

Research Article

An Opial-Type Inequality on Time Scales

Qiao-Luan Li¹ and Wing-Sum Cheung²

¹ College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang 050024, China

² Department of Mathematics, The University of Hong Kong, Hong Kong

Correspondence should be addressed to Qiao-Luan Li; qll71125@163.com

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We establish some new Opial-type inequalities involving higher order delta derivatives on time scales. These extend some known results in the continuous case in the literature and provide new estimates in the setting of time scales.

1. Introduction

Opial's inequality appeared for the first time in 1960 in [1] and has been receiving continual attention throughout the years (cf., e.g., [2–7]). The inequality together with its numerous generalizations, extensions, and discretizations has been playing a fundamental role in the study of the existence and uniqueness properties of solutions of initial and boundary value problems for differential equations as well as difference equations [8, 9]. Two excellent surveys on these inequalities can be found in [10, 11].

In 1960, Opial established the following integral inequality.

Theorem A (see [1]). *If $f \in C^1[0, h]$ satisfies $f(0) = f(h) = 0$ and $f(x) > 0$ for all $x \in (0, h)$, then*

$$\int_0^h |f(x)f'(x)| dx \leq \frac{h}{4} \int_0^h |f'(x)|^2 dx. \quad (1)$$

Shortly after the publication of Opial's paper, Olech provided a modified version of Theorem A. His result is stated in the following.

Theorem B (see [12]). *If f is absolutely continuous on $[0, h]$ with $f(0) = 0$, then*

$$\int_0^h |f(x)f'(x)| dx \leq \frac{h}{2} \int_0^h f'(x)^2 dx. \quad (2)$$

The equality in (2) holds if and only if $f(x) = cx$, where c is a constant.

The first natural extension of Opial's inequality (1) involving higher order derivatives $x^{(n)}(s)$ ($n \geq 1$) is embodied in the following.

Theorem C (see [10]). *Let $x(t) \in C^{(n)}[0, a]$ be such that $x^{(i)}(0) = 0$, $0 \leq i \leq n - 1$ ($n \geq 1$). Then the following inequality holds:*

$$\int_0^a |x(t)x^{(n)}(t)| dt \leq \frac{1}{2} a^n \int_0^a |x^{(n)}(t)|^2 dt. \quad (3)$$

In 1997, Alzer [13] considered Opial-type inequalities which involve higher-order derivatives of two functions. These generalize earlier results of Agarwal and Pang [14].

In this paper, we consider the Opial-type inequality which involves higher-order delta derivatives of two functions on time scales. Our results in special cases yield some of the recent results on Opial's inequality and provide some new estimates on such types of inequalities in this general setting.

2. Main Results

Let \mathbb{T} be a time scale; that is, \mathbb{T} is an arbitrary nonempty closed subset of real numbers. Let $a, b \in \mathbb{T}$. We suppose that the reader is familiar with the basic features of calculus on time scales for dynamic equations. Otherwise one can consult

Bohner and Peterson's book [15] for most of the materials needed.

We first quote the following elementary lemma and the delta time scales Taylor formula.

Lemma 1 (see [16]). *Let $a \geq 0$, $p \geq 1$ be real constants. Then*

$$a^{1/p} \leq \frac{1}{p}k^{(1-p)/p}a + \frac{p-1}{p}k^{1/p} \tag{4}$$

for any $k > 0$.

Lemma 2 (see [17]). *Let $f \in C_{rd}^m(\mathbb{T}) =$ the set of functions that are m times differentiable with rd-continuous derivatives on \mathbb{T} , $m \in \mathbb{N}$. Then for any $a, b \in \mathbb{T}$ and $t \in [a, b] \cap \mathbb{T}$,*

$$f(t) = \sum_{k=0}^{m-1} h_k(t, a) f^{\Delta^k}(a) + \int_a^t h_{m-1}(t, \sigma(\tau)) f^{\Delta^m}(\tau) \Delta\tau, \tag{5}$$

where $h_0(t, s) := 1$, $h_{n+1}(t, s) := \int_s^t h_n(\tau, s) \Delta\tau$, $n \in \mathbb{N}$.

Our main results are given in the following theorems.

Theorem 3. *Let $0 \leq r \leq s < t$, $s > 0$, $t > 1$ be real numbers, and let m, k be integers with $0 \leq k \leq m-1$. Let $p > 0$ and $q \geq 0$ be measurable functions on $Y := [a, b] \cap \mathbb{T}$. Further, let $f, g \in C_{rd}^{m-1}(Y)$ with $f^{\Delta^i}(a) = g^{\Delta^i}(a) = 0$, $i = 0, 1, \dots, m-1$, and let $f^{\Delta^{m-1}}, g^{\Delta^{m-1}}$ be absolutely continuous on Y such that the integrals $\int_a^b p(x)|f^{\Delta^m}(x)|^t \Delta x$ and $\int_a^b p(x)|g^{\Delta^m}(x)|^t \Delta x$ exist. Then one has*

$$\int_a^b q(x) \left[|f^{\Delta^k}(x)|^r |g^{\Delta^m}(x)|^s + |g^{\Delta^k}(x)|^r |f^{\Delta^m}(x)|^s \right] \Delta x \leq 2^{1-\alpha} \left(\int_a^b h(x)^{t/(t-s)} \Delta x \right)^{1-\alpha} \cdot [\beta k^{\beta-1} F(b) G(b) + (1-\beta) k^\beta (F(b) + G(b))]^\alpha, \tag{6}$$

where $\alpha := s/t$, $\beta := r/s$,

$$P(x) := \int_a^x h_{m-k-1}^{t/(t-1)}(x, \sigma(\tau)) p(\tau)^{1/(1-t)} \Delta\tau, \tag{7}$$

$$h(x) := q(x) p(x)^{-\alpha} P(x)^{r(t-1)/t},$$

$$F(x) := \int_a^x p(\tau) |f^{\Delta^m}(\tau)|^t \Delta\tau,$$

$$G(x) := \int_a^x p(\tau) |g^{\Delta^m}(\tau)|^t \Delta\tau.$$

Proof. Since $f^{\Delta^i}(a) = 0$, $i = 0, 1, \dots, m-1$, we obtain from Taylor's theorem that for all $x \in Y$,

$$f(x) = \int_a^x h_{m-1}(x, \sigma(\tau)) f^{\Delta^m}(\tau) \Delta\tau, \tag{8}$$

and hence

$$f^{\Delta^k}(x) = \int_a^x h_{m-k-1}(x, \sigma(\tau)) f^{\Delta^m}(\tau) \Delta\tau. \tag{9}$$

From (9) and Hölder's inequality we get

$$\begin{aligned} |f^{\Delta^k}(x)| &\leq \int_a^x h_{m-k-1}(x, \sigma(\tau)) |f^{\Delta^m}(\tau)| \Delta\tau \\ &= \int_a^x h_{m-k-1}(x, \sigma(\tau)) p(\tau)^{-1/t} p(\tau)^{1/t} |f^{\Delta^m}(\tau)| \Delta\tau \\ &\leq \left[\int_a^x h_{m-k-1}^{t/(t-1)}(x, \sigma(\tau)) p(\tau)^{1/(1-t)} \Delta\tau \right]^{1-1/t} \\ &\quad \cdot \left(\int_a^x p(\tau) |f^{\Delta^m}(\tau)|^t \Delta\tau \right)^{1/t} \\ &= P(x)^{1-1/t} F(x)^{1/t}, \end{aligned} \tag{10}$$

where $P(x) := \int_a^x h_{m-k-1}^{t/(t-1)}(x, \sigma(\tau)) p(\tau)^{1/(1-t)} \Delta\tau$, $F(x) := \int_a^x p(\tau) |f^{\Delta^m}(\tau)|^t \Delta\tau$.

Let

$$G(x) := \int_a^x p(\tau) |g^{\Delta^m}(\tau)|^t \Delta\tau. \tag{11}$$

Then we have

$$|g^{\Delta^m}(x)|^s = G^\Delta(x)^{s/t} p(x)^{-s/t}. \tag{12}$$

So (10) together with (12) implies

$$q(x) |f^{\Delta^k}(x)|^r |g^{\Delta^m}(x)|^s \leq h(x) F(x)^{r/t} G^\Delta(x)^{s/t}, \tag{13}$$

where $h(x) := q(x) p(x)^{-s/t} P(x)^{r(t-1)/t}$. Integrating both sides of (13) over Y and making use of Hölder's inequality, we obtain

$$\begin{aligned} &\int_a^b q(x) |f^{\Delta^k}(x)|^r |g^{\Delta^m}(x)|^s \Delta x \\ &\leq \int_a^b h(x) F(x)^{r/t} G^\Delta(x)^{s/t} \Delta x \\ &\leq \left(\int_a^b h(x)^{t/(t-s)} \Delta x \right)^{1-s/t} \left(\int_a^b F(x)^{r/s} G^\Delta(x) \Delta x \right)^{s/t}. \end{aligned} \tag{14}$$

Similarly, we get

$$\begin{aligned} &\int_a^b q(x) |g^{\Delta^k}(x)|^r |f^{\Delta^m}(x)|^s \Delta x \\ &\leq \left(\int_a^b h(x)^{t/(t-s)} \Delta x \right)^{1-s/t} \cdot \left(\int_a^b G(x)^{r/s} F^\Delta(x) \Delta x \right)^{s/t}. \end{aligned} \tag{15}$$

Recall the elementary inequalities

$$c_\alpha(A + B)^\alpha \leq A^\alpha + B^\alpha \leq d_\alpha(A + B)^\alpha, \quad (A, B \geq 0), \quad (16)$$

where

$$c_\alpha := \begin{cases} 1, & 0 \leq \alpha \leq 1, \\ 2^{1-\alpha}, & \alpha \geq 1, \end{cases} \quad (17)$$

$$d_\alpha := \begin{cases} 2^{1-\alpha}, & 0 \leq \alpha \leq 1, \\ 1, & \alpha \geq 1. \end{cases}$$

Let $\beta = r/s$. Since $\alpha = s/t \in (0, 1)$ and F is nondecreasing, from (14)–(16), we have

$$\begin{aligned} & \int_a^b q(x) \left[\left| f^{\Delta^k}(x) \right|^r \left| g^{\Delta^m}(x) \right|^s + \left| g^{\Delta^k}(x) \right|^r \left| f^{\Delta^m}(x) \right|^s \right] \Delta x \\ & \leq \left(\int_a^b h(x)^{t/(t-s)} \Delta x \right)^{1-\alpha} \\ & \quad \cdot \left[\left(\int_a^b F^\Delta(x) G^\beta(x) \Delta x \right)^\alpha + \left(\int_a^b G^\Delta(x) F^\beta(x) \Delta x \right)^\alpha \right] \\ & \leq \left(\int_a^b h(x)^{t/(t-s)} \Delta x \right)^{1-\alpha} \\ & \quad \cdot 2^{1-\alpha} \left(\int_a^b [F^\Delta(x) G^\beta(x) + G^\Delta(x) F^\beta(x)] \Delta x \right)^\alpha \\ & \leq \left(\int_a^b h(x)^{t/(t-s)} \Delta x \right)^{1-\alpha} \\ & \quad \cdot 2^{1-\alpha} \left(\int_a^b [F^\Delta(x) G^\beta(x) + G^\Delta(x) F^\beta(\sigma(x))] \Delta x \right)^\alpha. \end{aligned} \quad (18)$$

By Lemma 1, we get

$$\begin{aligned} & \int_a^b [F^\Delta(x) G^\beta(x) + G^\Delta(x) F^\beta(\sigma(x))] \Delta x \\ & \leq \int_a^b [\beta k^{\beta-1} G(x) F^\Delta(x) + (1-\beta) k^\beta F^\Delta(x) \\ & \quad + \beta k^{\beta-1} F(\sigma(x)) G^\Delta(x) + (1-\beta) k^\beta G^\Delta(x)] \Delta x \\ & = \beta k^{\beta-1} \int_a^b [G(x) F^\Delta(x) + F(\sigma(x)) G^\Delta(x)] \Delta x \\ & \quad + (1-\beta) k^\beta \left[\int_a^b F^\Delta(x) \Delta x + \int_a^b G^\Delta(x) \Delta x \right] \\ & = \beta k^{\beta-1} F(b) G(b) + (1-\beta) k^\beta [F(b) + G(b)]. \end{aligned} \quad (19)$$

From (18) and (19), we conclude

$$\begin{aligned} & \int_a^b q(x) \left[\left| f^{\Delta^k}(x) \right|^r \left| g^{\Delta^m}(x) \right|^s + \left| g^{\Delta^k}(x) \right|^r \left| f^{\Delta^m}(x) \right|^s \right] \Delta x \\ & \leq 2^{1-\alpha} \left(\int_a^b h(x)^{t/(t-s)} \Delta x \right)^{1-\alpha} \\ & \quad \cdot [\beta k^{\beta-1} F(b) G(b) + (1-\beta) k^\beta (F(b) + G(b))]^\alpha. \end{aligned} \quad (20)$$

The proof is complete. \square

Theorem 4. Let $r \geq 0, s > 0, s < t, t > 1$ be real numbers, and let m, k be integers with $0 \leq k \leq m - 1$. Let $p > 0$, and $q \geq 0$ be measurable functions on $Y := [a, b] \cap \mathbb{T}$. Further, let $f, g \in C_{rd}^{m-1}(Y)$ with let $f^{\Delta^i}(a) = g^{\Delta^i}(a) = 0, i = 0, 1, \dots, m - 1$, and $f^{\Delta^{m-1}}, g^{\Delta^{m-1}}$ be absolutely continuous on Y such that the integrals $\int_a^b p(x) |f^{\Delta^m}(x)|^t \Delta x$ and $\int_a^b p(x) |g^{\Delta^m}(x)|^t \Delta x$ exist. Then one has

$$\begin{aligned} & \int_a^b q(x) \left[\left| f^{\Delta^k}(x) \right|^r \left| g^{\Delta^m}(x) \right|^s + \left| g^{\Delta^k}(x) \right|^r \left| f^{\Delta^m}(x) \right|^s \right] \Delta x \\ & \leq 2^{1-\alpha} \left(\int_a^b h(x)^{t/(t-s)} \Delta x \right)^{1-\alpha} \\ & \quad \cdot [d_\beta \Gamma(G + F) - \Gamma(G) - \Gamma(F)]^\alpha, \end{aligned} \quad (21)$$

where $\Gamma(H) := \int_a^b H^\beta \Delta H, \beta := r/s, \alpha := s/t, h(x) := q(x)p(x)^{-\alpha} P(x)^{r(t-1)/t}, P(x) := \int_a^x h_m^{t/(t-1)}(x, \sigma(\tau)) p(\tau)^{1/(1-t)} \Delta \tau$, and

$$d_\beta := \begin{cases} 2^{1-\beta}, & 0 \leq \beta \leq 1, \\ 1, & \beta \geq 1. \end{cases} \quad (22)$$

Proof. Following the proof of Theorem 3, we obtain

$$\begin{aligned} & \int_a^b q(x) \left[\left| f^{\Delta^k}(x) \right|^r \left| g^{\Delta^m}(x) \right|^s + \left| g^{\Delta^k}(x) \right|^r \left| f^{\Delta^m}(x) \right|^s \right] \Delta x \\ & \leq \left(\int_a^b h(x)^{t/(t-s)} \Delta x \right)^{1-\alpha} \\ & \quad \cdot 2^{1-\alpha} \left(\int_a^b [F^\Delta(x) G^\beta(x) + G^\Delta(x) F^\beta(x)] \Delta x \right)^\alpha. \end{aligned} \quad (23)$$

Using (16),

$$\begin{aligned}
 & \int_a^b \left[F^\Delta(x) G^\beta(x) + G^\Delta(x) F^\beta(x) \right] \Delta x \\
 &= \int_a^b \left(G^\beta(x) + F^\beta(x) \right) \left(G^\Delta(x) + F^\Delta(x) \right) \Delta x \\
 &\quad - \int_a^b \left(G^\beta(x) G^\Delta(x) + F^\beta(x) F^\Delta(x) \right) \Delta x \\
 &\leq d_\beta \int_a^b \left(G(x) + F(x) \right)^\beta \Delta \left(G(x) + F(x) \right) \\
 &\quad - \int_a^b G^\beta(x) \Delta G(x) - \int_a^b F^\beta(x) \Delta F(x) \\
 &= d_\beta \Gamma(G + F) - \Gamma(G) - \Gamma(F).
 \end{aligned} \tag{24}$$

The proof is complete. □

Remark 5. In the special case where $\mathbb{T} = \mathbb{R}$, Theorem 4 reduces to Theorem 1 of [13].

Theorem 6. Let $f \in C_{rd}^{m-1}(\Upsilon)$, $\Upsilon := [a, b] \cap \mathbb{T}$ be such that $f^{\Delta^k}(a) = 0, 0 \leq k \leq m-1$, let $f^{\Delta^{m-1}}(x)$ be absolutely continuous on Υ , and let $\int_a^b |f^{\Delta^m}(x)|^2 \Delta x < \infty$. Then

$$\begin{aligned}
 & \int_a^b \left| f^{\Delta^k}(x) f^{\Delta^m}(x) \right| \Delta x \\
 &\leq \left(\int_a^b \int_a^x h_{m-k-1}^2(x, \sigma(\tau)) \Delta \tau \Delta x \right)^{1/2} \\
 &\quad \cdot \left(\int_a^b \left| f^{\Delta^m}(\tau) \right|^2 \Delta \tau \right)^{1/2}.
 \end{aligned} \tag{25}$$

Proof. From the hypotheses, we have

$$\left| f^{\Delta^k}(x) \right| \leq \int_a^x h_{m-k-1}(x, \sigma(\tau)) \left| f^{\Delta^m}(\tau) \right| \Delta \tau. \tag{26}$$

Multiplying (26) by $|f^{\Delta^m}(x)|$ and using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
 & \left| f^{\Delta^k}(x) f^{\Delta^m}(x) \right| \\
 &\leq \left| f^{\Delta^m}(x) \right| \int_a^x h_{m-k-1}(x, \sigma(\tau)) \left| f^{\Delta^m}(\tau) \right| \Delta \tau \\
 &\leq \left| f^{\Delta^m}(x) \right| \left(\int_a^x h_{m-k-1}^2(x, \sigma(\tau)) \Delta \tau \right)^{1/2} \\
 &\quad \cdot \left(\int_a^x \left| f^{\Delta^m}(\tau) \right|^2 \Delta \tau \right)^{1/2}.
 \end{aligned} \tag{27}$$

Integrating both sides over x from a to b and using Cauchy-Schwarz inequality, we observe

$$\begin{aligned}
 & \int_a^b \left| f^{\Delta^k}(x) f^{\Delta^m}(x) \right| \Delta x \\
 &\leq \left[\int_a^b \int_a^x h_{m-k-1}^2(x, \sigma(\tau)) \Delta \tau \Delta x \right]^{1/2} \\
 &\quad \cdot \left[\int_a^b \left| f^{\Delta^m}(x) \right|^2 \int_a^x \left| f^{\Delta^m}(\tau) \right|^2 \Delta \tau \Delta x \right]^{1/2} \\
 &= \left[\int_a^b \int_a^x h_{m-k-1}^2(x, \sigma(\tau)) \Delta \tau \Delta x \right]^{1/2} \\
 &\quad \cdot \left[\int_a^b \left| f^{\Delta^m}(\tau) \right|^2 \Delta \tau \right]^{1/2}.
 \end{aligned} \tag{28}$$

The proof is complete. □

Theorem 7. Let $p(x) > 0, q(x)$ be nonnegative and measurable on $\Upsilon = [a, b] \cap \mathbb{T}$, and let $f \in C_{rd}^{m-1}(\Upsilon)$ be such that $f^{\Delta^k}(a) = 0, 0 \leq k \leq m-1$. If $f^{\Delta^{m-1}}(x)$ is absolutely continuous on Υ , then for $r > 1, r_k > 0$, and any $0 \leq r_m < r$,

$$\begin{aligned}
 & \int_a^b q(x) \left| f^{\Delta^m}(x) \right|^{r_m} \left| f^{\Delta^k}(x) \right|^{r_k} \Delta x \\
 &\leq \left[\int_a^b q(x)^{r/(r-r_m)} p(x)^{-r_m/(r-r_m)} \right. \\
 &\quad \left. \times Q(x)^{(r-1)r_m/(r-r_m)} \Delta x \right]^{(r-r_m)/r} \Phi(y)^{r_m/r},
 \end{aligned} \tag{29}$$

where $Q(x) := \int_a^x h_{m-k-1}^{r/(r-1)}(x, \sigma(\tau)) p(\tau)^{-r/(r-1)} \Delta \tau$, $\Phi(y) := \int_a^b y^{r_k/r_m} \Delta y(x)$, $y(x) := \int_a^x p(\tau) \left| f^{\Delta^m}(\tau) \right|^r \Delta \tau$.

Proof. Following the hypotheses, it is easy to see that (26) holds. By using Hölder's inequality with indices r and $r/(r-1)$, we obtain

$$\begin{aligned}
 & \left| f^{\Delta^k}(x) \right| \\
 &\leq \int_a^x h_{m-k-1}(x, \sigma(\tau)) p(\tau)^{-1/r} p(\tau)^{1/r} \left| f^{\Delta^m}(\tau) \right| \Delta \tau \\
 &\leq \left[\int_a^x h_{m-k-1}^{r/(r-1)}(x, \sigma(\tau)) p(\tau)^{-r/(r-1)} \Delta \tau \right]^{(r-1)/r} \\
 &\quad \cdot \left[\int_a^x p(\tau) \left| f^{\Delta^m}(\tau) \right|^r \Delta \tau \right]^{1/r} \\
 &= Q(x)^{(r-1)/r} y(x)^{1/r},
 \end{aligned} \tag{30}$$

where $Q(x) := \int_a^x h_{m-k-1}^{r/(r-1)}(x, \sigma(\tau)) p(\tau)^{-r/(r-1)} \Delta \tau$, $y(x) := \int_a^x p(\tau) \left| f^{\Delta^m}(\tau) \right|^r \Delta \tau$. So we get

$$y^\Delta(x) = p(x) \left| f^{\Delta^m}(x) \right|^r, \tag{31}$$

and hence for any r_m ,

$$\left| f^{\Delta^m}(x) \right|^{r_m} = (p(x))^{-r_m/r} (y^\Delta(x))^{r_m/r}. \tag{32}$$

Thus for $r_k > 0$,

$$\begin{aligned} q(x) \left| f^{\Delta^m}(x) \right|^{r_m} \left| f^{\Delta^k}(x) \right|^{r_k} \\ \leq q(x) (p(x))^{-r_m/r} (y^\Delta(x))^{r_m/r} \\ \times Q(x)^{(r-1)r_k/r} y(x)^{r_k/r}. \end{aligned} \tag{33}$$

Integrating both sides of (33) from a to b and applying Hölder's inequality with indices r/r_m and $r/(r-r_m)$, we obtain

$$\begin{aligned} & \int_a^b q(x) \left| f^{\Delta^m}(x) \right|^{r_m} \left| f^{\Delta^k}(x) \right|^{r_k} \Delta x \\ & \leq \int_a^b q(x) p(x)^{-r_m/r} (y^\Delta(x))^{r_m/r} Q(x)^{(r-1)r_k/r} y(x)^{r_k/r} \Delta x \\ & \leq \left[\int_a^b q(x)^{r/(r-r_m)} p(x)^{-r_m/(r-r_m)} Q(x)^{(r-1)r_k/(r-r_m)} \Delta x \right]^{(r-r_m)/r} \\ & \quad \cdot \left[\int_a^b y^\Delta(x) y(x)^{r_k/r_m} \Delta x \right]^{r_m/r} \\ & = \left[\int_a^b q(x)^{r/(r-r_m)} p(x)^{-r_m/(r-r_m)} \right. \\ & \quad \left. \times Q(x)^{(r-1)r_k/(r-r_m)} \Delta x \right]^{(r-r_m)/r} \Phi(y)^{r_m/r}, \end{aligned} \tag{34}$$

where $\Phi(y) := \int_a^b y(x)^{r_k/r_m} \Delta y(x)$. The proof is complete. \square

Remark 8. In the special case where $\mathbb{T} = \mathbb{R}$, Theorems 6 and 7 reduce to Theorems 1 and 2 of [18].

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