, \\ &G(x_n, v_{n-1}(x_n), u_{n-1}(qx_n)) \\ &\rightarrow G(y, \bar{v}(y), \bar{u}(qy)) \quad \text{as } n \rightarrow \infty. \end{aligned} \tag{56}$$

Theorem 15. Let $\{x_i\}_{i=1}^{\infty}$ be dense in $[0, +\infty)$, and $F(x, y, z)$ and $G(x, y, z)$ satisfy the conditions of Lemma 8, then the n -term approximate solutions $u_n(x)$ and $v_n(x)$ in (35) converge to the exact solution $u(x)$ and $v(x)$ of (2), respectively, and $u(x) = \sum_{i=1}^{\infty} \bar{A}_i \bar{\psi}_i(x)$, $v(x) = \sum_{i=1}^{\infty} \bar{B}_i \bar{\psi}_i(x)$, where \bar{A}_i and \bar{B}_i are given, respectively, by (36), and (37).

Proof. (1) Firstly, we will prove the convergence of $u_n(x), v_n(x)$.

By (35), we infer that

$$u_{n+1}(x) = u_n(x) + \bar{A}_n \bar{\psi}_n(x), \tag{57}$$

$$v_{n+1}(x) = v_n(x) + \bar{B}_n \bar{\psi}_n(x). \tag{58}$$

From the orthogonality of $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$, it follows that

$$\begin{aligned} \|u_{n+1}\|_{W_2}^2 &= \|u_n\|_{W_2}^2 + (\bar{A}_n)^2 \\ &= \|u_{n-1}\|_{W_2}^2 + (\bar{A}_{n-1})^2 + (\bar{A}_n)^2 \\ &\vdots \\ &= \|u_1\|_{W_2}^2 + \sum_{i=1}^n (\bar{A}_i)^2, \\ \|v_{n+1}\|_{W_2}^2 &= \|v_n\|_{W_2}^2 + (\bar{B}_n)^2 \\ &= \|v_{n-1}\|_{W_2}^2 + (\bar{B}_{n-1})^2 + (\bar{B}_n)^2 \\ &\vdots \\ &= \|v_1\|_{W_2}^2 + \sum_{i=1}^n (\bar{B}_i)^2. \end{aligned} \tag{59}$$

From (59) and (60), we know sequence $\|u_n\|_{W_2}$ and $\|v_n\|_{W_2}$ is increasing. By Theorem 12, $\|u_n\|_{W_2}$ and $\|v_n\|_{W_2}$ are bounded; hence $\|u_n\|_{W_2}$ and $\|v_n\|_{W_2}$ are convergent such that

$$\begin{aligned} \sum_{i=1}^{\infty} (\bar{A}_i)^2 &< \infty, \\ \sum_{i=1}^{\infty} (\bar{B}_i)^2 &< \infty. \end{aligned} \tag{61}$$

This implies that

$$\bar{A}_i = \sum_{k=1}^i \beta_{ik} F(x_k, u_k(x_k), v_k(px_k)) \in l^2 \quad (i = 1, 2, \dots), \tag{62}$$

$$\bar{B}_i = \sum_{k=1}^i \beta_{ik} G(x_k, v_k(x_k), u_k(qx_k)) \in l^2 \quad (i = 1, 2, \dots). \tag{63}$$

Without loss of generality, assume $m > n$; we have

$$\begin{aligned} \|u_m(x) - u_n(x)\|_{W_2}^2 &= \|u_m(x) - u_{m-1}(x) \\ &\quad + u_{m-1}(x) - u_{m-2}(x) \\ &\quad + \dots + u_{n+1}(x) - u_n(x)\|_{W_2}^2 \\ &\leq \|u_m(x) - u_{m-1}(x)\|_{W_2}^2 \\ &\quad + \dots + \|u_{n+1}(x) - u_n(x)\|_{W_2}^2 \\ &= \sum_{i=n+1}^m (\bar{A}_i)^2 \rightarrow 0, \quad (n \rightarrow \infty), \end{aligned} \tag{64}$$

$$\begin{aligned} \|v_m(x) - v_n(x)\|_{W_2}^2 &= \|v_m(x) - v_{m-1}(x) \\ &\quad + v_{m-1}(x) - v_{m-2}(x) \\ &\quad + \dots + v_{n+1}(x) - v_n(x)\|_{W_2}^2 \\ &\leq \|v_m(x) - v_{m-1}(x)\|_{W_2}^2 \\ &\quad + \dots + \|v_{n+1}(x) - v_n(x)\|_{W_2}^2 \\ &= \sum_{i=n+1}^m (\bar{B}_i)^2 \rightarrow 0, \quad (n \rightarrow \infty). \end{aligned}$$

Considering the completeness of $W_2[0, +\infty)$, there exist $\bar{u}(x)$ and $\bar{v}(x)$ in $W_2[0, +\infty)$ such that

$$\begin{aligned} u_n(x) &\xrightarrow{\|\cdot\|} \bar{u}(x) \quad \text{as } n \rightarrow \infty, \\ v_n(x) &\xrightarrow{\|\cdot\|} \bar{v}(x) \quad \text{as } n \rightarrow \infty. \end{aligned} \tag{65}$$

(2) Secondly, we will prove that $\bar{u}(x)$ and $\bar{v}(x)$ are the solutions of (2).

By Lemma 14 and the proof of (1), we may know that $u_n(x)$, and $v_n(x)$ respectively, converge uniformly to $\bar{u}(x)$ and $\bar{v}(x)$ ($n \rightarrow \infty$). Taking limits in (35), we have

$$\begin{aligned} \bar{u}(x) &= \sum_{i=1}^{\infty} \bar{A}_i \bar{\psi}_i(x), \\ \bar{v}(x) &= \sum_{i=1}^{\infty} \bar{B}_i \bar{\psi}_i(x). \end{aligned} \tag{66}$$

Since

$$\begin{aligned} (\mathcal{L}\bar{u})(x_j) &= \sum_{i=1}^{\infty} \bar{A}_i \langle \mathcal{L}\bar{\psi}_i, \varphi_j \rangle \\ &= \sum_{i=1}^{\infty} \bar{A}_i \langle \bar{\psi}_i, \mathcal{L}^* \varphi_j \rangle \\ &= \sum_{i=1}^{\infty} \bar{A}_i \langle \bar{\psi}_i, \psi_j \rangle, \end{aligned} \tag{67}$$

in the same way, we have

$$(\mathcal{L}\bar{v})(x_j) = \sum_{i=1}^{\infty} \bar{B}_i \langle \bar{\psi}_i, \psi_j \rangle, \tag{68}$$

it follows that

$$\begin{aligned} \sum_{j=1}^n \beta_{nj} (\mathcal{L}\bar{u})(x_j) &= \sum_{i=1}^{\infty} \bar{A}_i \left\langle \bar{\psi}_i, \sum_{j=1}^n \beta_{nj} \psi_j \right\rangle \\ &= \sum_{i=1}^{\infty} \bar{A}_i \langle \bar{\psi}_i, \bar{\psi}_n \rangle \\ &= \bar{A}_n, \\ \sum_{j=1}^n \beta_{nj} (\mathcal{L}\bar{v})(x_j) &= \sum_{i=1}^{\infty} \bar{B}_i \left\langle \bar{\psi}_i, \sum_{j=1}^n \beta_{nj} \psi_j \right\rangle \\ &= \sum_{i=1}^{\infty} \bar{B}_i \langle \bar{\psi}_i, \bar{\psi}_n \rangle \\ &= \bar{B}_n. \end{aligned} \tag{69}$$

If $n = 1$, then

$$(\mathcal{L}\bar{u})(x_1) = F(x_1, u_1(x_1), v_1(px_1)), \tag{70}$$

$$(\mathcal{L}\bar{v})(x_1) = G(x_1, v_1(x_1), u_1(qx_1)). \tag{71}$$

If $n = 2$, then

$$\begin{aligned} &\beta_{21} (\mathcal{L}\bar{u})(x_1) + \beta_{22} (\mathcal{L}\bar{u})(x_2) \\ &= \beta_{21} F(x_1, u_1(x_1), v_1(px_1)) \\ &\quad + \beta_{22} F(x_2, u_2(x_2), v_2(px_2)), \\ &\beta_{21} (\mathcal{L}\bar{v})(x_1) + \beta_{22} (\mathcal{L}\bar{v})(x_2) \\ &= \beta_{21} G(x_1, v_1(x_1), u_1(qx_1)) \\ &\quad + \beta_{22} G(x_2, v_2(x_2), u_2(qx_2)). \end{aligned} \tag{72}$$

From (70) and (71), it is clear that

$$\begin{aligned} (\mathcal{L}\bar{u})(x_2) &= F(x_2, u_2(x_2), v_2(px_2)), \\ (\mathcal{L}\bar{v})(x_2) &= G(x_2, v_2(x_2), u_2(qx_2)). \end{aligned} \tag{73}$$

Furthermore, it is easy to see by induction that

$$(\mathcal{L}\bar{u})(x_j) = F(x_j, u_j(x_j), v_j(px_j)), \quad j = 1, 2, \dots, \quad (74)$$

$$(\mathcal{L}\bar{v})(x_j) = G(x_j, v_j(x_j), u_j(qx_j)), \quad j = 1, 2, \dots \quad (75)$$

Since $\{x_i\}_{i=1}^\infty$ is dense in $[0, +\infty)$, for any $y \in [0, +\infty)$, there exists subsequence $\{x_{n_j}\}$ such that

$$x_{n_j} \rightarrow y \quad \text{as } j \rightarrow \infty. \quad (76)$$

Hence, let $j \rightarrow \infty$ in (74), and (75); by Lemma 14 and the continuity of $F(x, u(x), v(px))$ and $G(x, v(x), u(qx))$, we have

$$(\mathcal{L}\bar{u})(y) = F(y, \bar{u}(y), \bar{v}(py)), \quad (77)$$

$$(\mathcal{L}\bar{v})(y) = G(y, \bar{v}(y), \bar{u}(qy)). \quad (78)$$

That is, $\bar{u}(x)$ and $\bar{v}(x)$ are the solutions of (2) and

$$\bar{u}(x) = \sum_{i=1}^\infty \bar{A}_i \bar{\psi}_i(x), \quad (79)$$

$$\bar{v}(x) = \sum_{i=1}^\infty \bar{B}_i \bar{\psi}_i(x), \quad (80)$$

where $\bar{A}_i = \sum_{k=1}^i \beta_{ik} F(x_k, \bar{u}_k(x_k), \bar{v}_k(px_k))$ and $\bar{B}_i = \sum_{k=1}^i \beta_{ik} G(x_k, \bar{v}_k(x_k), \bar{u}_k(qx_k))$. \square

In the proof of the convergence in Theorem 15 we only use $\|u_n\| \leq C, \|v_n\| \leq D$; thus we obtain the following corollary.

Corollary 16. *Suppose $\|u_n\|$ and $\|v_n\|$ are bounded; then the iterative sequence (35) is convergent to the exact solution of (2).*

Theorem 17. *Assume $u(x)$ and $v(x)$ are the solutions of (2), $r_n^1 = \|u(x) - u_n(x)\|_{W_2}$ and $r_n^2 = \|v(x) - v_n(x)\|_{W_2}$ are the approximate errors of $u_n(x), v_n(x)$, where $u_n(x)$ and $v_n(x)$ are given by (35). Then the errors r_n^1, r_n^2 are monotone decreasing in the sense of $\|\cdot\|_{W_2}$.*

Proof. From (35), and (79), it follows that

$$\begin{aligned} \|r_n^1(x)\|_{W_2}^2 &= \left\| \sum_{i=n+1}^\infty \bar{A}_i \bar{\psi}_i(x) \right\|_{W_2}^2 \\ &= \sum_{i=n+1}^\infty (\bar{A}_i)^2. \end{aligned} \quad (81)$$

In the same way, we obtain from (35), (80)

$$\begin{aligned} \|r_n^2(x)\|_{W_2}^2 &= \left\| \sum_{i=n+1}^\infty \bar{B}_i \bar{\psi}_i(x) \right\|_{W_2}^2 \\ &= \sum_{i=n+1}^\infty (\bar{B}_i)^2. \end{aligned} \quad (82)$$

Equations (81) and (82) show that the errors r_n^1 and r_n^2 are monotone decreasing in the sense of $\|\cdot\|_{W_2}$. \square

4. Numerical Examples

In order to demonstrate the efficiency of our algorithm for solving (2), we will present two numerical examples in the reproducing kernel space $W_2[0, +\infty)$. Let n be the number of discrete points in $[0, +\infty)$. Denote $E_n^u(x) \stackrel{\text{def}}{=} |(u(x) - u_n(x))/u(x)|, E_n^v(x) \stackrel{\text{def}}{=} |(v(x) - v_n(x))/v(x)|$. All computations are performed by the Mathematica 5.0 software package. Results obtained by the method are compared with the exact solution of each example and are found to be in good agreement with each other.

Example 18. In this example we consider the problem

$$\begin{aligned} u'(x) &= \frac{3}{1+u(x)^2} + 2x \cos\left(v\left(\frac{1}{3}x\right)\right) + f(x), \\ v'(x) &= 2e^{-v(x)} + 4x^3 \sin\left(u\left(\frac{1}{2}x\right)\right) + g(x), \\ u(0) &= 0, \quad v(0) = 0, \end{aligned} \quad (83)$$

with exact solution $u(x) = xe^{-x}, v(x) = \sin x$, and $f(x) = e^{-x} - e^{-x}x - (3/(1+e^{-2}x^2)) - 2x \cos(\sin(x/3)), g(x) = -2e^{-\sin x} + \cos x - 4x^3 \sin((1/2)xe^{-x/2})$. Applying the presented method in Section 3, we calculate the approximate solution $u_{50}(x)$ and $v_{50}(x)$ in $[0, 1]$ as follows.

Step 1. By the method of the appendix, the corresponding reproducing kernel functions can be obtained.

Step 2. Choosing a dense subset in $[0, 1]$, then we get the orthogonalization coefficients β_{ik} .

Step 3. According to (10), we can get the normal orthogonal systems $\{\bar{\psi}_i(x)\}_{i=1}^{50}$.

Step 4. Selecting the initial value $u_0(x) = v_0(x) = 0$, we obtain $u_1(x), u_2(x), \dots, u_{50}(x)$ and $v_1(x), v_2(x), \dots, v_{50}(x)$ by (19) developed in the paper.

The graphs of the superimposed image emerge in Figure 1. At the same time, we have computed the approximate solutions $u_n(x)$ and $v_n(x)$ ($n = 300, 900$) in $[0, 3]$ and also calculated the relative errors E_n^u and E_n^v in Table 1. The root mean square errors (RMSE) about $u(x)$ with u_n and $v(x)$ with v_n are shown in Table 2.

Example 19. Considering equations

$$\begin{aligned} u'(x) &= 3 \cos(u(x)) + 2x \sin\left(v\left(\frac{1}{5}x\right)\right) + f(x), \\ v'(x) &= 2 \sin(v(x)) + 4x^3 \cos\left(u\left(\frac{1}{3}x\right)\right) + g(x), \\ u(0) &= 0, \quad v(0) = 0. \end{aligned} \quad (84)$$

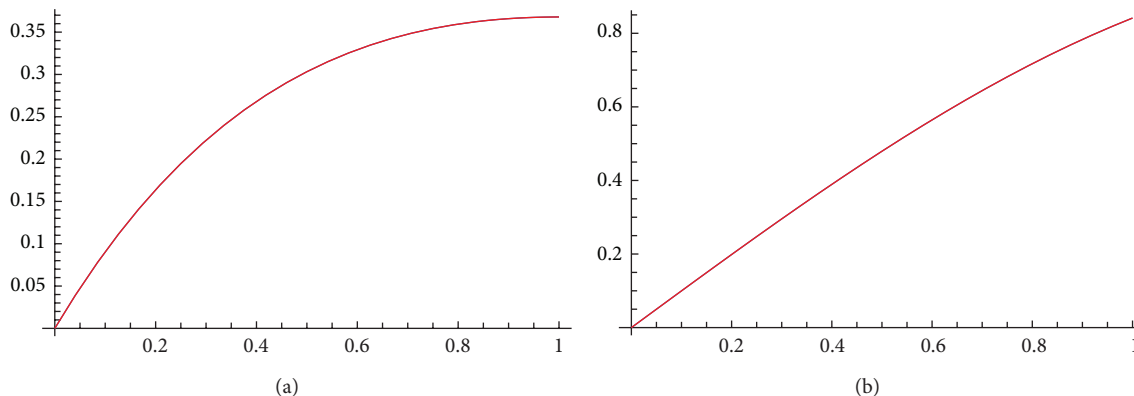


FIGURE 1: The left is the superimposed image of $u(x)$ with $u_{50}(x)$ in $[0, 1]$. The right is the superimposed image of $v(x)$ with $v_{50}(x)$ in $[0, 1]$.

TABLE 1: Relative errors in $[0, 3]$ for Example 18.

Node	E_{300}^u	E_{300}^v	E_{900}^u	E_{300}^v
0.3	$4.97833e-5$	$8.4527e-5$	$5.53163e-6$	$9.39806e-6$
0.6	$4.43080e-5$	$7.3330e-5$	$4.92288e-6$	$8.15664e-6$
0.9	$3.86801e-5$	$6.6731e-5$	$4.29733e-6$	$7.42512e-6$
1.2	$3.41875e-5$	$6.1845e-5$	$3.79797e-6$	$6.88400e-6$
1.5	$3.09088e-5$	$5.7095e-5$	$3.43349e-6$	$6.35872e-6$
1.8	$2.87391e-5$	$5.1303e-5$	$3.19221e-6$	$5.71898e-6$
2.1	$2.75649e-5$	$4.3153e-5$	$3.06149e-6$	$4.81984e-6$
2.4	$2.7288e-5$	$3.0422e-5$	$3.03034e-6$	$3.41736e-6$
2.7	$2.7829e-5$	$6.7566e-5$	$3.09008e-6$	$8.12936e-7$
3.0	$2.90670e-5$	$7.5017e-5$	$3.23206e-6$	$8.72219e-6$

TABLE 2: The RMS errors in $[0, 3]$ for Example 18.

n	$\sqrt{\sum_{i=1}^n (u_n(x_i) - u(x_i))^2/n}$	$\sqrt{\sum_{i=1}^n (v_n(x_i) - v(x_i))^2/n}$
300	$9.95194e-6$	$4.00007e-5$
900	$1.10551e-6$	$4.45554e-6$

TABLE 3: Relative errors in $[0, 4]$ for Example 19.

Node	E_{800}^u	E_{800}^v	E_{1200}^u	E_{1200}^v
0.4	$8.06470e-5$	$4.94886e-5$	$3.58334e-5$	$2.20033e-5$
0.8	$1.07473e-4$	$5.27211e-5$	$4.77632e-5$	$2.34475e-5$
1.2	$2.56877e-4$	$5.32904e-5$	$1.14199e-4$	$2.37040e-5$
1.6	$1.19110e-4$	$5.01608e-5$	$5.29170e-5$	$2.23098e-5$
2.0	$8.56122e-5$	$4.35915e-5$	$3.80383e-5$	$1.93810e-5$
2.4	$2.27843e-4$	$3.52666e-5$	$1.01280e-4$	$1.56705e-5$
2.8	$1.38237e-4$	$2.74018e-5$	$6.13912e-5$	$1.21676e-5$
3.2	$7.26035e-5$	$2.15572e-5$	$3.22429e-5$	$9.56745e-6$
3.6	$1.86684e-5$	$1.80587e-5$	$8.29627e-5$	$8.01449e-6$
4.0	$1.50042e-4$	$1.64230e-5$	$6.66348e-5$	$7.14957e-6$

TABLE 4: The RMS errors in $[0, 4]$ for Example 19.

n	$\sqrt{\sum_{i=1}^n (u_n(x_i) - u(x_i))^2/n}$	$\sqrt{\sum_{i=1}^n (v_n(x_i) - v(x_i))^2/n}$
800	$1.87807e-3$	$7.85856e-5$
1200	$8.34276e-4$	$3.48826e-5$

The true solutions are $u(x) = 5x$, $v(x) = 4 \tan(x)$, $f(x) = 5 - 3 \cos(5x) - 2x \sin(4 \tan(1/5))$, $g(x) = -4x^3 \cos(5x/3) + 4 \sec x^2 - 2 \sin(4 \tan x)$. The numerical results are given in Figure 2 and Tables 3 and 4. The figures and tables illustrate that the method given in the paper is efficient.

5. Conclusion

In this paper, RKHSM has been successfully applied to find the solutions of systems of nonlinear IDDEs with proportional delays. The efficiency and accuracy of the proposed decomposition method were demonstrated by two test problems. It is concluded from above tables and figures that the RKHSM is an accurate and efficient method to solve IDDEs with proportional delays. Moreover, the method is also effective for solving some nonlinear initial-boundary value problems and nonlocal boundary value problems.

Appendix

The Reproducing Kernel Space $W_2[0, m]$

$W_2[0, m]$ is defined as $W_2[0, m] = \{u(x) \mid u' \text{ are absolutely continuous real value functions, } u'' \in L^2[0, m], u(0) = 0\}$. The inner product in $W_2[0, m]$ is given by

$$\langle u(y), v(y) \rangle_{W_2} = \int_0^m (4uv + 5u'v' + u''v'') dy, \quad (A.1)$$

where $u, v \in W_2[0, m]$ and the norm $\|u\|_{W_2}$ is denoted by $\|u\|_{W_2} = \sqrt{\langle u, u \rangle_{W_2}}$.

Theorem A.1. *The space $W_2[0, m]$ is a reproducing kernel space; that is, for any $u(y) \in W_2[0, m]$ and each fixed $x \in [0, m]$, there exists $R_x(y) \in W_2[0, m]$, $y \in [0, m]$, such that*

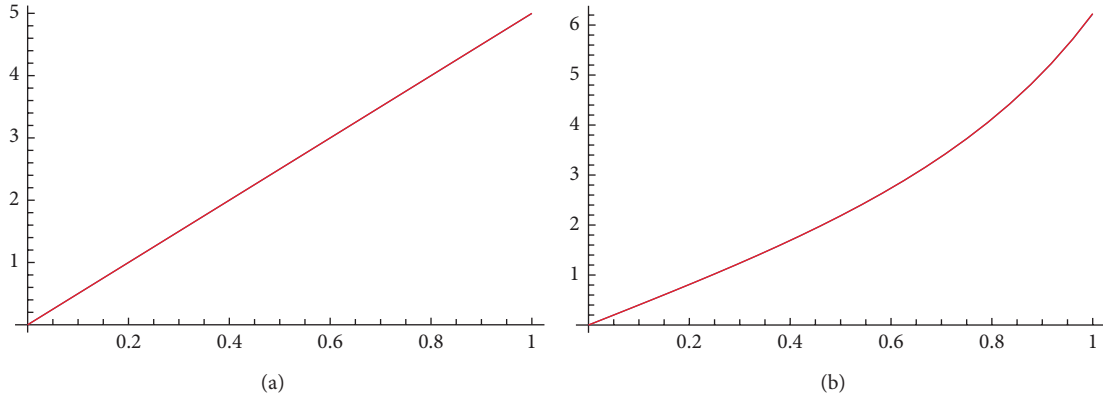


FIGURE 2: The left is the superimposed image of $u(x)$ with $u_{50}(x)$ in $[0, 1]$. The right is the superimposed image of $v(x)$ with $v_{50}(x)$ in $[0, 1]$.

$\langle u(y), R_x(y) \rangle_{W_2[0,m]} = u(x)$. The reproducing kernel $R_x(y)$ can be denoted by

$$R_x(y) = \begin{cases} c_1 e^y + c_2 e^{-y} + c_3 e^{2y} + c_4 e^{-2y}, & y \leq x, \\ d_1 e^y + d_2 e^{-y} + d_3 e^{2y} + d_4 e^{-2y}, & y > x. \end{cases} \quad (A.2)$$

Proof. Applying to the integrations by parts for (A.1), we have

$$\begin{aligned} &\langle u(y), R_x(y) \rangle_{W_2[0,m]} \\ &= \int_0^m u(y) [4R_x(y) - 5R_x''(y) + R_x^{(4)}(y)] dy \\ &\quad + 5u(y) R_x'(y) \Big|_0^m + u'(y) R_x''(y) \Big|_0^m \\ &\quad - u(y) R_x^{(3)}(y) \Big|_0^m. \end{aligned} \quad (A.3)$$

Since $R_x(y) \in W_2[0, m]$, it follows that

$$R_x(0) = 0. \quad (A.4)$$

For $u(y) \in W_2[0, m]$, thus, $u(0) = 0$.

Suppose that $R_x(y)$ satisfies the following generalized differential equations:

$$\begin{aligned} 4R_x(y) - 5R_x''(y) + R_x^{(4)}(y) &= \delta(y - x), \\ 5R_x'(m) - R_x^{(3)}(m) &= 0, \\ R_x''(m) &= 0, \\ R_x''(0) &= 0. \end{aligned} \quad (A.5)$$

Then $\langle u(y), R_x(y) \rangle_{W_2[0,m]} = \int_0^m u(y) \delta(y - x) dy = u(x)$. Hence, $R_x(y)$ is the reproducing kernel of space $W_2[0, m]$.

In the following, we will get the expression of the reproducing kernel $R_x(y)$.

The characteristic equation of $R_x(y) - 5R_x''(y) + R_x^{(4)}(y) = \delta(y - x)$ is given by $\lambda^4 - 5\lambda^2 + 4 = 0$, and the characteristic roots are $\lambda_{1,2} = \pm 1, \lambda_{3,4} = \pm 2$.

We denote $R_x(y)$ by

$$R_x(y) = \begin{cases} c_1 e^y + c_2 e^{-y} + c_3 e^{2y} + c_4 e^{-2y}, & y \leq x, \\ d_1 e^y + d_2 e^{-y} + d_3 e^{2y} + d_4 e^{-2y}, & y > x. \end{cases} \quad (A.6)$$

By the definition of space $W_2[0, m]$, coefficients $c_1, \dots, c_4, d_1, \dots, d_4$ satisfy

$$\begin{aligned} R_x^{(k)}(x+0) &= R_x^{(k)}(x-0) \quad (k = 0, 1, 2), \\ R_x^{(3)}(m) - 5R_x'(m) &= 0, \\ R_x^{(3)}(x+0) - R_x^{(3)}(x-0) &= 1, \\ R_x''(m) &= 0, \\ R_x''(0) &= 0, \\ R_x(0) &= 0, \end{aligned} \quad (A.7)$$

from which, the unknown coefficients of (A.6) can be obtained:

$$\begin{aligned} c_1 &= \frac{e^{-2x} (4e^{3m} - 7e^{3x} - 9e^{2m+x} + 7e^{6m+x} + 9e^{4m+3x} - 4e^{3m+4x})}{6(-7 - 9e^{2m} + 9e^{4m} + 7e^{6m})} \\ c_2 &= \frac{e^{-2x} (-4e^{3m} + 7e^{3x} + 9e^{2m+x} - 7e^{6m+x} - 9e^{4m+3x} + 4e^{3m+4x})}{6(-7 - 9e^{2m} + 9e^{4m} + 7e^{6m})} \end{aligned}$$

$$\begin{aligned}
c_3 &= \frac{e^{-2x} \left(-9e^{4m} - 7e^{6m} + 7e^{4x} + 8e^{3m+x} - 8e^{3(m+x)} + 9e^{2m+4x} \right)}{12 \left(-7 - 9e^{2m} + 9e^{4m} + 7e^{6m} \right)} \\
c_4 &= \frac{-e^{-2x} \left(-9e^{4m} - 7e^{6m} + 7e^{4x} + 8e^{3m+x} - 8e^{3(m+x)} + 9e^{2m+4x} \right)}{12 \left(-7 - 9e^{2m} + 9e^{4m} + 7e^{6m} \right)} \\
d_1 &= -\frac{e^{-2x} \left(-1 + e^{2x} \right) \left(4e^{3m} + 7e^x - 9e^{4m+x} + 4e^{3m+2x} \right)}{6 \left(-7 - 9e^{2m} + 9e^{4m} + 7e^{6m} \right)} \\
d_2 &= \frac{e^{2m-2x} \left(-1 + e^{2x} \right) \left(4e^m + 7e^{4m+x} - 9e^x + 4e^{m+2x} \right)}{6 \left(-7 - 9e^{2m} + 9e^{4m} + 7e^{6m} \right)} \\
d_3 &= \frac{e^{-2x} \left(-1 + e^{2x} \right) \left(7 + 9e^{2m} + 7e^{2x} + 9e^{2(m+x)} - 8e^{3m+x} \right)}{12 \left(-7 - 9e^{2m} + 9e^{4m} + 7e^{6m} \right)} \\
d_4 &= -\frac{e^{3m-2x} \left(-1 + e^{2x} \right) \left(9e^m + 7e^{3m} + 9e^{m+2x} - 8e^x + 7e^{3m+2x} \right)}{12 \left(-7 - 9e^{2m} + 9e^{4m} + 7e^{6m} \right)}.
\end{aligned} \tag{A.8}$$

□

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