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Research Article

Coincidence Points of Weaker Contractions in Partially Ordered Metric Spaces

Kuo-Ching Jen,[1] Ing-Jer Lin,[2] and Chi-Ming Chen[3]

[1] *General Education Center, St. John's University, Taiwan*
[2] *Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 824, Taiwan*
[3] *Department of Applied Mathematics, National Hsinchu University of Education, Taiwan*

Correspondence should be addressed to Chi-Ming Chen; ming@mail.nhcue.edu.tw

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We prove new coincidence point theorems for the $(\varphi, \psi, \phi, \xi)$-contractions and generalized Meir-Keeler-type α-ψ-contractions in partially ordered metric spaces. Our results generalize many recent coincidence point theorems in the literature.

1. Introduction and Preliminaries

Throughout this paper, by $\mathbb{R}^+$, we denote the set of all nonnegative real numbers, while $\mathbb{N}$ is the set of all natural numbers. Let (X, d) be a metric space, D a subset of X, and $f : D \to X$ a map. We say f is contractive if there exists $\alpha \in [0, 1)$ such that for all $x, y \in D$,

$$d(fx, fy) \le \alpha \cdot d(x, y). \tag{1}$$

The well-known Banach's fixed point theorem asserts that if $D = X$, f is contractive and (X, d) is complete, then f has a unique fixed point in X. It is well known that the Banach contraction principle [1] is a very useful and classical tool in nonlinear analysis. Also, this principle has many generalizations. For instance, a mapping $f : X \to X$ is called a quasicontraction if there exists $k < 1$ such that

$$\begin{aligned} d(fx, fy) \le k \cdot \max\{&d(x, y), d(x, fx), \\ &d(y, fy), d(x, fy), d(y, fx)\}, \end{aligned} \tag{2}$$

for any $x, y \in X$. In 1974, Ćirić [2] introduced these maps and proved an existence and uniqueness fixed point theorem.

Recently, Eslamian and Abkar proved the following theorem.

Theorem 1 (see [3]). *Let (X, d) be a complete metric space and $f : X \to X$ be such that*

$$\begin{aligned} \psi(d(fx, fy)) \le \alpha(d(x, y)) - \beta(d(x, y)), \\ \text{for each } x, y \in X, \end{aligned} \tag{3}$$

where $\psi, \alpha, \beta : \mathbb{R}^+ \to \mathbb{R}^+$ are as follows: ψ is continuous and nondecreasing, α is continuous, β is lower semicontinuous, and

$$\begin{gathered} \psi(t) - \alpha(t) + \beta(t) > 0 \quad \forall t > 0, \\ \psi(t) = 0 \quad iff \;\; t = 0, \quad \alpha(0) = \beta(0) = 0. \end{gathered} \tag{4}$$

Then f has a fixed point in X.

Recently, fixed point theory has developed rapidly in partially ordered metric spaces (e.g., [4–8]).

In 2012, Choudhury and Kundu [9] proved the following coincidence theorem as a generalization of Theorem 1.

Theorem 2 (see [9]). *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space and $f, g : X \to X$ be such that $fX \subset gX$, f is g-nondecreasing, gX is closed, and*

$$\begin{aligned} \psi(d(fx, fy)) \le \alpha(d(gx, gy)) - \beta(d(gx, gy)), \\ \text{for each } x, y \in X \text{ such that } gx \sqsubseteq gy, \end{aligned} \tag{5}$$

where $\psi, \alpha, \beta : \mathbb{R}^+ \to \mathbb{R}^+$ are such that ψ is continuous and nondecreasing, α is continuous, β is lower semicontinuous, and

$$\psi(t) - \alpha(t) + \beta(t) > 0 \quad \forall t > 0,$$
$$\psi(t) = 0 \quad iff \ t = 0, \quad \alpha(0) = \beta(0) = 0. \tag{6}$$

Also, if any nondecreasing sequence $\{x_n\}$ in X converges to ν, then we assume that

$$x_n \sqsubseteq \nu \quad \forall n \in \mathbb{N}. \tag{7}$$

If there exists $x_0 \in X$ with $gx_0 \sqsubseteq fx_0$, then f and g have a coincidence point in X.

In this paper, we prove new coincidence point theorems for the $(\varphi, \psi, \phi, \xi)$-contractions and generalized Meir-Keeler-type α-ψ-contractions in partially ordered metric spaces. Our results generalize many recent coincidence point theorems in the literature.

2. Main Results

We start with the following definition.

Definition 3 (g-nondecreasing mapping [4]). Let $(X, \sqsubseteq)$ be a partially ordered set and $f, g : X \to X$. Then f is said to be g-nondecreasing if, for $x, y \in X$,

$$gx \sqsubseteq gy \Longrightarrow fx \sqsubseteq fy. \tag{8}$$

In the sequel, we denote by Ψ the class of functions $\psi : \mathbb{R}^{+5} \to \mathbb{R}^+$ satisfying the following conditions:

(ψ_1) ψ is an increasing, continuous function in each coordinate,

(ψ_2) for all $t \in \mathbb{R}^+$, $\psi(t,t,t,0,2t) \le t$, $\psi(t,t,t,2t,0) \le t$, $\psi(0,0,t,t,0) \le t$, and $\psi(t,0,0,t,t) \le t$,

(ψ_3) $\psi(t_1,t_2,t_3,t_4,t_5) = 0$ if and only if $t_1 = t_2 = t_3 = t_4 = t_5 = 0$.

Next, we denote by Φ the class of functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

(ϕ_1) ϕ is a continuous function and monotone nondecreasing;

(ϕ_2) $\phi(t) > 0$ for $t > 0$ and $\phi(0) = 0$;

(ϕ_3) ϕ is subadditive, that is, $\phi(t_1 + t_2) \le \phi(t_1) + \phi(t_2)$ for all $t_1, t_2 > 0$.

And, we denote the following sets of functions:

$$\Theta = \{\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \text{ such that } \varphi \text{ is continuous}\},$$
$$\Xi = \{\xi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \text{ such that } \xi \text{ is lower continuous}\}. \tag{9}$$

Let X be a nonempty set and $(X, \sqsubseteq)$ be a partially ordered set endowed with a metric d. Then, the triple $(X, \sqsubseteq, d)$ is called a partially ordered metric space.

We now state the $(\varphi, \psi, \phi, \xi)$-contraction and the main fixed point theorem for the $(\varphi, \psi, \phi, \xi)$-contraction in partially ordered metric spaces, as follows.

Definition 4. Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, and let $f, g : X \to X$. Then the pair (f, g) is called a $(\varphi, \psi, \phi, \xi)$-contraction if the following inequality holds:

$$\begin{aligned} &\varphi(d(fx, fy)) \\ &\quad \le \psi(\phi(d(gx, gy)), \phi(d(gx, fx)), \\ &\qquad \phi(d(gy, fy)), \phi(d(gx, fy)), \phi(d(gy, fx))) \\ &\qquad - \xi(\max\{d(gx, gy), d(gx, fx), d(gy, fy)\}), \end{aligned} \tag{10}$$

for all $x, y \in X$ with $gx \sqsubseteq gy$, where $\varphi \in \Theta$, $\psi \in \Psi$, $\phi \in \Phi$ and $\xi \in \Xi$.

We now state the main fixed point theorem for the $(\varphi, \psi, \phi, \xi)$-contraction in partially ordered metric spaces, as follows.

Theorem 5. *Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, and let $f, g : X \to X$ be such that $fX \subset gX$, f is g-nondecreasing and gX is closed. Suppose the pair (f, g) is a $(\varphi, \psi, \phi, \xi)$-contraction, and*

$$\varphi(t) - \phi(t) + \xi(t) > 0 \quad \forall t > 0,$$
$$\varphi(t) = 0 \quad iff \ t = 0, \quad \phi(0) = \xi(0) = 0. \tag{11}$$

Also, if any nondecreasing sequence $\{x_n\}$ in X converges to ν, then we assume that

$$x_n \sqsubseteq \nu \quad \forall n \in \mathbb{N}. \tag{12}$$

If there exists $x_0 \in X$ with $gx_0 \sqsubseteq fx_0$, then f and g have a coincidence point in X.

Proof. Since $fX \subset gX$ and there exists $x_0 \in X$ with $gx_0 \sqsubseteq fx_0$, we can choose $x_1 \in X$ such that $gx_1 = fx_0$. Since f is g-nondecreasing, we have $fx_0 \sqsubseteq fx_1$. In this process, we construct the sequence $\{x_n\}$ recursively as

$$fx_n = gx_{n+1} \quad \forall n \in \mathbb{N}. \tag{13}$$

Thus, we also conclude that

$$\begin{aligned} gx_0 \sqsubseteq fx_0 = gx_1 \sqsubseteq fx_1 = gx_2 \sqsubseteq \cdots \sqsubseteq fx_{n-1} \\ = gx_n \sqsubseteq fx_n = gx_{n+1} \sqsubseteq \cdots. \end{aligned} \tag{14}$$

If any two consecutive terms in (14) are equal, then the conclusion of the theorem follows. So we may assume that

$$d(fx_{n-1}, fx_n) \ne 0, \quad \forall n \in \mathbb{N}. \tag{15}$$

Now, we claim that $d(fx_n, fx_{n+1}) \le d(fx_{n-1}, fx_n)$ for all $n \in \mathbb{N}$. If not, we assume that $d(fx_{n-1}, fx_n) < d(fx_n, fx_{n+1})$ for some $n \in \mathbb{N}$, substituting $x = x_n$ and $y = x_{n+1}$ in (10) and using the definition of the function ψ, we have

$$\begin{aligned}
&\psi(\phi(d(gx_n, gx_{n+1})), \phi(d(gx_n, fx_n)),\\
&\quad \phi(d(gx_{n+1}, fx_{n+1})), \phi(d(gx_n, fx_{n+1})),\\
&\quad \phi(d(gx_{n+1}, fx_n)))\\
&\quad = \psi(\phi(d(fx_{n-1}, fx_n)), \phi(d(fx_{n-1}, fx_n)),\\
&\qquad \phi(d(fx_n, fx_{n+1})), \phi(d(fx_{n-1}, fx_{n+1})),\\
&\qquad \phi(d(fx_n, fx_n)))\\
&\quad \le \psi(\phi(d(fx_n, fx_{n+1})), \phi(d(fx_n, fx_{n+1})),\\
&\qquad \phi(d(fx_n, fx_{n+1})), 2\phi(d(fx_n, fx_{n+1})), \phi(0))\\
&\quad \le \phi(d(fx_n, fx_{n+1})),\\
&\xi(\max\{d(gx_n, gx_{n+1}), d(gx_n, fx_n),\\
&\qquad d(gx_{n+1}, fx_{n+1})\})\\
&\quad = \xi(\max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n),\\
&\qquad d(fx_n, fx_{n+1})\})\\
&\quad = \xi(d(fx_n, fx_{n+1})),
\end{aligned} \tag{16}$$

and hence

$$\varphi(d(fx_n, fx_{n+1})) \le \phi(d(fx_n, fx_{n+1})) - \xi(d(fx_n, fx_{n+1})). \tag{17}$$

Since $\varphi(t) - \phi(t) + \xi(t) > 0$ for all $t > 0$, we have that $d(fx_n, fx_{n+1}) = 0$, which contradicts to (15). Therefore, we conclude that

$$d(fx_n, fx_{n+1}) \le d(fx_{n-1}, x_n) \quad \forall n \in \mathbb{N}. \tag{18}$$

From above argument, we also have that for each $n \in \mathbb{N}$

$$\varphi(d(fx_n, fx_{n+1})) \le \phi(d(fx_{n-1}, fx_n)) - \xi(d(fx_{n-1}, fx_n)). \tag{19}$$

It follows (18) that the sequence $\{d(fx_n, fx_{n+1})\}$ is monotone decreasing, it must converge to some $\eta \ge 0$. Taking limit as $n \to \infty$ in (19) and using the continuities of φ and ϕ and the lower semicontinuity of ξ, we get

$$\varphi(\eta) \le \phi(\eta) - \xi(\eta), \tag{20}$$

which implies that $\eta = 0$. So we conclude that

$$\lim_{n\to\infty} d(fx_n, fx_{n+1}) = 0. \tag{21}$$

We next claim that $\{fx_n\}$ is a Cauchy sequence, that is, for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that if $p, q \ge n$, then $d(fx_p, fx_q) < \varepsilon$.

Suppose the above statement is false. Then there exists $\epsilon > 0$ such that for any $n \in \mathbb{N}$, there are $p_n, q_n \in \mathbb{N}$ with $p_n > q_n \ge n$ satisfying

$$d(fx_{q_n}, fx_{p_n}) \ge \epsilon. \tag{22}$$

Further, corresponding to $q_n \ge n$, we can choose p_n in such a way that it is the smallest integer with $p_n > q_n \ge n$ and $d(fx_{q_n}, fx_{p_n}) \ge \epsilon$. Therefore $d(fx_{q_n}, fx_{p_n-1}) < \epsilon$. Now we have that for all $n \in \mathbb{N}$

$$\begin{aligned}
\epsilon &\le d(fx_{p_n}, fx_{q_n})\\
&\le d(fx_{p_n}, fx_{p_n-1}) + d(fx_{p_n-1}, fx_{q_n})\\
&< d(fx_{p_n}, fx_{p_n-1}) + \epsilon.
\end{aligned} \tag{23}$$

Letting $n \to \infty$, then we get

$$\lim_{n\to\infty} d(fx_{p_n}, fx_{q_n}) = \epsilon. \tag{24}$$

On the other hand, we have

$$\begin{aligned}
d(fx_{p_n}, fx_{q_n}) &\le d(fx_{p_n}, fx_{p_n-1}) + d(fx_{p_n-1}, fx_{q_n-1})\\
&\quad + d(fx_{q_n-1}, fx_{q_n}),\\
d(fx_{p_n-1}, fx_{q_n-1}) &\le d(fx_{p_n-1}, fx_{p_n}) + d(fx_{p_n}, fx_{q_n})\\
&\quad + d(fx_{q_n}, fx_{q_n-1}).
\end{aligned} \tag{25}$$

Letting $n \to \infty$, then we get

$$\lim_{n\to\infty} d(fx_{p_n-1}, fx_{q_n-1}) = \epsilon. \tag{26}$$

By (14), we have that the elements gx_{p_n} and gx_{q_n} are comparable. Substituting $x = x_{p_n}$ and $y = x_{q_n}$ in (10), we have that for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\psi(\phi(d(gx_{p_n}, gx_{q_n})), \phi(d(gx_{p_n}, fx_{p_n})),\\
&\quad \phi(d(gx_{q_n}, fx_{q_n})), \phi(d(gx_{p_n}, fx_{q_n})),\\
&\quad \phi(d(gx_{q_n}, fx_{p_n})))\\
&\le \psi(\phi(d(fx_{p_n-1}, fx_{q_n-1})), \phi(d(fx_{p_n-1}, fx_{p_n})),\\
&\quad \phi(d(fx_{q_n-1}, fx_{q_n})), \phi(d(fx_{p_n-1}, fx_{q_n})),\\
&\quad \phi(d(fx_{q_n-1}, fx_{p_n})))\\
&\le \psi(\phi(d(fx_{p_n-1}, fx_{q_n-1})), \phi(d(fx_{p_n-1}, fx_{p_n})),\\
&\quad \phi(d(fx_{q_n-1}, fx_{q_n})), \phi(d(fx_{p_n-1}, fx_{p_n}))\\
&\qquad + \phi(d(fx_{p_n}, fx_{q_n})), \phi(d(fx_{q_n-1}, fx_{q_n}))\\
&\qquad + \phi(d(fx_{q_n}, fx_{p_n}))),\\
&M(x_{p_n}, x_{q_n}) = \max\{d(gx_{p_n}, gx_{q_n}),\\
&\qquad d(gx_{p_n}, fx_{p_n}), d(gx_{q_n}, fx_{q_n})\}\\
&\qquad = \max\{d(fx_{p_n-1}, fx_{q_n-1}), d(fx_{p_n-1}, fx_{p_n}),\\
&\qquad d(fx_{q_n-1}, fx_{q_n})\}.
\end{aligned} \tag{27}$$

By above argument and using inequality (10), we can conclude that

$$\begin{aligned}\varphi(\epsilon) &\le \psi(\phi(\epsilon), 0, 0, \phi(\epsilon), \phi(\epsilon)) - \xi(\epsilon) \\ &\le \phi(\epsilon) - \xi(\epsilon),\end{aligned} \tag{28}$$

which implies that $\epsilon = 0$, a contradiction. Therefore, the sequence $\{fx_n\}$ is a Cauchy sequence.

Since X is complete and gX is closed, there exists $v \in X$ such that

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = gv. \tag{29}$$

Later, we prove that v is a coincidence point of f and g. From (14) and (29), we deduce that

$$gx_n \sqsubseteq gv, \quad \forall n \in \mathbb{N}. \tag{30}$$

Substituting $x = x_n$ and $y = v$ in (10), we have that

$$\begin{aligned}&\varphi(d(fx_n, fv)) \\ &\quad \le \psi(\phi(d(gx_n, gv)), \phi(d(gx_n, fx_n)), \\ &\qquad \phi(d(gv, fv)), \phi(d(gx_n, fv)), \\ &\qquad \phi(d(gv, fx_n))) \\ &\quad - \xi(\max\{d(gx_n, gv), d(gx_n, fx_n), d(gv, fv)\}).\end{aligned} \tag{31}$$

Taking $n \to \infty$ in the above inequality, we have

$$\begin{aligned}\varphi(d(gv, fv)) &\le \psi(0, 0, \phi(d(gv, fv)), \phi(d(gv, fv)), 0) \\ &\quad - \xi(d(gv, fv)) \le \phi(d(gv, fv)) \\ &\quad - \xi(d(gv, fv)),\end{aligned} \tag{32}$$

which implies that $d(gv, fv) = 0$, that is, $gv = fv$. So we complete the proof. □

We give the following example to illustrate Theorem 5.

Example 6. Let $X = [0, 1]$. We define a partial order "$\sqsubseteq$" on X as $x \sqsubseteq y$ if and only if $x \ge y$ for all $x, y \in X$. We take the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Let $f, g : X \to X$ be defined as

$$f(x) = \frac{1}{16}x^2, \qquad g(x) = \frac{1}{4}x^2. \tag{33}$$

Let $\varphi, \phi, \xi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined as

$$\varphi(t) = \phi(t) = t, \qquad \xi(t) = \frac{t}{8} \quad \forall t \in [0, 1], \tag{34}$$

and let $\psi : \mathbb{R}^{+5} \to \mathbb{R}^+$ denote

$$\psi(t_1, t_2, t_3, t_4, t_5) = \frac{1}{2} \cdot \max\left\{t_1, t_2, t_3, \frac{t_4}{2}, \frac{t_5}{2}\right\}. \tag{35}$$

Without loss of generality, we assume that $x > y$ and verity inequality (10).

For all $x, y \in [0, 1]$ with $x > y$, we have

$$\begin{aligned}\varphi(d(fx, fy)) &= \frac{1}{16}(x^2 - y^2), \\ \phi(d(gx, gy)) &= \frac{1}{4}(x^2 - y^2), \\ \phi(d(gx, fx)) &= \frac{1}{4}x^2 - \frac{1}{16}x^2 = \frac{3}{16}x^2, \\ \phi(d(gy, fy)) &= \frac{1}{4}y^2 - \frac{1}{16}y^2 = \frac{3}{16}y^2, \\ \phi(d(gx, fy)) &= \frac{1}{4}x^2 - \frac{1}{16}y^2 > \frac{3}{16}x^2, \\ \phi(d(fx, gy)) &= \left|\frac{1}{16}x^2 - \frac{1}{4}y^2\right|, \\ &\xi(\max d(gx, gy), d(gx, fx), d(gy, fy)) \\ &= \begin{cases}\frac{1}{4}(x^2 - y^2), & \text{if } x > 2y, \\ \frac{3}{16}x^2, & \text{if } x \le 2y,\end{cases} \\ &\psi(\phi(d(gx, gy)), \phi(d(gx, fx)), \phi(d(gy, fy)), \\ &\quad \phi(d(gx, fy)), \phi(d(fx, gy))) \\ &= \frac{1}{8}x^2 - \frac{1}{32}y^2.\end{aligned} \tag{36}$$

Therefore, inequality (10) is satisfied and all the conditions of Theorem 5 are satisfied, and we obtained that 0 is a coincidence point of f and g.

Applying Definition 4, Theorem 5, and Example 6, if we let

$$\begin{aligned}&\psi(\phi(d(gx, gy)), \phi(d(gx, fx)), \phi(d(gy, fy)), \\ &\quad \phi(d(gx, fy)), \phi(d(gy, fx))) \\ &\quad = \max\Big\{\phi(d(gx, gy)), \phi(d(gx, fx)), \phi(d(gy, fy)), \\ &\qquad \frac{1}{2}\phi(d(gx, fy)), \frac{1}{2}\phi(d(fx, gy))\Big\},\end{aligned} \tag{37}$$

we are easy to get the following theorem.

Theorem 7. *Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, and let $f, g : X \to X$ be such that $fX \subset gX$, f is g-nondecreasing, gX is closed, and*

$$\begin{aligned}&\varphi(d(fx, fy)) \\ &\quad \le \max\{\phi(d(gx, gy)), \phi(d(gx, fx)), \\ &\qquad \phi(d(gy, fy)), \phi(d(gx, fy)), \psi(d(fx, gy))\} \\ &\quad - \xi(\max\{d(gx, gy), d(gx, fx), d(gy, fy)\}),\end{aligned} \tag{38}$$

for all $x, y \in X$ *such that* $gx \sqsubseteq gy$, *where* $\varphi \in \Theta$, $\psi \in \Psi$, $\phi \in \Phi$ *and* $\xi \in \Xi$, *and*

$$\begin{gathered}\varphi(t) - \phi(t) + \xi(t) > 0 \quad \forall t > 0, \\ \varphi(t) = 0 \quad iff\ t = 0, \quad \phi(0) = \xi(0) = 0.\end{gathered} \tag{39}$$

Also, if any nondecreasing sequence $\{x_n\}$ *in* X *converges to* ν, *then one assumes that*

$$x_n \sqsubseteq \nu \quad \forall n \in \mathbb{N}. \tag{40}$$

If there exists $x_0 \in X$ with $gx_0 \sqsubseteq fx_0$, then f and g have a coincidence point in X.

In the other research of this paper, we recall the Meir-Keeler-type contraction [10] and α-admissible mapping [11]. In 1969, Meir and Keeler [10] introduced the following notion of Meir-Keeler-type contraction in a metric space (X, d).

Definition 8. Let (X, d) be a metric space, $f : X \to X$. Then f is called a Meir-Keeler-type contraction whenever for each $\eta > 0$ there exists $\gamma > 0$ such that

$$\eta \leq d(x, y) < \eta + \gamma \Longrightarrow d(fx, fy) < \eta. \tag{41}$$

And, the following definition was introduced in [11].

Definition 9. Let $f : X \to X$ be a self-mapping of a set X and $\alpha : X \times X \to \mathbb{R}^+$. Then f is called a α-admissible mapping if

$$x, y \in X, \qquad \alpha(x, y) \geq 1 \Longrightarrow \alpha(fx, fy) \geq 1. \tag{42}$$

We introduce the notion of α-g-admissible mapping, as follows.

Definition 10. Let $f, g : X \to X$ be a self-mapping of a set X and $\alpha : X \times X \to \mathbb{R}^+$. Then f is called a α-g-admissible mapping if

$$x, y \in X, \qquad \alpha(gx, gy) \geq 1 \Longrightarrow \alpha(fx, fy) \geq 1. \tag{43}$$

We give the following example to illustrate Definition 10.

Example 11. Let $X = \mathbb{R}^+$ and we define

$$\begin{gathered} g(x) = x + \frac{1}{2}, \qquad f(x) = g(x) + \frac{1}{x+1}, \\ \alpha(x, y) = x + y. \end{gathered} \tag{44}$$

Then f is a α-g-admissible mapping.

We now state the new notions of generalized Meir-Keeler-type ψ-contractions and generalized Meir-Keeler-type α-ψ-contractions in partially ordered complete metric spaces, as follows.

Definition 12. Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, and let $f, g : X \to X$. Then the pair (f, g) is called a generalized Meir-Keeler-type ψ-contraction whenever for each $\eta > 0$, there exists $\delta > 0$ such that

$$\begin{aligned} \eta \leq \psi(&d(gx, gy), d(gx, fx), d(gy, fy), \\ &d(gx, fy), d(gy, fx)) \\ &< \eta + \delta \Longrightarrow d(fx, fy) < \eta, \end{aligned} \tag{45}$$

for all $x, y \in X$ with $gx \sqsubseteq gy$, where $\psi \in \Psi$.

Definition 13. Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, let $f, g : X \to X$, and $\alpha : X \times X \to \mathbb{R}^+$. Then (f, g) is called a generalized Meir-Keeler-type α-ψ-contraction if the following conditions hold:

(1) f is α-g-admissible;

(2) for each $\eta > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} \eta \leq \psi(&d(gx, gy), d(gx, fx), d(gy, fy), \\ &d(gx, fy), d(gy, fx)) \\ &< \eta + \delta \Longrightarrow \alpha(fx, fx)\alpha(gy, gy) d(fx, fy) < \eta, \end{aligned} \tag{46}$$

for all $x, y \in X$ with $gx \sqsubseteq gy$, where $\psi \in \Psi$.

Remark 14. Note that if f is a generalized Meir-Keeler-type α-ψ-contraction, then we have that for all $x, y \in X$

$$\begin{aligned} &\alpha(fx, fx)\alpha(gy, gy) d(fx, fy) \\ &\quad \leq \psi(d(gx, gy), d(gx, fx), d(gy, fy), \\ &\qquad d(gx, fy), d(gy, fx)). \end{aligned} \tag{47}$$

Further, if

$$\begin{aligned} \psi(&d(gx, gy), d(gx, fx), d(gy, fy), \\ &d(gx, fy), d(gy, fx)) = 0, \end{aligned} \tag{48}$$

then $d(fx, fy) = 0$.

On the other hand, if

$$\begin{aligned} \psi(&d(gx, gy), d(gx, fx), d(gy, fy), \\ &d(gx, fy), d(gy, fx)) > 0, \end{aligned} \tag{49}$$

then

$$\begin{aligned} &\alpha(fx, fx)\alpha(gy, gy) d(fx, fy) \\ &\quad < \psi(d(gx, gy), d(gx, fx), d(gy, fy), \\ &\qquad d(gx, fy), d(gy, fx)). \end{aligned} \tag{50}$$

We now state our main result for the generalized Meir-Keeler-type α-ψ-contraction, as follows.

Theorem 15. *Let* $(X, \sqsubseteq, d)$ *be a partially ordered complete metric space, let* $\alpha : X \times X \to \mathbb{R}^+$ *be continuous in each coordinate, and let* $f, g : X \to X$ *be such that* $fX \subset gX$, f *is* g*-nondecreasing, and* gX *is closed. Suppose the pair* (f, g) *is a generalized Meir-Keeler-type* α-ψ*-contraction and the following conditions hold.*

(i) *If any nondecreasing sequence* $\{x_n\}$ *in* X *converges to* ν, *then we assume that*

$$x_n \sqsubseteq \nu \quad \forall n \in \mathbb{N}. \tag{51}$$

(ii) *There exists* $x_0 \in X$ *with* $gx_0 \sqsubseteq fx_0$ *and* $\alpha(fx_0, fx_0) \geq 1$.

(iii) *If* $\alpha(fx_n, fx_n) \geq 1$ *for all* $n \in \mathbb{N}$, *then* $\lim_{n\to\infty}\alpha(fx_n, fx_n) \geq 1$.

Then f *and* g *have a coincidence point in* X.

Proof. Since $fX \subset gX$ and by (ii), there exists $x_0 \in X$ with $gx_0 \sqsubseteq fx_0$ and $\alpha(fx_0, fx_0) \geq 1$, we can choose $x_1 \in X$ such that $gx_1 = fx_0$. Since f is g-nondecreasing, we have $fx_0 \sqsubseteq fx_1$. In this process, we construct the sequence $\{x_n\}$ recursively as

$$fx_n = gx_{n+1} \quad \forall n \in \mathbb{N}. \tag{52}$$

Thus, we also conclude that

$$\begin{aligned} gx_0 \sqsubseteq fx_0 = gx_1 \sqsubseteq fx_1 = gx_2 \sqsubseteq \cdots \sqsubseteq fx_{n-1} \\ = gx_n \sqsubseteq fx_n = gx_{n+1} \sqsubseteq \cdots . \end{aligned} \tag{53}$$

If any two consecutive terms in (53) are equal, then the conclusion of the theorem follows. So we may assume that

$$d(fx_{n-1}, fx_n) \neq 0, \quad \forall n \in \mathbb{N}. \tag{54}$$

On the other hand, since f is α-g-admissible and $\alpha(fx_0, fx_0) = \alpha(gx_1, gx_1) \geq 1$, we have

$$\alpha(fx_1, fx_1) = \alpha(gx_2, gx_2) \geq 1. \tag{55}$$

By continuing this process, we get

$$\alpha(fx_n, fx_n) = \alpha(gx_{n+1}, gx_{n+1}) \geq 1 \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{56}$$

By (53), (54), and (56), substituting $x = x_n$ and $y = x_{n+1}$ in (50), we have

$$\begin{aligned} &d(fx_n, fx_{n+1}) \\ &\quad \leq \alpha(fx_n, fx_n)\alpha(gx_{n+1}, gx_{n+1}) d(fx_n, fx_{n+1}) \\ &\quad < \psi(d(gx_n, gx_{n+1}), d(gx_n, fx_n), d(gx_{n+1}, fx_{n+1}), \\ &\qquad (gx_n, fx_{n+1}), d(gx_{n+1}, fx_n)) \\ &\quad = \psi(d(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \\ &\qquad d(fx_{n-1}, fx_{n+1}), d(fx_n, fx_n)). \end{aligned} \tag{57}$$

If $d(fx_{n-1}, fx_n) \leq d(fx_n, fx_{n+1})$, then the inequality (57) becomes

$$\begin{aligned} &d(fx_n, fx_{n+1}) \\ &\quad < \psi(d(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \\ &\qquad d(fx_{n-1}, fx_{n+1}), d(fx_n, fx_n)) \\ &\quad \leq \psi(d(fx_n, fx_{n+1}), d(fx_n, fx_{n+1}), d(fx_n, fx_{n+1}), \\ &\qquad 2d(fx_n, fx_{n+1}), 0) \\ &\quad \leq d(fx_n, fx_{n+1}), \end{aligned} \tag{58}$$

which implies a contradiction, and we get that $d(fx_n, fx_{n+1}) < d(fx_{n-1}, fx_n)$.

From the argument above, we have that the sequence $\{d(fx_n, fx_{n+1})\}$ is decreasing, and it must converge to some $\eta \geq 0$, that is,

$$\lim_{n\to\infty} d(fx_n, fx_{n+1}) = \eta. \tag{59}$$

It follow from that (57) and (59), we have

$$\begin{aligned} &\lim_{n\to\infty} \psi(d(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \\ &\qquad d(fx_{n-1}, fx_{n+1}), d(fx_n, fx_n)) = \eta. \end{aligned} \tag{60}$$

Notice that $\eta = \inf\{d(fx_n, fx_{n+1}) : n \in \mathbb{N} \cup \{0\}\}$. We claim that $\eta = 0$. Suppose, to the contrary, that $\eta > 0$. Since (f, g) is a generalized Meir-Keeler-type α-ψ-contraction, corresponding to η use, and taking into account the above inequality (60), there exist $\delta > 0$ and a natural number k such that

$$\begin{aligned} \eta &\leq \psi(d(fx_{k-1}, fx_k), d(fx_{k-1}, fx_k), \\ &\qquad d(fx_k, fx_{k+1}), d(fx_{k-1}, fx_{k+1}), \\ &\qquad d(fx_k, fx_k)) \\ &< \eta + \delta \Longrightarrow \alpha(fx_k, fx_k)\alpha(gx_{k+1}, gx_{k+1}) \\ &\qquad \times d(fx_k, fx_{k+1}) < \eta, \end{aligned} \tag{61}$$

which implies

$$\begin{aligned} d(fx_k, fx_{k+1}) &\leq \alpha(fx_k, fx_k) \\ &\quad \times \alpha(gx_{k+1}, gx_{k+1}) d(fx_k, fx_{k+1}) < \eta. \end{aligned} \tag{62}$$

So we get a contradiction, since $\eta = \inf\{d(fx_n, fx_{n+1}) : n \in \mathbb{N} \cup \{0\}\}$. Thus we have that

$$\lim_{n\to\infty} d(fx_n, fx_{n+1}) = 0. \tag{63}$$

We next claim that $\{fx_n\}$ is a Cauchy sequence, that is, for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that if $p, q \geq n$, then $d(fx_p, fx_q) < \varepsilon$.

Suppose the above statement is false. Then there exists $\epsilon > 0$ such that for any $n \in \mathbb{N}$, there are $p_n, q_n \in \mathbb{N}$ with $p_n > q_n \geq n$ satisfying

$$d(fx_{q_n}, fx_{p_n}) \geq \epsilon. \tag{64}$$

Further, corresponding to $q_n \geq n$, we can choose p_n in such a way that is it the smallest integer with $p_n > q_n \geq n$ and $d(fx_{q_n}, fx_{p_n}) \geq \epsilon$. Therefore $d(fx_{q_n}, fx_{p_n-1}) < \epsilon$. Now we have that for all $n \in \mathbb{N}$

$$\begin{aligned} \epsilon \leq d(fx_{p_n}, fx_{q_n}) &\leq d(fx_{p_n}, fx_{p_n-1}) \\ &\quad + d(fx_{p_n-1}, fx_{q_n}) \\ &< d(fx_{p_n}, fx_{p_n-1}) + \epsilon. \end{aligned} \tag{65}$$

Letting $n \to \infty$, then we get

$$\lim_{n\to\infty} d\left(fx_{p_n}, fx_{q_n}\right) = \epsilon. \quad (66)$$

On the other hand, we have

$$\begin{aligned} d\left(fx_{p_n}, fx_{q_n}\right) &\leq d\left(fx_{p_n}, fx_{p_n-1}\right) + d\left(fx_{p_n-1}, fx_{q_n-1}\right) \\ &\quad + d\left(fx_{q_n-1}, fx_{q_n}\right), \\ d\left(fx_{p_n-1}, fx_{q_n-1}\right) &\leq d\left(fx_{p_n-1}, fx_{p_n}\right) + d\left(fx_{p_n}, fx_{q_n}\right) \\ &\quad + d\left(fx_{q_n}, fx_{q_n-1}\right). \end{aligned} \quad (67)$$

Letting $n \to \infty$, then we get

$$\lim_{n\to\infty} d\left(fx_{p_n-1}, fx_{q_n-1}\right) = \epsilon. \quad (68)$$

By (53), we have that the elements gx_{p_n} and gx_{q_n} are comparable. Substituting $x = x_{p_n}$ and $y = x_{q_n}$ in (50), we have that for all $n \in \mathbb{N}$,

$$\begin{aligned} & d\left(fx_{p_n}, fx_{q_n}\right) \\ &\quad \leq \alpha\left(fx_{p_n}, fx_{p_n}\right)\alpha\left(gx_{q_n}, gx_{q_n}\right) d\left(fx_{p_n}, fx_{q_n}\right) \\ &\quad < \psi\left(d\left(gx_{p_n}, gx_{q_n}\right), d\left(gx_{p_n}, fx_{p_n}\right), d\left(gx_{q_n}, fx_{q_n}\right),\right. \\ &\qquad \left. d\left(gx_{p_n}, fx_{q_n}\right), d\left(gx_{q_n}, fx_{p_n}\right)\right) \\ &\quad \leq \psi\left(d\left(fx_{p_n-1}, fx_{q_n-1}\right), d\left(fx_{p_n-1}, fx_{p_n}\right),\right. \\ &\qquad d\left(fx_{q_n-1}, fx_{q_n}\right), d\left(fx_{p_n-1}, fx_{q_n}\right), \\ &\qquad \left. d\left(fx_{q_n-1}, fx_{p_n}\right)\right) \\ &\quad \leq \psi\left(d\left(fx_{p_n-1}, fx_{q_n-1}\right), d\left(fx_{p_n-1}, fx_{p_n}\right),\right. \\ &\qquad d\left(fx_{q_n-1}, fx_{q_n}\right), d\left(fx_{p_n-1}, fx_{p_n}\right) + d\left(fx_{p_n}, fx_{q_n}\right), \\ &\qquad \left. d\left(fx_{q_n-1}, fx_{q_n}\right) + d\left(fx_{q_n}, fx_{p_n}\right)\right). \end{aligned} \quad (69)$$

Letting $n \to \infty$ in (69), then we get

$$\epsilon < \psi(\epsilon, 0, 0, \epsilon, \epsilon) \leq \epsilon, \quad (70)$$

which implies a contradiction. Thus, $\{fx_n\}$ is a Cauchy sequence.

Since X is complete and gX is closed, there exists $\nu \in X$ such that

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = g\nu. \quad (71)$$

Since α is continuous in each coordinate and by the condition (iii), we have

$$\alpha(g\nu, g\nu) = \lim_{n\to\infty} \alpha(fx_n, fx_n) \geq 1. \quad (72)$$

Later, we prove that ν is a coincidence point of f and g. From (53) and (71), we deduce that

$$gx_n \sqsubseteq g\nu, \quad \forall n \in \mathbb{N}. \quad (73)$$

By (72) and substituting $x = x_n$ and $y = \nu$ in (50), we have that

$$\begin{aligned} d(fx_n, f\nu) &\leq \alpha(fx_n, fx_n)\alpha(g\nu, g\nu) d(fx_n, f\nu) \\ &< \psi(d(gx_n, g\nu), d(gx_n, fx_n), \\ &\qquad d(g\nu, f\nu), d(gx_n, f\nu), d(g\nu, fx_n)). \end{aligned} \quad (74)$$

Taking $n \to \infty$ in the above inequality, we have

$$\begin{aligned} d(g\nu, f\nu) &< \psi(d(g\nu, g\nu), d(g\nu, g\nu), \\ &\qquad d(g\nu, f\nu), d(g\nu, f\nu), d(g\nu, g\nu)) \\ &\leq d(g\nu, f\nu). \end{aligned} \quad (75)$$

This implies that $g\nu = f\nu$. So we complete the proof. □

Apply Theorem 15, we are easy to get the following theorem.

Theorem 16. *Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, and let $f, g : X \to X$ be such that $fX \subset gX$, f is g-nondecreasing, and gX is closed. Suppose the pair (f, g) is a generalized Meir-Keeler-type ψ-contraction and the following conditions hold.*

(i) *If any nondecreasing sequence $\{x_n\}$ in X converges to ν, then we assume that*

$$x_n \sqsubseteq \nu \quad \forall n \in \mathbb{N}. \quad (76)$$

(ii) *There exists $x_0 \in X$ with $gx_0 \sqsubseteq fx_0$.*

Then f and g have a coincidence point in X.

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