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Research Article

Algebraic Structures Based on a Classifying Space of a Compact Lie Group

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We analyze the algebraic structures based on a classifying space of a compact Lie group. We construct the connected graded free Lie algebra structure by considering the rationally nontrivial indecomposable and decomposable generators of homotopy groups and the cohomology cup products, and we show that the homomorphic image of homology generators can be expressed in terms of the Lie brackets in rational homology. By using the Milnor-Moore theorem, we also investigate the concrete primitive elements in the Pontrjagin algebra.

1. Introduction

A Lie group is a differentiable manifold M with a group structure in which the multiplication $M \times M \to M$ and the inversive map $M \to M$ $(m \mapsto m^{-1})$ are differentiable. Therefore, it can be studied using differential calculus in contrast with the case of more general topological groups as a special case of H-spaces. It is well known that the only spheres that are connected H-spaces are S^1, S^3, and S^7. We note that the first two spheres are Lie groups while the last one is just an H-space which is not an A_3-space but just an A_2-space in the sense of Stasheff [1]. Lie groups play an enormous role in algebraic topology as well as modern differential geometry on several different levels. The presence of continuous symmetries expressed via a Lie group action on a manifold places strong constraints on its geometry and facilitates analysis on the manifold. Moreover, linear actions of Lie groups are especially important and are studied in representation theory.

As usual we let Σ and Ω be the suspension and loop functors in the (pointed) homotopy category, respectively. It is well known that the functors Σ and Ω are examples of adjoint functors. Moreover, co-H-spaces and H-spaces are important objects of research in homotopy theory and they are the dual notions in the sense of Eckmann and Hilton. We refer to Arkowitz's paper [2] and Scheerer's article [3] for a survey of the vast literature about co-H-spaces, H-spaces, and related topics.

Let $SNT(X)$ denote the set of all homotopy types $[Y]$ such that X and Y have the same n-type for each nonnegative integer n (see [4–6]). McGibbon and Møller [7] showed that if G is a connected compact Lie group, then its classifying space usually has an uncountable $SNT(BG)$ except for several cases and gave an excellent set of examples. Furthermore, in [8] the classical projective n-spaces (real, complex, and quaternionic) were studied in terms of their self-maps from a homotopy point of view. Recently, some common fixed point results for single as well as set valued mappings involving certain rational expressions in complete partial metric spaces were obtained in [9]. Moreover, some fixed point and common fixed point theorems on ordered cone b-metric spaces were also established in [10].

In this paper all spaces are based and have the based homotopy type of based, connected CW-complexes. All maps and homotopies preserve the base point. Unless otherwise stated, we do not distinguish notationally between a map and its homotopy class.

The main purpose of this paper is to investigate the algebraic explanation based on a classifying space of the compact Lie group $U(1)$. After constructing self-maps using the suspension structure, we define a useful commutator of self-maps on the suspension of a classifying space of the compact

Lie group. We construct the connected graded free Lie algebra structure by considering the rationally nontrivial indecomposable and decomposable generators of homotopy groups and the cohomology cup products. We show that the homomorphic image of homology generators can be expressed in terms of the Lie brackets in rational homology. By using the Milnor-Moore theorem, we also investigate the concrete primitive elements as the images of the Hurewicz homomorphisms in the Pontrjagin algebra.

2. Preliminaries

Let $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the ring of integers, the fields of rational, real, and complex numbers, respectively. The *unitary group* $U(n)$ is the group of $n \times n$ unitary matrices with the group operation that of matrix multiplication as a subgroup of the general linear group $\mathrm{GL}(n, \mathbb{C})$. The unitary group $U(n)$ is a real Lie group of dimension n^2. The Lie algebra of $U(n)$ consists of $n \times n$ skew-Hermitian matrices with the Lie bracket given by the commutator. The *first unitary group* $U(1)$ is canonically the circle group consisting of all complex numbers with absolute value 1 under multiplication; that is, the set of all 1×1 unitary matrices. The first unitary group shows up in a variety of forms in mathematics. Some of the more common forms are as follows:

$$U(1) \cong T \cong \mathbb{R}/\mathbb{Z} \cong \mathrm{SO}(2), \tag{1}$$

where T is a one torus and SO(2) is the orthogonal 2×2 matrices with determinant 1. The latter is said to be the *special orthogonal group*.

A *classifying space* BG of a topological group G is the quotient of a weakly contractible space EG by a free action of G. It has the property that any principal G-bundle over a paracompact manifold is isomorphic to a pullback of the principal bundle $EG \to BG$. We note that the classifying space functor B is essentially inverse to the loop space functor Ω in algebraic topology.

If X is a co-H-group with comultiplication

$$\varphi : X \longrightarrow X \vee X \tag{2}$$

and homotopy inverse, then for every pointed space Y the set $[X, Y]$ of homotopy classes from X to Y can be given the structure of a group under the addition as follows: for $a, b \in [X, Y]$, the binary operation, denoted "$a + b$", is defined by the homotopy class of the composition of maps

$$X \xrightarrow{\varphi} X \vee X \xrightarrow{a \vee b} Y \vee Y \xrightarrow{\nabla} Y, \tag{3}$$

where ∇ is a folding map.

The Eckmann-Hilton dual of a co-H-group is an H-group (see [11, 12]). As an adjointness, if X is any pointed space and (Y, m) is an H-group, then the set $[X, Y]$ becomes a group if we define the product "$c \cdot d$" to be the homotopy class of the composition of maps

$$X \xrightarrow{\Delta} X \times X \xrightarrow{c \times d} Y \times Y \xrightarrow{m} Y, \tag{4}$$

where $\Delta : X \to X \times X$ is the diagonal map, $c, d \in [X, Y]$, and $m : X \times X \to X$ is a multiplication.

The principle examples of a co-H-group and an H-group are the suspension ΣX and the loop space ΩX of a space X, respectively.

We note that $BU(1)$ has a CW-decomposition as follows:

$$BU(1) = S^2 \cup_{\gamma_1} e^4 \cup_{\gamma_2} e^6 \cup_{\gamma_3} \cdots \cup_{\gamma_{n-1}} e^{2n} \cup_{\gamma_n} e^{2(n+1)} \cup_{\gamma_{n+1}} \cdots, \tag{5}$$

where γ_n is an attaching map for $n = 1, 2, 3, \ldots$.

From now on, we denote $BU(1)_k$ by the k-skeleton of a CW-complex $BU(1)$. We define maps $f_n : BU(1) \to \Omega\Sigma BU(1)$ for $n = 1, 2, 3, \ldots$ as follows.

Definition 1. The cofibration sequence

$$BU(1)_{2n-1} \overset{i_n}{\hookrightarrow} BU(1) \xrightarrow{p_n} \frac{BU(1)}{BU(1)_{2n-1}} \tag{6}$$

induces an exact sequence of groups

$$\begin{aligned} \left[\frac{BU(1)}{BU(1)_{2n-1}}, \Omega\Sigma BU(1)\right] &\xrightarrow{p_n^\sharp} [BU(1), \Omega\Sigma BU(1)] \\ &\xrightarrow{i_n^\sharp} \left[BU(1)_{2n-1}, \Omega\Sigma BU(1)\right], \end{aligned} \tag{7}$$

for each $n = 1, 2, 3, \ldots$. We now take an essential map

$$f_n \in {i_n^\sharp}^{-1}(*) = \ker\left(i_n^\sharp\right) \subset [BU(1), \Omega\Sigma BU(1)], \tag{8}$$

for each $n = 1, 2, 3, \ldots$. Similarly, by using the above exact sequences, we can choose an essential map

$$\widetilde{f}_n : \frac{BU(1)}{BU(1)_{2n-1}} \longrightarrow \Omega\Sigma BU(1), \tag{9}$$

such that $p_n^\sharp(\widetilde{f}_n) = f_n$ for $n = 1, 2, 3, \ldots$.

In the above definition, we note that $BU(1)_1 = *$, and thus $p_1^\sharp$ is an isomorphism. We also note that

$$\frac{BU(1)}{BU(1)_{2n-1}} = S^{2n} \cup \text{higher cells}. \tag{10}$$

We now define the following.

Definition 2. We define a rationally nontrivial homotopy element

$$\xi_n : S^{2n} \longrightarrow \Omega\Sigma BU(1) \tag{11}$$

of the homotopy groups modulo torsions $\pi_{2n}(\Omega\Sigma BU(1))/\mathrm{torsion}$ by $\xi_n = \widetilde{f}_n|_{S^{2n}}$ for each $n = 1, 2, 3, \ldots$.

Now we take the self-map

$$F_n : \Sigma BU(1) \longrightarrow \Sigma BU(1) \tag{12}$$

by the adjointness of

$$f_n : BU(1) \longrightarrow \Omega\Sigma BU(1) \tag{13}$$

for each $n = 1, 2, 3, \ldots$.

We recall that

$$\widetilde{H}_*(BU(1);\mathbb{Q}) \cong \mathbb{Q}\{u_1, u_2, \ldots, u_n, \ldots\} \tag{14}$$

as a graded $\mathbb{Q}$-module, where u_n is the standard generator of $H_{2n}(BU(1);\mathbb{Q})$ for each $n = 1, 2, 3, \ldots$. The salient Bott-Samelson theorem [13] says that the Pontrjagin algebra $H_*(\Omega\Sigma BU(1);\mathbb{Q})$ is isomorphic to the tensor algebra $TH_*(BU(1);\mathbb{Q})$. In other words, the rational homology of $\Omega\Sigma K$ is the tensor algebra $T\{u_1, u_2, \ldots, u_n, \ldots\}$ generated by $\{u_1, u_2, \ldots, u_n, \ldots\}$, where u_n is a rational homology generator with diagonal

$$\Delta(u_k) = \sum_{i+j=k} u_i \otimes u_j \tag{15}$$

and $u_n = E_*(u_n)$. Here, E is the canonical inclusion map $E : BU(1) \to \Omega\Sigma BU(1)$ defined by $E(s)(t) = \langle s, t\rangle \in \Sigma BU(1)$, and u_0 means 1. From the Serre spectral sequence of a fibration $\Omega BU(1) \to PBU(1) \to BU(1)$, we have an algebra isomorphism

$$H^*(BU(1);\mathbb{Q}) \cong \mathbb{Q}[\alpha]. \tag{16}$$

Here $\mathbb{Q}[\alpha]$ is the polynomial algebra over $\mathbb{Q}$ generated by α of degree 2; that is, α is a generator of $H^2(BU(1);\mathbb{Q})$ with the Kronecker index $\langle \alpha^i, u_j\rangle = \delta_{ij}$.

We now consider a wedge of spheres $X = S^{n_1} \vee S^{n_2} \vee \cdots \vee S^{n_k}$ for $2 \le n_1 \le n_2 \le \cdots \le n_k$. Let $r_t : S^{n_t} \to X$ be the tth inclusion for $t = 1, 2, \ldots, k$. We then inductively define and order *basic (Whitehead) products* as follows. Basic products of weight 1 are (in order) $r_1, r_2, \ldots, r_k$. Assume basic products of weight $< n$ have been defined and ordered so that if $r < s < n$, any basic product of weight r is less than all basic products of weight s. Then a basic product of weight n is a Whitehead product $[a, b]$, where a is a basic product of weight m and b is a basic product of weight l, $m + l = n$, $a < b$. Furthermore, if b is a Whitehead product $[c, d]$ of basic products c and d, then we require that $c \le a$. The basic products of weight n are ordered arbitrarily among themselves and are greater than any basic product of weight $< n$. Note that to a basic product of weight n we can associate a string of distinct symbols $r_{v_1}, \ldots, r_{v_s}$, for $1 \le v_i \le k$, which are the elements which appear in the basic products. Suppose in the basic product w_s, r_p occurs l_p times, $l_p \ge 1$. Then the *height* of the basic product is $\sum l_p(r_p - 1) + 1$ and the *length* is $\sum l_p - 1$. Clearly if w_s has height h_s, then $w_s \in \pi_{h_s}(X)$.

We end this section with the following Hilton's formula [14].

Theorem 3. *Let the ordered basic products of $X = S^{n_1} \vee S^{n_2} \vee \cdots \vee S^{n_k}$ be $w_1, w_2, \ldots, w_s, \ldots$ with the height of $w_s = h_s$. Then for every m,*

$$\pi_m(X) \approx \bigoplus_{s=1}^{\infty} \pi_m\left(S^{h_s}\right). \tag{17}$$

The isomorphism $\theta : \bigoplus_{s=1}^{\infty} \pi_m(S^{h_s}) \to \pi_m(X)$ is defined by

$$\theta|\,\pi_m\left(S^{h_s}\right) = w_{s*} : \pi_m\left(S^{h_s}\right) \longrightarrow \pi_m(X). \tag{18}$$

We note that the direct sum is finite for each m since $h_s \to \infty$.

3. Commutators and Lie Algebra Structures

By using the addition of a co-H-group in Section 2, we define the following.

Definition 4. We define a *commutator*

$$C\left(F_{i_1}, F_{i_2}\right) : \Sigma BU(1) \longrightarrow \Sigma BU(1) \tag{19}$$

of F_{i_1} and F_{i_2} in $[\Sigma BU(1), \Sigma BU(1)]$ by

$$C\left(F_{i_1}, F_{i_2}\right) = F_{i_1} + F_{i_2} - F_{i_1} - F_{i_2}, \tag{20}$$

where the operations are the suspension additions on $\Sigma BU(1)$, and

$$-F : \Sigma BU(1) \longrightarrow \Sigma BU(1) \tag{21}$$

is the suspension inverse defined by

$$-F(\langle s, t\rangle) = F(\langle s, 1 - t\rangle) \tag{22}$$

for $s \in BU(1)$ and $t \in I$.

Let

$$l : \Omega\Sigma BU(1) \longrightarrow \Omega\Sigma BU(1) \tag{23}$$

be the map of loop inverse given by $l(\omega) = \omega^{-1}$, where $\omega^{-1}(t) = \omega(1 - t)$, $t \in I$.

Similarly, we define the following.

Definition 5. The map $C(f_{i_1}, f_{i_2}) : BU(1) \to \Omega\Sigma BU(1)$ is a commutator of f_{i_1} and f_{i_2} in $[BU(1), \Omega\Sigma BU(1)]$ defined by

$$C\left(f_{i_1}, f_{i_2}\right)(s) = f_{i_1}(s) \cdot f_{i_2}(s) \cdot l\left(f_{i_1}(s)\right) \cdot l\left(f_{i_2}(s)\right), \tag{24}$$

where $s \in BU(1)$, and the multiplication is the loop multiplication.

Remark 6. Let $c(f_{i_1}, f_{i_2}) : BU(1) \times BU(1) \to \Omega\Sigma BU(1)$ be a map given by

$$c\left(f_{i_1}, f_{i_2}\right)(s, t) = f_{i_1}(s) \cdot f_{i_2}(t) \cdot l\left(f_{i_1}(s)\right) \cdot l\left(f_{i_2}(t)\right). \tag{25}$$

Then we get

$$C\left(f_{i_1}, f_{i_2}\right) = c\left(f_{i_1}, f_{i_2}\right) \circ \Delta, \tag{26}$$

where $\Delta : BU(1) \to BU(1) \times BU(1)$ is the diagonal map.

Note that the weak category of $\Omega\Sigma BU(1)$ is not finite because there are infinitely many nonzero cohomology cup products in it, and thus it has the infinite Lusternik-Schnirelmann category [12, Chapter X]. Moreover, Arkowitz and Curjel [15, Theorem 5] showed that the n-fold commutator is of finite order if and only if all n-fold cup products of any positive dimensional rational cohomology classes of a space vanish (see also [16]). Therefore, we can consider the iterated commutators which are nontrivial in $\Omega\Sigma BU(1)$.

We recall that the Samelson product gives $\pi_*(\Omega X)$, $* \geq 1$, the structure of graded Lie algebra (see [17, 18]); that is, if $x \in \pi_p(\Omega X)$, $y \in \pi_q(\Omega X)$ and $z \in \pi_r(\Omega X)$, then

$$\begin{gathered}\langle x, y\rangle = (-1)^{p(q-1)} \langle y, x\rangle, \\ (-1)^{pr} \langle x, \langle y, z\rangle\rangle + (-1)^{pq} \langle y, \langle z, x\rangle\rangle \\ + (-1)^{qr} \langle z, \langle x, y\rangle\rangle = 0.\end{gathered} \tag{27}$$

Let $k_1 : X \to X\times X$ and $k_2 : X \to X\times X$ be the first and second inclusions between based spaces, respectively; that is, $k_1(x) = (x, x_0)$ and $k_2(x) = (x_0, x)$, where x_0 is the base point of X. Recall that an element $z \in H_*(X)$ is said to be *primitive* if and only if $\Delta_*(z) = k_{1*}(z) + k_{2*}(z) = z \otimes 1 + 1 \otimes z$ in homology, where $\Delta : X \to X \times X$ is the diagonal map.

Let $h : \pi_*(\Omega X) \to H_*(\Omega X;\mathbb{Q})$ be the Hurewicz homomorphism. In 1965, Milnor and Moore [19] proved the following salient theorem (see also [18, page 293]).

Theorem 7. *If X is a simply connected topological space and if $\mathbb{F}$ is a field of characteristic zero, then*

(1) *the Samelson product makes $\pi_*(\Omega X)\otimes\mathbb{F}$ into a graded Lie algebra denoted by L_X;*

(2) *the Hurewicz homomorphism for ΩX is an isomorphism of L_X onto the Lie algebra $P_*(\Omega X;\mathbb{F})$ of primitive elements in $H_*(\Omega X;\mathbb{F})$;*

(3) *the Hurewicz homomorphism extends to an isomorphism of graded Hopf algebras $UL_X \cong H_*(\Omega X;\mathbb{F})$, where UL_X is the universal enveloping algebra of L_X.*

It is natural to ask what are the rationally nontrivial indecomposable and decomposable generators of the graded Lie algebra for $\Omega\Sigma BU(1)$? The following gives an answer to this question.

Theorem 8. *The connected graded Lie algebra for $\Omega\Sigma BU(1)$ with the Samelson products modulo torsions is as follows:*

$$\frac{\pi_*(\Omega\Sigma BU(1))}{\textit{torsion}} = L\{\xi_1, \xi_2, \dots, \xi_n, \dots\}, \tag{28}$$

where the dimension of ξ_n is equal to $2n$ in the graded homotopy groups for each $n = 1, 2, 3, \dots$.

Proof. It suffices to show that the iterated Samelson products $\langle\xi_{i_k}, \langle\xi_{i_{k-1}}, \dots, \langle\xi_{i_1}, \xi_{i_2}\rangle\cdots\rangle\rangle$ in homotopy groups $\pi_{2(i_1+\cdots+i_k)}(\Omega\Sigma BU(1))$ are rationally nontrivial decomposable generators, where the ξ_{i_j} are indecomposable generators in dimension $2i_j$, for $i_j = 1, 2, 3, \dots$ and $k \geq 2$.

We first note that the Eckmann-Hilton dual of the Hopf-Thom theorem (see [11, pages 263–269] and [12, Chapter III]) says that $\Sigma BU(1)$ has the rational homotopy type of the wedge products of infinitely many spheres; that is,

$$\Sigma BU(1) \simeq_{\mathbb{Q}} S^3 \vee S^5 \vee S^7 \vee \cdots \vee S^{2n+1} \vee \cdots. \tag{29}$$

Let $\hat{\xi}_{i_j} : S^{2i_j+1} \to \Sigma BU(1)$ be the adjoint of $\xi_{i_j} : S^{2i_j} \to \Omega\Sigma BU(1)$. We prove the result in the case of twofold Samelson products. Suppose that $\langle\xi_{i_1}, \xi_{i_2}\rangle$ is rationally trivial in $\pi_{2(i_1+i_2)}(\Omega\Sigma BU(1))$. It follows by the adjointness and the Hilton's formula that the twofold Whitehead products $[\hat{\xi}_{i_1}, \hat{\xi}_{i_2}]$ have a finite order in $\pi_{2(i_1+i_2)+1}(S^{2i_1+1} \vee S^{2i_2+1})$ which is a subgroup of $\pi_{2(i_1+i_2)+1}(\Sigma BU(1))$. By using a cofibration sequence

$$\begin{gathered}S^{2(i_1+i_2)+1} \xrightarrow{[\hat{\xi}_{i_1},\hat{\xi}_{i_2}]} S^{2i_1+1} \vee S^{2i_2+1} \longrightarrow S^{2i_1+1} \times S^{2i_2+1} \\ \longrightarrow S^{2i_1+1} \wedge S^{2j_1+1} \longrightarrow \cdots,\end{gathered} \tag{30}$$

we have

$$S^{2i_1+1} \times S^{2i_2+1} \simeq \left(S^{2i_1+1} \vee S^{2i_2+1}\right) \cup_{[\hat{\xi}_{i_1},\hat{\xi}_{i_2}]} e^{2(i_1+i_2)+2}, \tag{31}$$

where $[\hat{\xi}_{i_1}, \hat{\xi}_{i_2}] : S^{2(i_1+i_2)+1} \to S^{2i_1+1} \vee S^{2i_2+1}$ is the attaching map. The assumption shows that $[\hat{\xi}_{i_1}, \hat{\xi}_{i_2}]_{\mathbb{Q}} \simeq *$ when $[\hat{\xi}_{i_1}, \hat{\xi}_{i_2}]$ is rationalized, and thus

$$S^{2i_1+1} \times S^{2i_2+1} \simeq_{\mathbb{Q}} S^{2i_1+1} \vee S^{2i_2+1} \vee S^{2(i_1+i_2)+2}, \tag{32}$$

where "$\simeq_{\mathbb{Q}}$" is a rational homotopy equivalence. We also have a contradiction by applying the cohomology cup products to the above rational homotopy equivalence.

For induction, we now suppose that the $(k-1)$-fold Samelson products $\langle\xi_{i_{k-1}}, \dots, \langle\xi_{i_1}, \xi_{i_2}\rangle\cdots\rangle$ are rationally nontrivial in the homotopy group $\pi_{2(i_1+\cdots+i_{k-1})}(\Omega\Sigma BU(1))$. By Theorem 3 and adjointness again, we can consider the iterated Whitehead product $[\hat{\xi}_{i_{k-1}}, \dots, [\hat{\xi}_{i_1}, \hat{\xi}_{i_2}]\cdots]$ as a rational generator of

$$\begin{gathered}\pi_{2(i_1+\cdots+i_{k-1})+1}\left(S^{2(i_1+\cdots+i_{k-1})+1}\right) \otimes \mathbb{Q} \\ \subset \pi_{2(i_1+\cdots+i_{k-1})+1}(\Sigma BU(1)) \otimes \mathbb{Q}.\end{gathered} \tag{33}$$

Similarly, a cofibration shows that

$$\begin{aligned}&S^{2(i_1+\cdots+i_{k-1})+1} \times S^{2i_k+1} \\ &\quad\simeq \left(S^{2(i_1+\cdots+i_{k-1})+1} \vee S^{2i_k+1}\right) \cup_{\zeta} e^{2(i_1+\cdots+i_{k-1}+i_k)+2},\end{aligned} \tag{34}$$

where $\zeta = [\hat{\xi}_{i_k}, [\hat{\xi}_{i_{k-1}}, \dots, [\hat{\xi}_{i_1}, \hat{\xi}_{i_2}]\cdots]] : S^{2(i_1+\cdots+i_k)+1} \to S^{2(i_1+\cdots+i_{k-1})+1} \vee S^{2i_k+1}$ is the attaching map. If this map has a finite order in homotopy groups, then we have a contradiction again by the same argument of the cohomology cup products.

Finally by taking the adjointness, we complete the proof. □

Let $\bar{\xi}_n : S^{2n} \to \Omega\Sigma BU(1)_{\mathbb{Q}}$ be the composition $r \circ \xi_n$ of the rationally nontrivial indecomposable element $\xi_n : S^{2n} \to \Omega\Sigma BU(1)$ of $\pi_{2n}(\Omega\Sigma BU(1))$ with the rationalization $r : \Omega\Sigma BU(1) \to \Omega\Sigma BU(1)_{\mathbb{Q}}$ for each $n = 1, 2, \dots$. Since there is a one-to-one correspondence between the integral generators of homotopy groups modulo torsions and rational generators of rational homotopy groups; that is, $\mathrm{rank}_{\mathbb{Z}}(\pi_{2n+1}(\Sigma BU(1))/\text{torsion}) = \mathrm{rank}_{\mathbb{Q}}(\pi_{2n+1}(\Sigma BU(1)) \otimes \mathbb{Q})$, by using the Milnor-Moore theorem and Theorem 8, we have the following.

Corollary 9. *The graded rational homotopy group $\pi_*(\Omega\Sigma BU(1)) \otimes \mathbb{Q}$ with the Samelson products becomes* *a connected graded free Lie algebra generated by* $\{\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_n, \ldots\}$*; that is,*

$$\pi_* (\Omega\Sigma BU(1)) \otimes \mathbb{Q} \cong L\left\{\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_n, \ldots\right\}. \tag{35}$$

We note that the iterated Samelson products $\langle \bar{\xi}_{i_k}, \langle \bar{\xi}_{i_{k-1}}, \ldots, \langle \bar{\xi}_{i_1}, \bar{\xi}_{i_2}\rangle \cdots \rangle\rangle$ are decomposable generators in the above free Lie algebra.

It is well known that the Hurewicz homomorphism $h : \pi_*(\Omega\Sigma BU(1)) \to H_*(\Omega\Sigma BU(1); \mathbb{Q})$ carries the Samelson product into the Lie bracket defined by

$$[z, w] = zw - (-1)^{|z||w|} wz, \tag{36}$$

where $z, w \in H_*(\Omega\Sigma BU(1); \mathbb{Q})$, and the multiplication is the Pontrjagin multiplication.

Theorem 10. *Let* $\widehat{C(F_{i_1}, F_{i_2})} : BU(1) \to \Omega\Sigma BU(1)$ *be the adjoint of* $C(F_{i_1}, F_{i_2}) : \Sigma BU(1) \to \Sigma BU(1)$*. Then one has the following:*

(1) $\widehat{C(F_{i_1}, F_{i_2})} = C(f_{i_1}, f_{i_2})$*;*

(2) $C(f_{i_1}, f_{i_2})_*(u_{i_1+i_2}) = [h(\xi_{i_1}), h(\xi_{i_2})]$*, where* $u_{i_1+i_2}$ *is a rational homology generator in dimension* $2(i_1 + i_2)$*,* $[h(\xi_{i_1}), h(\xi_{i_2})]$ *is the Lie bracket, and* ξ_{i_1} *and* ξ_{i_2} *are homotopy elements of* $\pi_*(\Omega\Sigma BU(1))$*.*

Proof. (1) The adjointness shows that, for $F \in [\Sigma BU(1), \Sigma BU(1)]$, $s \in BU(1)$, $t \in I$, and $\langle s, t\rangle \in \Sigma BU(1)$, the map

$$\kappa = \widehat{\ } : [\Sigma BU(1), \Sigma BU(1)] \longrightarrow [BU(1), \Omega\Sigma BU(1)], \tag{37}$$

defined by

$$(\kappa F)(s)(t) = \widehat{F}(s)(t) = F(\langle s, t\rangle), \tag{38}$$

is a group isomorphism. It thus follows that

$$\widehat{C(F_{i_1}, F_{i_2})} = C\left(\widehat{F}_{i_1}, \widehat{F}_{i_2}\right) = C\left(f_{i_1}, f_{i_2}\right), \tag{39}$$

which is the commutator of f_{i_1} and f_{i_2} in $[BU(1), \Omega\Sigma BU(1)]$.

(2) For the second part, if $p : BU(1)_{2(i_1+i_2)} \to S^{2i_1+2i_2}$ is the projection map to the top cell and if $\langle , \rangle$ is the Samelson product in $\pi_*(\Omega\Sigma BU(1))$, then the following diagram is commutative up to homotopy (see also [20, Theorem 1.4]):

$$\begin{tikzcd} BU(1)_{2(i_1+i_2)} \arrow[r, hook, "i"] \arrow[d, "p"] & BU(1) \arrow[d, "C(f_{i_1}, f_{i_2})"] \\ S^{2i_1+2i_2} \arrow[r, "\langle \xi_{i_1}, \xi_{i_2}\rangle"] & \Omega\Sigma BU(1). \end{tikzcd} \tag{40}$$

By applying the rational homology to the above diagram, we have

$$\begin{aligned} C\left(f_{i_1}, f_{i_1}\right)_* \left(u_{(i_1+i_2)}\right) &= C\left(f_{i_1}, f_{i_1}\right)_* i_* \left(u_{(i_1+i_2)}\right) \\ &= \left\langle \xi_{i_1}, \xi_{i_2}\right\rangle_* p_* \left(u_{(i_1+i_2)}\right) \\ &= \left\langle \xi_{i_1}, \xi_{i_2}\right\rangle_* \left(u'_{(i_1+i_2)}\right) \\ &= h\left(\left\langle \xi_{i_1}, \xi_{i_2}\right\rangle\right) \\ &= \left[h\left(\xi_{i_1}\right), h\left(\xi_{i_2}\right)\right]. \end{aligned} \tag{41}$$

Here,

(i) $u_{i_1+i_2}$ is also used as a generator of

$$H_{2(i_1+i_2)}\left(BU(1)_{2(i_1+i_2)}; \mathbb{Q}\right) \cong H_{2(i_1+i_2)}(BU(1); \mathbb{Q}); \tag{42}$$

(ii) $i_*(u_{(i_1+i_2)}) = u_{(i_1+i_2)}$;

(iii)

$$p_* : H_{2(i_1+i_2)}\left(BU(1)_{2(i_1+i_2)}; \mathbb{Q}\right) \longrightarrow H_{2(i_1+i_2)}\left(S^{2(i_1+i_2)}; \mathbb{Q}\right) \tag{43}$$

is an isomorphism sending the generator $u_{(i_1+i_2)}$ to the fundamental homology class $u'_{(i_1+i_2)}$.

□

Theorem 11. *Let* $E : BU(1) \to \Omega\Sigma BU(1)$ *be the canonical inclusion and let* $p_{i_1+i_2} : BU(1) \to BU(1)/BU(1)_{2i_1+2i_2-1}$ *be the projection map. Then for a given commutator* $C(F_{i_1}, F_{i_2}) : \Sigma BU(1) \to \Sigma BU(1)$*, there exists a map*

$$\overline{\Omega C(F_{i_1}, F_{i_2}) \circ E} : \frac{BU(1)}{BU(1)_{2i_1+2i_2-1}} \longrightarrow \Omega\Sigma BU(1) \tag{44}$$

such that $\overline{\Omega C(F_{i_1}, F_{i_2}) \circ E} \circ p_{i_1+i_2} = \Omega C(F_{i_1}, F_{i_2}) \circ E$*.*

Proof. We first show that the following diagram is strictly commutative:

$$\begin{tikzcd} BU(1) \arrow[r, "E"] \arrow[d, "C(f_{i_1}, f_{i_2})"'] & \Omega\Sigma BU(1) \arrow[dl, "\Omega C(F_{i_1}, F_{i_2})"] \\ \Omega\Sigma BU(1). & \end{tikzcd} \tag{45}$$

Indeed, the composition $\Omega C(F_{i_1}, F_{i_2}) \circ E : BU(1) \to \Omega\Sigma BU(1)$ induces a map

$$\left(\Omega C\left(F_{i_1}, F_{i_2}\right) \circ E\right)(s) : I \longrightarrow \Sigma BU(1) \tag{46}$$

sending $t \in I$ to

$$\left(\Omega C\left(F_{i_1}, F_{i_2}\right) \circ E\right)(s)(t) = C\left(F_{i_1}, F_{i_2}\right)(\langle s, t\rangle). \tag{47}$$

On the other hand, $C(f_{i_1}, f_{i_2}) : BU(1) \to \Omega\Sigma BU(1)$ sends $s \in BU(1)$ to

$$C\left(f_{i_1}, f_{i_2}\right)(s) : I \longrightarrow \Sigma BU(1). \tag{48}$$

By adjointness, we have

$$\begin{aligned}\left(C\left(f_{i_1}, f_{i_2}\right)(s)\right)(t) &= \left(C\left(\widehat{F_{i_1}, F_{i_2}}\right)(s)\right)(t)\\ &= C\left(F_{i_1}, F_{i_2}\right)(\langle s, t\rangle),\end{aligned} \tag{49}$$

where $\langle s, t\rangle$ is an element of $\Sigma BU(1)$. Therefore, we get

$$\Omega C\left(F_{i_1}, F_{i_2}\right) \circ E = C\left(f_{i_1}, f_{i_2}\right). \tag{50}$$

We now consider the cell structure on $BU(1)\times BU(1)$ with

$$(BU(1) \times BU(1))_{2i_1+2i_2-1} = \bigcup_{m+n=2i_1+2i_2-1} BU(1)_m \times BU(1)_n. \tag{51}$$

Since the restrictions $f_{i_1}|_{BU(1)_{2i_1-1}}$ and $f_{i_2}|_{BU(1)_{2i_2-1}}$ are inessential from our construction of $f_n : BU(1) \to \Omega\Sigma BU(1)$ for each $n = 1, 2, 3, \ldots$, we see that if $m \le 2i_1 - 1$ or $n \le 2i_2 - 1$, then $c(f_{i_1}, f_{i_2})|_{BU(1)_m\times BU(1)_n}$ is null homotopic. By using the homotopy extension property, we can extend the null homotopy to all of $(BU(1) \times BU(1))_{2i_1+2i_2-1}$. Thus, by the cellular approximation theorem, we have

$$\begin{aligned}\Omega C\left(F_{i_1}, F_{i_2}\right) \circ E\Big|_{BU(1)_{2i_1+2i_2-1}} &= C\left(f_{i_1}, f_{i_2}\right)\Big|_{BU(1)_{2i_1+2i_2-1}}\\ &= c\left(f_{i_1}, f_{i_2}\right) \circ \Delta|_{BU(1)_{2i_1+2i_2-1}}\\ &\simeq c\left(f_{i_1}, f_{i_2}\right) \circ \Delta'\Big|_{BU(1)_{2i_1+2i_2-1}}\\ &\simeq *,\end{aligned} \tag{52}$$

where $\Delta' : BU(1) \to BU(1) \times BU(1)$ is a cellular map. Therefore there exists a map

$$\overline{\Omega C(F_{i_1}, F_{i_2}) \circ E} : \frac{BU(1)}{BU(1)_{2i_1+2i_2-1}} \longrightarrow \Omega\Sigma BU(1) \tag{53}$$

such that the following diagram is commutative up to homotopy:

$$\begin{tikzcd} BU(1)_{2(i_1+i_2)-1} \arrow[r, hook, "i_{i_1+i_2}"] & BU(1) \arrow[r, "p_{i_1+i_2}"] \arrow[d, "\Omega C(F_{i_1}, F_{i_2})\circ E"'] & BU(1)/BU(1)_{2(i_1+i_2)-1} \arrow[dl, "\overline{\Omega C(F_{i_1}, F_{i_2})\circ E}"] \\ & \Omega\Sigma BU(1), & \end{tikzcd} \tag{54}$$

where the top row is a cofibration sequence as required. □

The Milnor-Moore theorem asserts that the image of the Hurewicz homomorphism is primitive. The following is another expression of the primitive elements in the Pontrjagin algebra.

Theorem 12. *The image of a homomorphism*

$$f_{n_*} : H_{2n}(BU(1)) \longrightarrow H_{2n}(\Omega\Sigma BU(1)) \tag{55}$$

is primitive for each $n = 1, 2, 3, \ldots$.

Proof. Since $f_n : BU(1) \to \Omega\Sigma BU(1)$, $n = 1, 2, 3, \ldots$ can be factored as

$$BU(1) \xrightarrow{p_n} \frac{BU(1)}{BU(1)_{2n-1}} \xrightarrow{\tilde{f}_n} \Omega\Sigma BU(1) \tag{56}$$

so that the restriction to the bottom sphere of the map $\tilde{f}_n$ coincides with the map $\xi_n : S^{2n} \to \Omega\Sigma BU(1)$, and since $BU(1)/BU(1)_{2n-1}$ is $(2n - 1)$-connected, by the Hurewicz isomorphism theorem, every class in $H_{2n}(BU(1)/BU(1)_{2n-1})$ is spherical, and thus primitive. We therefore know that the image of $\tilde{f}_{n_*}$ lies in the set of primitives $P_{2n}(\Omega\Sigma BU(1))$ in $H_{2n}(\Omega\Sigma BU(1))$, so does the image of f_{n_*} for each $n = 1, 2, 3, \ldots$. □

Since the restriction to the skeleton

$$\Omega C\left(F_{i_k}, C\left(F_{i_{k-1}}, \ldots, C\left(F_{i_1}, F_{i_2}\right)\cdots\right)\right) \circ E\Big|_{BU(1)_{2(i_1+i_2+\cdots+i_k)-1}} \tag{57}$$

is null homotopic, by Theorem 12, we have the following.

Corollary 13.

$$\Omega C\left(F_{i_k}, C\left(F_{i_{k-1}}, \ldots, C\left(F_{i_1}, F_{i_2}\right)\cdots\right)\right)_*\left(u_{i_1+i_2+\cdots+i_k}\right) \tag{58}$$

is in the Lie subalgebra $P_*(\Omega\Sigma BU(1); \mathbb{Q})$ of primitive elements in $H_*(\Omega\Sigma BU(1); \mathbb{Q})$.

Remark 14. We note that the self-map

$$\begin{aligned}\Omega C\left(F_{i_k}, C\left(F_{i_{k-1}}, \ldots, C\left(F_{i_1}, F_{i_2}\right)\cdots\right)\right) &: \Omega\Sigma BU(1)\\ &\longrightarrow \Omega\Sigma BU(1)\end{aligned} \tag{59}$$

is a loop map; thus it is an H-map. It is well known in [3, page 75] that there is a bijection between the groups $[\Sigma BU(1), \Sigma BU(1)]$ and $[\Omega\Sigma BU(1), \Omega\Sigma BU(1)]_H$ of homotopy classes of H-maps $\Omega\Sigma BU(1) \to \Omega\Sigma BU(1)$. We thus get the corresponding self-map $C(F_{i_k}, C(F_{i_{k-1}}, \ldots, C(F_{i_1}, F_{i_2})\cdots))$ in the group $[\Sigma BU(1), \Sigma BU(1)]$. Moreover, by using the classical Whitehead theorem, we obtain a homotopy self-equivalence of the form $I + C(F_{i_k}, C(F_{i_{k-1}}, \ldots, C(F_{i_1}, F_{i_2})\cdots))$; that is, $I + C(F_{i_k}, C(F_{i_{k-1}}, \ldots, C(F_{i_1}, F_{i_2})\cdots)) \in \mathrm{Aut}(\Sigma BU(1))$, where I is the identity map and "+" is the suspension addition on $\Sigma BU(1)$.

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