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Research Article

On the Domain of the Triangle $A(\lambda)$ on the Spaces of Null, Convergent, and Bounded Sequences

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We introduce the spaces of $A(\lambda)$-null, $A(\lambda)$-convergent, and $A(\lambda)$-bounded sequences. We examine some topological properties of the spaces and give some inclusion relations concerning these sequence spaces. Furthermore, we compute α-, β-, and γ-duals of these spaces. Finally, we characterize some classes of matrix transformations from the spaces of $A(\lambda)$-bounded and $A(\lambda)$-convergent sequences to the spaces of bounded, almost convergent, almost null, and convergent sequences and present a Steinhaus type theorem.

1. Introduction

By ω, we denote the space of all complex sequences. If $x \in \omega$, then we simply write $x = (x_k)$ instead of $x = (x_k)_{k=0}^{\infty}$. Also, we will use the conventions that $e = (1, 1, \ldots)$, and $e^{(n)}$ is the sequence whose only nonzero term is 1 in the nth place for each $n \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. Any vector subspace of ω is called a sequence space. We will write ℓ_∞, c and c_0 for the sequence spaces of all bounded, convergent, and null sequences, respectively. Further, by ℓ_p with $1 \leq p < \infty$, we denote the sequence space of all p-absolutely convergent series, that is, $\ell_p = \{x = (x_k) \in \omega : \sum_k |x_k|^p < \infty\}$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. Moreover, we write bs and cs for the spaces of all bounded and convergent series, respectively. A sequence space μ is called an FK-space if it is a complete linear metric space with continuous coordinates $p_n : \mu \to \mathbb{C}$, where $\mathbb{C}$ denotes the complex field and $p_n(x) = x_n$ for all $x = (x_n) \in \mu$ and every $n \in \mathbb{N}$. A normed FK-space is called a BK-space, that is, a BK-space is a Banach space with continuous coordinates. The sequence spaces c_0 and c are BK-spaces with the usual sup-norm given by $\|x\|_\infty = \sup_{n\in\mathbb{N}}|x_n|$. Also, the space ℓ_p is a BK-space with the usual norm $\|\cdot\|_p$ defined by

$$\|x\|_p = \left(\sum_n |x_n|^p\right)^{1/p}, \tag{1}$$

where $1 \leq p < \infty$. A sequence (y_n) in a normed space X is called a Schauder basis for X if for every $x \in X$ there is a unique sequence (α_n) of scalars such that $x = \sum_n \alpha_n y_n$, that is,

$$\lim_{n\to\infty} \left\| x - (\alpha_0 y_0 + \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n) \right\| = 0. \tag{2}$$

The alpha-, beta-, and gamma-duals μ^α, μ^β, and μ^γ of a sequence space μ are, respectively, defined by

$$\begin{aligned} \mu^\alpha &= \{a = (a_k) \in \omega : ax = (a_k x_k) \in \ell_1 \ \forall x = (x_k) \in \mu\}, \\ \mu^\beta &= \{a = (a_k) \in \omega : ax = (a_k x_k) \in cs \ \forall x = (x_k) \in \mu\}, \\ \mu^\gamma &= \{a = (a_k) \in \omega : ax = (a_k x_k) \in bs \ \forall x = (x_k) \in \mu\}. \end{aligned} \tag{3}$$

If A is an infinite matrix with complex entries a_{nk}, where $k, n \in \mathbb{N}$, then we write $A = (a_{nk})$ instead of $A = (a_{nk})_{n,k=0}^{\infty}$. Also, we write A_n for the sequence in the nth row of the matrix A, that is, $A_n = (a_{nk})_{k=0}^{\infty}$ for every $n \in \mathbb{N}$. Further, if $x = (x_k) \in \omega$ then we define the A-transform of x as the sequence $Ax = \{(Ax)_n\}_{n=0}^{\infty}$, where

$$(Ax)_n = \sum_k a_{nk} x_k \tag{4}$$

provided the series on the right hand side of (4) convergent for each $n \in \mathbb{N}$.

Furthermore, the sequence x is said to be A-summable to $l \in \mathbb{C}$ if Ax converges to l which is called the A-limit of x. In addition, let μ and ν be sequence spaces. Then, we say that A defines a matrix mapping from μ into ν if for every sequence $x \in \mu$ the A-transform of x exists and is in ν. Moreover, we write $(\mu : \nu)$ for the class of all infinite matrices that map μ into ν. Thus, $A \in (\mu : \nu)$ if and only if $A_n \in \mu^{\beta}$ for all $n \in \mathbb{N}$ and $Ax \in \nu$ for all $x \in \mu$. The matrix domain μ_A of an infinite matrix A in a sequence space μ is defined by

$$\mu_A = \{x \in \omega : Ax \in \mu\} \tag{5}$$

which is a sequence space. The approach constructing a new sequence space by means of the matrix domain of a triangle matrix was employed by several authors, see for instance [1–4]. In this paper, we introduce the spaces of $A(\lambda)$-null, $A(\lambda)$-convergent, and $A(\lambda)$-bounded sequences which generalize the results given in [2]. Further, we define some related BK-spaces and construct their bases. Moreover, we establish some inclusion relations concerning those spaces and determine their alpha-, beta-, and gamma-duals. Finally, we characterize some classes of matrix transformations on these sequence spaces.

2. Notion of $A(\lambda)$-Null, $A(\lambda)$-Convergent, and $A(\lambda)$-Bounded Sequences

Let $\lambda = (\lambda_k)$ be a strictly increasing sequence of positive real numbers tending to infinity, as $k \to \infty$ and $\lambda_{n+1} \geq 2\lambda_n$ for each $n \in \mathbb{N}$. From this last relation, it follows that $\Delta^2 \lambda_n \geq 0$. The first and second differences are defined as follows: $\Delta\lambda_k = \lambda_k - \lambda_{k-1}$ and $\Delta^2\lambda_k = \Delta(\Delta\lambda_k) = \lambda_k - 2\lambda_{k-1} + \lambda_{k-2}$ for all $k \in \mathbb{N}$, where $\lambda_{-1} = \lambda_{-2} = 0$.

Let $x = (x_k)$ be a sequence of complex numbers, such that $x_{-1} = x_{-2} = 0$. We say that the sequence $x = (x_k)$ is $A(\lambda)$-strongly convergent to a number l if

$$\lim_{n\to\infty} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} \left| (\lambda_k - 2\lambda_{k-1} + \lambda_{k-2})(x_k - l) \right| = 0. \tag{6}$$

This generalizes the concept of Λ-strong convergence (see [5]).

Lemma 1 (see [5]). *A sequence $x = (x_n)$ of complex numbers λ-strongly converges to a number l if and only if $x = (x_n)$ converges to l in the ordinary sense and*

$$\lim_{n\to\infty} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} \lambda_{k-1} \left| x_k - x_{k-1} \right| = 0. \tag{7}$$

Let us define the sequence $y = (y_n)$ by the $A(\lambda)$-transform of a sequence $x = (x_k)$, that is,

$$y_n = (A_\lambda x)_n = \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} (\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}) x_k \tag{8}$$

for all $n \in \mathbb{N}$. Throughout the text, we suppose that the terms of the sequences $x = (x_k)$ and $y = (y_k)$ are connected with the relation (8).

Lemma 2 (see [5]). *If a sequence (y_n) converges to l in the ordinary sense and condition* (7) *of Lemma 1 holds, then the sequence $x = (x_n)$ of complex numbers $A(\lambda)$-strongly converges to l.*

Remark 3 (see [5]). From above results, we can conclude the following. The sequence $x = (x_n)$ of complex numbers $A(\lambda)$-strongly converges to l if and only if the following relation holds:

$$\lim_{n\to\infty} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} (\lambda_k - 2\lambda_{k-1} + \lambda_{k-2})(x_k - l) = 0. \tag{9}$$

Now, we define the infinite matrix $A(\lambda) = \{a_{nk}(\lambda)\}_{n,k=0}^{\infty}$ by

$$a_{nk}(\lambda) = \begin{cases} \dfrac{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}}{\lambda_n - \lambda_{n-1}}, & 0 \leq k \leq n; \\ 0, & k > n \end{cases} \tag{10}$$

for all $n, k \in \mathbb{N}$. Then, $A(\lambda)$-transform of a sequence $x \in \omega$ is the sequence $A_\lambda x = \{(A_\lambda x)_n\}_{n=0}^{\infty}$, where $(A_\lambda x)_n$ is given by the relation (8) for every $n \in \mathbb{N}$. Thus, the sequence x is $A(\lambda)$-convergent if and only if x is $A(\lambda)$-summable. Further, if x is $A(\lambda)$-convergent then the $A(\lambda)$-limit of x exists and coincides with the ordinary limit of x, that is, to say that the method $A(\lambda)$ is regular.

3. The Spaces of $A(\lambda)$-Null, $A(\lambda)$-Convergent, and $A(\lambda)$-Bounded Sequences

We introduce the classes $A_\lambda(c_0)$, $A_\lambda(c)$, and $A_\lambda(\ell_\infty)$ of all $A(\lambda)$-null, $A(\lambda)$-convergent, and $A(\lambda)$-bounded sequences of complex numbers, that is,

$$\begin{aligned} A_\lambda(c_0) &= \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} (A_\lambda x)_n = 0 \right\}, \\ A_\lambda(c) &= \left\{ x = (x_k) \in \omega : \exists l \in \mathbb{C} \ni \lim_{n\to\infty} (A_\lambda x)_n = l \right\}, \\ A_\lambda(\ell_\infty) &= \left\{ x = (x_k) \in \omega : \sup_{n\in\mathbb{N}} \left| (A_\lambda x)_n \right| < \infty \right\}. \end{aligned} \tag{11}$$

Obviously, $A_\lambda(c_0)$, $A_\lambda(c)$, and $A_\lambda(\ell_\infty)$ are the linear spaces with respect to the usual operations coordinatewise addition and scalar multiplication of sequences. Here and after, by X we denote any of the spaces c_0, c, and ℓ_∞. It is not hard to see that the quantity

$$\|x\|_{A_\lambda(X)} := \sup_{n\in\mathbb{N}} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} \left| (\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}) x_k \right| \tag{12}$$

is finite for every $x = (x_k) \in A_\lambda(X)$, and $\|\cdot\|_{A_\lambda(X)}$ is a norm on $A_\lambda(X)$.

Denote by $\|\cdot\|_{bv}$ the usual bv-norm, that is, to say that

$$\|x\|_{bv} := \sum_k |x_k - x_{k-1}|. \tag{13}$$

With the notation of (5), we can redefine the spaces $A_\lambda(c_0)$, $A_\lambda(c)$, and $A_\lambda(\ell_\infty)$ as follows:

$$\begin{aligned} A_\lambda(c_0) &= (c_0)_{A(\lambda)}, \\ A_\lambda(c) &= (c)_{A(\lambda)}, \\ A_\lambda(\ell_\infty) &= (\ell_\infty)_{A(\lambda)}. \end{aligned} \tag{14}$$

Theorem 4. *The sequence spaces $A_\lambda(c_0)$, $A_\lambda(c)$, and $A_\lambda(\ell_\infty)$ are BK-spaces with the norm given by*

$$\|x\|_{A_\lambda(X)} = \|A_\lambda x\|_\infty = \sup_{n\in\mathbb{N}} |(A_\lambda x)_n|. \tag{15}$$

Proof. This follows from Theorem 4.3.12 given in [6] and the relations (14). □

Theorem 5. *The sequence spaces $A_\lambda(c_0)$, $A_\lambda(c)$, and $A_\lambda(\ell_\infty)$ are norm isomorphic to the spaces c_0, c, and ℓ_∞, respectively.*

Proof. Since the matrix $A(\lambda)$ is triangle, it has unique inverse which is also triangle matrix (see [6, 1.4.8]). Therefore, the linear operator, defined by $T : A_\lambda(X) \to X$, $Tx = A_\lambda x$ for all $x \in A_\lambda(X)$, is bijective and norm preserving by relation (15). □

As a consequence of Theorems 4 and 5, we get the following result.

Corollary 6. *Define the sequence $e^{(n)}(\lambda) \in A_\lambda(c_0)$ for every fixed $n \in \mathbb{N}$ by*

$$e_k^{(n)}(\lambda) = \begin{cases} (-1)^{k-n} \dfrac{\lambda_n - \lambda_{n-1}}{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}}, & n \le k \le n+1, \\ 0, & \text{otherwise}, \end{cases} \tag{16}$$

where $k \in \mathbb{N}$. Then, one has the following.

(1) *The sequence $\{e_k^{(0)}(\lambda), e_k^{(1)}(\lambda), e_k^{(2)}(\lambda), \ldots\}$ is a Schauder basis for the space $A_\lambda(c_0)$, and every $x \in A_\lambda(c_0)$ has a unique representation: $x = \sum_n (A_\lambda x)_n e_k^{(n)}(\lambda)$.*

(2) *The sequence $\{e, e_k^{(0)}(\lambda), e_k^{(1)}(\lambda), e_k^{(2)}(\lambda), \ldots\}$ is a Schauder basis for the space $A_\lambda(c)$, and every $x \in A_\lambda(c)$ has a unique representation: $x = le + \sum_n [(A_\lambda x)_n - l] e_k^{(n)}(\lambda)$, where $l = \lim_{n\to\infty} (A_\lambda x)_n$.*

4. Some Inclusion Relations Related to the New Spaces

In this section, we give some inclusion relations concerning the spaces $A_\lambda(c_0)$, $A_\lambda(c)$, and $A_\lambda(\ell_\infty)$.

Theorem 7. *The inclusions $A_\lambda(c_0) \subset A_\lambda(c) \subset A_\lambda(\ell_\infty)$ strictly hold.*

Proof. Let us suppose that $x = (x_n) \in A_\lambda(c_0)$, then it follows that $x = (x_n) \in A_\lambda(c)$ and $x = (x_n) \in A_\lambda(\ell_\infty)$. In what follows we show that these inclusions are strict. The first inclusion follows from the fact that every sequence, which converges in ordinary sense, converges in $A(\lambda)$-sense to the same limit. To prove the strictness of the inclusion $A_\lambda(c) \subset A_\lambda(\ell_\infty)$, define the sequence $x = (x_k)$ by

$$x_k = (-1)^k \frac{\lambda_k - \lambda_{k-2}}{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}} \tag{17}$$

for all $k \in \mathbb{N}$. Then, it follows that

$$\begin{aligned} (A_\lambda x)_n &= \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} (\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}) x_k \\ &= \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} (-1)^k (\lambda_k - \lambda_{k-2}) = (-1)^n. \end{aligned} \tag{18}$$

Therefore, it is trivial that $x = (x_k) \in A_\lambda(\ell_\infty) \setminus A_\lambda(c)$. □

Theorem 8. *The equality $A_\lambda(c_0) \cap c = c_0$ holds.*

Proof. First, we prove that $A_\lambda(c_0) \cap c \subset c_0$. If a sequence $x = (x_n)$ converges in the ordinary sense to l then it follows that $x_n \to l$ converges in $A(\lambda)$-sense, too. This gives the first inclusion. The converse inclusion follows from Lemma 1, in [5]. □

In what follows we describe some properties of the sequence (λ_n) in the space ℓ_∞.

Theorem 9. *For the sequence (λ_n) which is given in Section 2, the following relations are satisfied:*

(i) *$(\Delta\lambda_{n-1}/(\Delta\lambda_n - \Delta\lambda_{n-1}))_{n=1}^\infty \notin \ell_\infty$ if and only if $\liminf_{n\to\infty}((\Delta\lambda_{n+1} - \Delta\lambda_n)/\Delta\lambda_n) = 0$;*

(ii) *$(\Delta\lambda_{n-1}/(\Delta\lambda_n - \Delta\lambda_{n-1}))_{n=1}^\infty \in \ell_\infty$ if and only if $\liminf_{n\to\infty}((\Delta\lambda_{n+1} - \Delta\lambda_n)/\Delta\lambda_n) > 0$.*

Proof. (i) Let us start with the expression

$$\begin{aligned} s_n(x) &= x_n - (A_\lambda x)_n \\ &= \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} (\lambda_k - 2\lambda_{k-1} + \lambda_{k-2})(x_n - x_k). \end{aligned} \tag{19}$$

After some calculations, we get

$$\begin{aligned} s_n(x) &= \frac{\Delta\lambda_{n-1}}{\Delta\lambda_n} [s_n(x) + (A_\lambda x)_n + (A_\lambda x)_{n-1}] \\ &= \frac{\Delta\lambda_{n-1}}{\Delta\lambda_n - \Delta\lambda_{n-1}} [(A_\lambda x)_n - (A_\lambda x)_{n-1}]. \end{aligned} \tag{20}$$

On the other hand, from the definition of the sequence (λ_n) we have

$$\lambda_n \geq 2\lambda_{n-1} \Longrightarrow \Delta^2\lambda_n \geq 0$$
$$\Longrightarrow \Delta\lambda_n \geq \Delta\lambda_{n-1} \Longrightarrow \frac{\Delta\lambda_n}{\Delta\lambda_{n-1}} \geq 1. \tag{21}$$

From the last relation, we have following two possibilities:

(a) $\liminf_{n\to\infty}((\Delta\lambda_n - \Delta\lambda_{n-1})/\Delta\lambda_{n-1}) > 0$ or

(b) $\liminf_{n\to\infty}((\Delta\lambda_n - \Delta\lambda_{n-1})/\Delta\lambda_{n-1}) = 0$.

Part (a) is satisfied if and only if $(\Delta\lambda_{n-1}/(\Delta\lambda_n - \Delta\lambda_{n-1}))_{n=1}^{\infty}$ is bounded. Part (b) is satisfied if and only if $(\Delta\lambda_{n-1}/(\Delta\lambda_n - \Delta\lambda_{n-1}))_{n=1}^{\infty}$ is unbounded. □

Lemma 10. *The inclusions $c_0 \subset A_\lambda(c_0)$ and $c \subset A_\lambda(c)$ hold. Those spaces coincide if and only if $s(x) \in c_0$ for every $x \in A_\lambda(c_0)$, respectively, $A_\lambda(c)$, where $s(x) = \{s_n(x)\}_{n=0}^{\infty}$.*

Lemma 11. *The inclusion $\ell_\infty \subset A_\lambda(\ell_\infty)$ holds. Those spaces coincide if and only if $s(x) \in \ell_\infty$ for every $x \in A_\lambda(\ell_\infty)$.*

Theorem 12. *The inclusions $c_0 \subset A_\lambda(c_0)$, $c \subset A_\lambda(c)$ and $\ell_\infty \subset A_\lambda(\ell_\infty)$ strictly hold if and only if*

$$\liminf_{n\to\infty} \frac{\Delta\lambda_{n+1} - \Delta\lambda_n}{\Delta\lambda_n} > 0. \tag{22}$$

Proof. Let us suppose that $\ell_\infty \subset A_\lambda(\ell_\infty)$ is strict. Then, from Lemma 11, it follows that there exists a sequence $x = (x_n) \in A_\lambda(\ell_\infty)$ such that $s(x) = \{s_n(x)\}_{n=0}^{\infty} \notin \ell_\infty$. Since $x = (x_n) \in A_\lambda(\ell_\infty)$, we have $A_\lambda x \in \ell_\infty$ which leads us to the fact that $\{x_n - s_n(x)\} \in \ell_\infty$. On the other hand, from relation (20), it follows that $(\Delta\lambda_{n-1}/(\Delta\lambda_n - \Delta\lambda_{n-1})) \notin \ell_\infty$. The last relation is equivalent to

$$\liminf_{n\to\infty} \frac{\Delta\lambda_{n+1} - \Delta\lambda_n}{\Delta\lambda_n} = 0 \tag{23}$$

by part (i) of Theorem 9. In a similar way we can conclude that the inclusions $c_0 \subset A_\lambda(c_0)$, $c \subset A_\lambda(c)$ are strict. In what follows we prove the sufficiency. Let

$$\liminf_{n\to\infty} \frac{\Delta\lambda_{n+1} - \Delta\lambda_n}{\Delta\lambda_n} = 0. \tag{24}$$

Then, from, part (i) of Theorem 9, it follows that $(\Delta\lambda_{n-1}/(\Delta\lambda_n - \Delta\lambda_{n-1})) \notin \ell_\infty$ and $((\Delta\lambda_{n-1} + \Delta\lambda_n)/(\Delta\lambda_n - \Delta\lambda_{n-1})) \notin \ell_\infty$. Let us define the sequence $x = (x_n)$ by

$$x_n = (-1)^n \frac{\Delta\lambda_{n-1} + \Delta\lambda_n}{\Delta\lambda_n - \Delta\lambda_{n-1}} \tag{25}$$

for all $n \in \mathbb{N}$. Then, we get the following estimation:

$$\begin{aligned} \left|(A_\lambda x)_n\right| &= \frac{1}{\lambda_n - \lambda_{n-1}} \left| \sum_{k=0}^{n} (-1)^k \frac{\Delta\lambda_{k-1} + \Delta\lambda_k}{\Delta\lambda_k - \Delta\lambda_{k-1}} \times (\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}) \right| \\ &= \frac{1}{\lambda_n - \lambda_{n-1}} \left| \sum_{k=0}^{n} (-1)^k (\Delta\lambda_{k-1} + \Delta\lambda_k) \right| \\ &= \frac{1}{\lambda_n - \lambda_{n-1}} \left|(-1)^n \Delta\lambda_n\right| = 1. \end{aligned} \tag{26}$$

Hence, $A_\lambda x \in \ell_\infty$ which means that $x \in A_\lambda(\ell_\infty) \setminus \ell_\infty$. If $\liminf_{n\to\infty}((\Delta\lambda_{n+1} - \Delta\lambda_n)/\Delta\lambda_n) = 0$. Then, there exists a subsequence (n_r) such that

$$\lim_{r\to\infty} \frac{\Delta\lambda_{n_r+1} - \Delta\lambda_{n_r}}{\Delta\lambda_{n_r}} = 0. \tag{27}$$

Now, let us define the sequence $x = (x_n)$ by

$$x_n = \begin{cases} 1, & n = n_r, \\ -\dfrac{\Delta\lambda_{k-1} - \Delta\lambda_{k-2}}{\Delta\lambda_k - \Delta\lambda_{k-1}}, & n = n_r, \\ 0, & \text{otherwise}, \end{cases} \tag{28}$$

for all $n \in \mathbb{N}$. It follows from (28) that $x \notin c$. On the other hand,

$$(A_\lambda x)_n = \begin{cases} \dfrac{\Delta\lambda_k - \Delta\lambda_{k-1}}{\Delta\lambda_k}, & n = n_r, \\ 0, & \text{otherwise}. \end{cases} \tag{29}$$

Now, from the relations (27) and (29), we derive that $x = (x_n) \in A_\lambda(c_0) \subset A_\lambda(c)$. This completes the proof. □

As an immediate result of Theorem 12, we have the following.

Corollary 13. *The equalities $c_0 = A_\lambda(c_0)$, $c = A_\lambda(c)$, and $\ell_\infty = A_\lambda(\ell_\infty)$ are satisfied if and only if*

$$\liminf_{n\to\infty} \frac{\Delta\lambda_{n+1} - \Delta\lambda_n}{\Delta\lambda_n} > 0. \tag{30}$$

Proposition 14. *The following statements hold.*

(i) *Although c and $A_\lambda(c_0)$ overlap, the space $A_\lambda(c_0)$ does not include the space c.*

(ii) *Although ℓ_∞ and $A_\lambda(c)$ overlap, the space $A_\lambda(c)$ does not include the space ℓ_∞.*

Proposition 15. *If $\liminf_{n\to\infty}((\Delta\lambda_{n+1} - \Delta\lambda_n)/\Delta\lambda_n) = 0$, then the following statements hold.*

(i) *Neither of the spaces c and $A_\lambda(c_0)$ includes the other.*

(ii) *Neither of the spaces $A_\lambda(c_0)$ and ℓ_∞ includes the other.*

(iii) *Neither of the spaces $A_\lambda(c)$ and ℓ_∞ includes the other.*

5. The α-, β-, and γ-Duals of the Spaces $A_\lambda(c_0)$, $A_\lambda(c)$, and $A_\lambda(\ell_\infty)$

In this section, we determine the alpha-, beta-, and gamma-duals of the spaces $A_\lambda(c_0)$, $A_\lambda(c)$, and $A_\lambda(\ell_\infty)$.

We need the following lemma due to Stieglitz and Tietz [3] in proving Theorem 17.

Lemma 16. *$A = (a_{nk}) \in (c_0 : \ell_1) = (c : \ell_1)$ if and only if*

$$\sup_{K,N\in\mathcal{F}} \left| \sum_{n\in N} \sum_{k\in K} a_{nk} \right| < \infty. \tag{31}$$

Here and after, by $\mathcal{F}$ one denotes the collection of all finite subsets of $\mathbb{N}$.

Theorem 17. *The α-dual of the spaces $A_\lambda(c_0)$, $A_\lambda(c)$, and $A_\lambda(\ell_\infty)$ is the set*

$$a_1(\lambda) = \left\{ a = (a_n) \in \omega : \sum_n \frac{\lambda_n - \lambda_{n-1}}{\lambda_n - 2\lambda_{n-1} + \lambda_{n-2}} |a_n| < \infty \right\}. \tag{32}$$

Proof. Define the matrix $B = (b_{nk})$ with the aid of a sequence $a = (a_n)$ as follows:

$$b_{nk} = \begin{cases} (-1)^{n-k} \dfrac{\lambda_k - \lambda_{k-1}}{\lambda_n - 2\lambda_{n-1} + \lambda_{n-2}} a_n, & n-1 \le k \le n, \\ 0, & 0 \le k \le n-1 \text{ or } k > n. \end{cases} \tag{33}$$

Then, $x = (x_n) \in A_\lambda(c_0)$, we have from Theorem 5

$$a_n x_n = a_n \sum_{k=n-1}^{n} (-1)^{n-k} \frac{\lambda_k - \lambda_{k-1}}{\lambda_n - 2\lambda_{n-1} + \lambda_{n-2}} y_k = (By)_n \tag{34}$$

for all $n \in \mathbb{N}$. From the relation (34), it follows that $ax = (a_n x_n) \in \ell_1$ whenever $x \in A_\lambda(c_0)$ if and only if $By \in \ell_1$ whenever $y = (y_k) \in c_0$, that is, $a \in \{A_\lambda(c_0)\}^\alpha$ if and only if $B \in (c_0 : \ell_1)$. By Lemma 16, this is possible if and only if

$$\sup_{K,N\in\mathcal{F}} \left| \sum_{n\in N} \sum_{k\in K} b_{nk} \right| < \infty. \tag{35}$$

Now, from definition of the sets K, N and the matrix $B = (b_{nk})$, it follows that (35) holds if and only if

$$\sum_n \frac{\lambda_n - \lambda_{n-1}}{\lambda_n - 2\lambda_{n-1} + \lambda_{n-2}} |a_n| < \infty \tag{36}$$

which gives that $\{A_\lambda(c_0)\}^\alpha = a_1(\lambda)$.

In a similar way, one can show that $a_1(\lambda)$ is the α-dual of the spaces $A_\lambda(c)$ and $A_\lambda(\ell_\infty)$. So, we omit the details. □

Theorem 18. *Define the sets A, B, C, and D as follows:*

$$\begin{gathered} A = \left\{ a = (a_k) \in \omega : \sum_k \left| \Delta \left[\frac{a_k}{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}} (\lambda_k - \lambda_{k-1}) \right] \right| \right\}, \\ B = \left\{ a = (a_k) \in \omega : \sup_{k\in\mathbb{N}} \left| \frac{a_k (\lambda_k - \lambda_{k-1})}{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}} \right| \right\}, \\ C = \left\{ a = (a_k) \in \omega : \lim_{k\to\infty} \frac{a_k (\lambda_k - \lambda_{k-1})}{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}} \text{ exists} \right\}, \\ D = \left\{ a = (a_k) \in \omega : \lim_{k\to\infty} \frac{a_k (\lambda_k - \lambda_{k-1})}{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}} = 0 \right\}. \end{gathered} \tag{37}$$

Then, one has $\{A_\lambda(c_0)\}^\beta = A \cap B$, $\{A_\lambda(c)\}^\beta = A \cap C$ and $\{A_\lambda(\ell_\infty)\}^\beta = A \cap D$.

Proof. Since the proof is similar for the spaces $A_\lambda(c_0)$ and $A_\lambda(\ell_\infty)$, we consider only the space $A_\lambda(c)$. Let $u = (u_k) \in \omega$. Then, taking into account the relation (8) between the sequences $x = (x_k)$ and $y = (y_k)$, we obtain that

$$\begin{aligned} \sum_{k=0}^{n} u_k x_k &= \sum_{k=0}^{n} u_k \sum_{j=k-1}^{k} (-1)^{k-j} \frac{\lambda_j - \lambda_{j-1}}{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}} y_j \\ &= \sum_{k=0}^{n-1} (\lambda_k - \lambda_{k-1}) \Delta \left(\frac{u_k}{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}} \right) y_k \\ &\quad + \frac{(\lambda_n - \lambda_{n-1}) u_n y_n}{\lambda_n - 2\lambda_{n-1} + \lambda_{n-2}}, \\ &= (By)_n, \quad \forall n \in \mathbb{N}, \end{aligned} \tag{38}$$

where

$$\Delta \left(\frac{u_k}{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}} \right) = \frac{u_k}{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}} - \frac{u_{k+1}}{\lambda_{k+1} - 2\lambda_k + \lambda_{k-1}}, \tag{39}$$

and the matrix $B = (b_{nk})$ is defined by

$$b_{nk} = \begin{cases} (\lambda_k - \lambda_{k-1}) \times \Delta \left(\dfrac{u_k}{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}} \right), & 0 \le k \le n-1, \\ \dfrac{(\lambda_n - \lambda_{n-1}) u_n}{\lambda_n - 2\lambda_{n-1} + \lambda_{n-2}}, & k = n, \\ 0, & k > n \end{cases} \tag{40}$$

for all $k, n \in \mathbb{N}$. Therefore, one can easily see from (38) that $ux = (u_k x_k) \in cs$ with $x = (x_k) \in A_\lambda(c)$ if and only if $By \in c$ with $y = (y_k) \in c$, where $B = (b_{nk})$ is defined by (40). That is, to say that $u = (u_k) \in \{A_\lambda(c)\}^\beta$ if and only if B is a matrix satisfying the conditions of Kojima-Schur's theorem (cf. Başar [7, Theorem 3.3.3, page 35]). This leads to the fact that $\{A_\lambda(c)\}^\beta = A \cap C$. □

Theorem 19. *The γ-dual of the spaces $A_\lambda(c_0)$, $A_\lambda(c)$, and $A_\lambda(\ell_\infty)$ is the set $A \cap B$.*

Proof. This is similar to the proof of Theorem 18. So, we omit the details. □

6. Some Matrix Transformation Related to Sequence Spaces $A_\lambda(c_0)$, $A_\lambda(c)$, and $A_\lambda(\ell_\infty)$

In this section, we characterize the matrix transformations from the spaces $A_\lambda(\ell_\infty)$ and $A_\lambda(c)$ into the spaces ℓ_∞, f, f_0, c, and c_0 of bounded, almost convergent, almost null, convergent, and null sequences, respectively. We write throughout for brevity that

$$\begin{aligned}
\Delta a_{nk} &= a_{nk} - a_{n,k+1}, \\
\tilde{a}_{nk} &= (\lambda_k - \lambda_{k-1})\, \Delta \left(\frac{a_{nk}}{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}} \right), \\
a(n,k) &= \sum_{j=0}^{n} a_{jk}, \\
c_{nk} &= \sum_{j=0}^{n} \frac{1}{j+1} a_{jk}, \\
d_{nk} &= s a_{n-1,k} + r a_{nk}, \\
e_{nk} &= t a_{n-2,k} + s a_{n-1,k} + r a_{nk}, \\
a(n,k,m) &= \frac{1}{m+1} \sum_{j=0}^{m} a_{n+j,k}
\end{aligned} \tag{41}$$

for all $k, m, n \in \mathbb{N}$, and we use these abbreviations with other letters, where $r, s, t \in \mathbb{R} \setminus \{0\}$.

Theorem 20. $A = (a_{nk}) \in (A_\lambda(X) : \ell_\infty)$ *if and only if*

$$\left(\frac{\lambda_k - \lambda_{k-1}}{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}} a_{nk} \right)_{k \in \mathbb{N}} \in c_0 \quad \text{for each fixed } n \in \mathbb{N} \tag{42}$$

$$\sup_{n \in \mathbb{N}} \sum_k |\tilde{a}_{nk}| < \infty. \tag{43}$$

Proof. Suppose that the conditions (42) and (43) hold, and take any $x = (x_k) \in A_\lambda(X)$. Then, the sequence $(a_{nk})_{k \in \mathbb{N}} \in \{A_\lambda(X)\}^\beta$ for all $n \in \mathbb{N}$, and this implies the existence of the A-transform of x.

Let us now consider the following equality derived by using the relation (8) from the mth partial sum of the series $\sum_k a_{nk} x_k$:

$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m-1} \tilde{a}_{nk} y_k + \frac{\lambda_m - \lambda_{m-1}}{\lambda_m - 2\lambda_{m-1} + \lambda_{m-2}} a_{nm} y_m \tag{44}$$

for all $m, n \in \mathbb{N}$. Therefore, we obtain from (44) with (42), as $m \to \infty$, that

$$\sum_k a_{nk} x_k = \sum_k \tilde{a}_{nk} y_k, \quad \forall n \in \mathbb{N}. \tag{45}$$

Now, by taking the sup-norm in (45), we derive that

$$\sup_{n \in \mathbb{N}} |(Ax)_n| \le \|y\|_\infty \sup_{n \in \mathbb{N}} \sum_k |\tilde{a}_{nk}| < \infty \tag{46}$$

which shows the sufficiency of the conditions (42) and (43).

Conversely, suppose that $A = (a_{nk}) \in (A_\lambda(X) : \ell_\infty)$. Then, since $(a_{nk})_{k \in \mathbb{N}} \in \{A_\lambda(X)\}^\beta$ for all $n \in \mathbb{N}$ by the hypothesis, the necessity of (42) is trivial and (45) holds. Consider the continuous linear functionals f_n defined on $A_\lambda(X)$ by the sequences $a_n = (a_{nk})_{k \in \mathbb{N}}$ as

$$f_n(x) = \sum_k a_{nk} x_k, \quad \forall n \in \mathbb{N}. \tag{47}$$

Since $A_\lambda(\ell_\infty) \cong \ell_\infty$, $A_\lambda(c) \cong c$ and $A_\lambda(c_0) \cong c_0$, it should follow with (45) that $\|f_n\| = \|\tilde{a}_n\|_\infty$. This just says that the functionals defined by the rows of A on $A_\lambda(X)$ are pointwise bounded. Hence, by Banach-Steinhaus theorem, f_n's are uniformly bounded which gives that there exists a constant $K > 0$ such that $\|f_n\| \le K$ for all $n \in \mathbb{N}$. It therefore follows that $\sum_k |\tilde{a}_{nk}| = \|f_n\| \le K$ holds for all $n \in \mathbb{N}$ which shows the necessity of the condition (43).

This step completes the proof. □

Prior to characterizing the class of infinite matrices from the space $A_\lambda(\ell_\infty)$ into the space of almost convergent sequences, we give a short knowledge on the concept of *almost convergence*. The *shift operator* P is defined on ω by $P_n(x) = x_{n+1}$ for all $n \in \mathbb{N}$. A Banach limit L is defined on ℓ_∞, as a non-negative linear functional, such that $L(Px) = L(x)$ and $L(e) = 1$. A sequence $x = (x_k) \in \ell_\infty$ is said to be almost convergent to the generalized limit α if all Banach limits of x coincide and are equal to α [8] and is denoted by $f - \lim x_k = \alpha$. Let P^i be the composition of P with itself i times and write for a sequence $x = (x_k)$

$$t_{mn}(x) = \frac{1}{m+1} \sum_{i=0}^{m} P_n^i(x), \quad \forall m, n \in \mathbb{N}. \tag{48}$$

Lorentz [8] proved that $f - \lim x_k = \alpha$ if and only if $\lim_{m \to \infty} t_{mn}(x) = \alpha$, uniformly in n. It is well known that a convergent sequence is almost convergent such that its ordinary and generalized limits are equal. By f_0 and f, we denote the spaces of almost null and almost convergent sequences, respectively. Now, we can give the lemma characterizing the almost coercive matrices.

Lemma 21 (see [9, Theorem 1]). *$A = (a_{nk}) \in (\ell_\infty : f)$ if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty; \tag{49}$$

$$\exists \alpha_k \in \mathbb{C} \ni f - \lim a_{nk} = \alpha_k \quad \text{for each fixed } k \in \mathbb{N}; \tag{50}$$

$$\lim_{m\to\infty} \sum_k |a(n,k,m) - \alpha_k| = 0 \quad \text{uniformly in } n. \tag{51}$$

Theorem 22. *$A = (a_{nk}) \in (A_\lambda(\ell_\infty) : f)$ if and only if the conditions* (42) *and* (43) *hold, and*

$$\exists \alpha_k \in \mathbb{C} \ni f - \lim \tilde{a}_{nk} = \tilde{a}_k \quad \text{for each fixed } k \in \mathbb{N}; \tag{52}$$

$$\lim_{m\to\infty} \sum_k |\tilde{a}(n,k,m) - \tilde{a}_k| = 0 \quad \text{uniformly in } n. \tag{53}$$

Proof. Let $A \in (A_\lambda(\ell_\infty) : f)$. Then, since $f \subset \ell_\infty$, the necessity of (42) and (43) is immediately obtained from Theorem 20. To prove the necessity of (52), consider the sequence $e^{(n)}(\lambda) = \{e_k^{(n)}(\lambda)\}_{n\in\mathbb{N}} \in A_\lambda(\ell_\infty)$, defined by (16) for every fixed $k \in \mathbb{N}$. Since Ax exists and is in f for every $x \in A_\lambda(\ell_\infty)$, one can easily see that $Ae^{(n)}(\lambda) = (\tilde{a}_{nk})_{n\in\mathbb{N}} \in f$ for all $k \in \mathbb{N}$, that is, the condition (52) is necessary.

Define the matrix $B = (b_{nk})$ by $b_{nk} = \tilde{a}_{nk}$ for all $k, n \in \mathbb{N}$. Then, we derive from the equality (45) that $Ax = By$. Since $A = (a_{nk}) \in (A_\lambda(\ell_\infty) : f)$ by the hypothesis, we have $B \in (\ell_\infty : f)$. Therefore, the matrix B satisfies the condition (51) of Lemma 21 which is equivalent to the condition (53).

Conversely, suppose that the matrix A satisfies the conditions (42), (43), (52), and (53), and $x \in A_\lambda(\ell_\infty)$. Reconsider the equality $Ax = By$ obtained from (45) with b_{nk} instead of $\tilde{a}_{nk}$. Then, the conditions (49), (50), and (51) are satisfied by the matrix B. Hence, B is almost coercive by Lemma 21 and this gives by passing to f-limit in (45) that $Ax \in f$, that is, $A \in (A_\lambda(\ell_\infty) : f)$, as desired.

This concludes the proof. □

As a direct consequence of Theorem 22, we have the following.

Corollary 23. *$A = (a_{nk}) \in (A_\lambda(\ell_\infty) : f_0)$ if and only if the conditions* (42) *and* (43) *hold, and* (52) *and* (53) *hold with $\tilde{a}_k = 0$ for all $k \in \mathbb{N}$.*

Theorem 24. *$A = (a_{nk}) \in (A_\lambda(\ell_\infty) : c)$ if and only if the condition* (42) *holds, and the conditions*

$$\sum_k |\tilde{a}_{nk}| \text{ converge uniformly in } n \in \mathbb{N}; \tag{54}$$

$$\exists \alpha_k \in \mathbb{C} \ni \lim_{n\to\infty} a_{nk} = \alpha_k \quad \text{for each fixed } k \in \mathbb{N}. \tag{55}$$

Corollary 25. *$A = (a_{nk}) \in (A_\lambda(\ell_\infty) : c_0)$ if and only if the conditions* (42) *and* (54) *hold, and* (55) *also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$.*

Now, we give the following lemma due to King [10] characterizing the class of almost conservative matrices.

Lemma 26. *$A = (a_{nk}) \in (c : f)$ if and only if* (49) *and* (50) *hold, and*

$$\exists \alpha \in \mathbb{C} \ni f - \lim \sum_k a_{nk} = \alpha. \tag{56}$$

Theorem 27. *$A = (a_{nk}) \in (A_\lambda(c) : f)$ if and only if the conditions* (42), (43), *and* (52) *hold, and the condition*

$$\exists \alpha \in \mathbb{C} \ni f - \lim \sum_k \tilde{a}_{nk} = \tilde{a}. \tag{57}$$

also holds.

Proof. This is obtained by a similar way used in proving Theorem 22 with Lemma 26 instead of Lemma 21. So, to avoid the repetition of the similar statements we omit the details. □

Corollary 28. *$A = (a_{nk}) \in (A_\lambda(c) : f)_\rho$ if and only if the conditions* (42) *and* (43) *hold, and the conditions* (52) *and* (57) *also hold with $\alpha_k = 0$ for all $k \in \mathbb{N}$ and $\tilde{a} = \rho$, respectively; where by $(A_\lambda(c) : f)_\rho$, we denote the class of infinite matrices A such that $f - \lim Ax = \rho[A(\lambda) - \lim x]$ for all $x \in A_\lambda(c)$.*

Now, we give the following Steinhaus type theorem.

Theorem 29. *The classes $(A_\lambda(\ell_\infty) : f)$ and $(A_\lambda(c) : f)_\rho$ are disjoint, where $\rho \in \mathbb{R} \setminus \{0\}$.*

Proof. Suppose that the classes $(A_\lambda(\ell_\infty) : f)$ and $(A_\lambda(c) : f)_\rho$ are not disjoint. Then, there is at least one $A = (a_{nk})$ in the set $(A_\lambda(\ell_\infty) : f) \cap (A_\lambda(c) : f)_\rho$. Therefore, one can derive by combining (53) and (52) with $\alpha_k = 0$ for all $k \in \mathbb{N}$ that

$$\lim_{m\to\infty} \sum_k |\tilde{a}(n,k,m)| = 0 \quad \text{uniformly in } n \tag{58}$$

which is contrary to the condition (57) with $\tilde{a} = \rho \neq 0$. This completes the proof. □

Lemma 30 (see [11, Lemma 5.3]). *Let μ, ν be any two sequence spaces, A an infinite matrix, and B a triangle matrix. Then, $A \in (\mu : \nu_B)$ if and only if $BA \in (\mu : \nu)$.*

It is trivial that Lemma 30 has several consequences. Indeed, combining Lemma 30 with Theorems 20, 22, 24, and 27 and Corollaries 23, 25, and 28 by choosing B as one of the special matrices C_1, E^r, R^t, Δ, $\Delta^{(1)}$, A^r, or S, one can easily obtain the following results.

Corollary 31. *Let $A = (a_{nk})$ be an infinite matrix over the complex field. Then, the following statements hold.*

(i) *$E = (e_{nk}) \in (A_\lambda(X) : bv_\infty)$ if and only if* (42) *and* (43) *hold with $e_{nk} - e_{n-1,k}$ instead of a_{nk}, where bv_∞ denotes the space of all sequences $x = (x_k)$ such that $(x_k - x_{k-1}) \in \ell_\infty$ and was introduced by Başar and Altay [11].*

(ii) *$E = (e_{nk}) \in (A_\lambda(X) : e_\infty^r)$ if and only if* (42) *and* (43) *hold with $\sum_{j=0}^n \binom{n}{j} (1-r)^{n-j} r^j e_{jk}$ instead of a_{nk},* *where* e_∞^r *denotes the space of all sequences* $x = (x_k)$ *such that* $(\sum_{j=0}^{n} \binom{n}{j} (1-r)^{n-j} r^j e_{kj} x_j) \in \ell_\infty$ *and was introduced by Altay et al. [12].*

(iii) $E = (e_{nk}) \in (A_\lambda(X) : X_\infty)$ *if and only if* (42) *and* (43) *hold with* $e(n,k)/(n+1)$ *instead of* a_{nk}, *where* X_∞ *denotes the space of all sequences* $x = (x_k)$ *such that* $(\sum_{j=0}^{n} x_j/(n+1)) \in \ell_\infty$ *and was introduced by Ng and Lee [13].*

(iv) $E = (e_{nk}) \in (A_\lambda(X) : r_\infty^t)$ *if and only if* (42) *and* (43) *hold with* $\sum_{j=0}^{n} t_j e_{jk}/T_n$ *instead of* a_{nk}, *where* r_∞^t *denotes the space of all sequences* $x = (x_k)$ *such that* $(\sum_{j=0}^{n} t_j x_j/T_n) \in \ell_\infty$ *and was introduced by Altay and Başar [14].*

(v) $E = (e_{nk}) \in (A_\lambda(X) : bs)$ *if and only if* (42) *and* (43) *hold with* $e(n,k)$ *instead of* a_{nk}.

Corollary 32. *Let* $A = (a_{nk})$ *be an infinite matrix over the complex field. Then, the following statements hold.*

(i) $E = (e_{nk}) \in (A_\lambda(\ell_\infty) : c(\Delta))$ *if and only if* (42), (54), *and* (55) *hold with* $e_{nk} - e_{n+1,k}$ *instead of* a_{nk}, *where* $c(\Delta)$ *denotes the space of all sequences* $x = (x_k)$ *such that* $(x_k - x_{k+1}) \in c$ *and was introduced by Kızmaz [15].*

(ii) $E = (e_{nk}) \in (A_\lambda(\ell_\infty) : e_c^r)$ *if and only if* (42), (54), *and* (55) *hold with* $\sum_{j=0}^{n} \binom{n}{j} (1-r)^{n-j} r^j e_{jk}$ *instead of* a_{nk}, *where* e_c^r *denotes the space of all sequences* $x = (x_k)$ *such that* $E^r x \in c$ *and was introduced by Altay and Başar [16].*

(iii) $E = (e_{nk}) \in (A_\lambda(\ell_\infty) : \tilde{c})$ *if and only if* (42), (54), *and* (55) *hold with* $e(n,k)/(n+1)$ *instead of* a_{nk}, *where* $\tilde{c}$ *denotes the space of all sequences* $x = (x_k)$ *such that* $C_1 x \in c$ *and was introduced by Şengönül and Başar [17].*

(iv) $E = (e_{nk}) \in (A_\lambda(\ell_\infty) : r_c^t)$ *if and only if* (42), (54), *and* (55) *hold with* $\sum_{j=0}^{n} t_j e_{jk}/T_n$ *instead of* a_{nk}, *where* r_c^t *denotes the space of all sequences* $x = (x_k)$ *such that* $R^t x \in c$ *and was introduced by Altay and Başar [18].*

(v) $E = (e_{nk}) \in (A_\lambda(\ell_\infty) : cs)$ *if and only if* (42), (54) *and* (55) *hold with* $e(n,k)$ *instead of* a_{nk}.

Corollary 33. *Let* $A = (a_{nk})$ *be an infinite matrix over the complex field. Then, the following statements hold.*

(i) $A = (a_{nk}) \in (A_\lambda(\ell_\infty) : \hat{f})$ *if and only if* (42), (43), (52), *and* (53) *hold with* d_{nk} *instead of* a_{nk}, *where* $\hat{f}$ *denotes the space of all sequences* $x = (x_k)$ *such that* $B(r,s)x \in f$ *and was introduced by Başar and Kirişçi [19].*

(ii) $A = (a_{nk}) \in (A_\lambda(\ell_\infty) : \hat{f}_0)$ *if and only if* (42) *and* (43) *hold, and* (52) *and* (53) *hold with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$ *and* d_{nk} *instead of* a_{nk}, *where* $\hat{f}_0$ *denotes the space of all sequences* $x = (x_k)$ *such that* $B(r,s)x \in f_0$ *and was introduced by Başar and Kirişçi [19].*

(iii) $A = (a_{nk}) \in (A_\lambda(c) : \hat{f})$ *if and only if* (42), (43), (52), *and* (53) *hold with* d_{nk} *instead of* a_{nk}.

(iv) $A = (a_{nk}) \in (A_\lambda(c) : \hat{f}_0)$ *if and only if* (42) *and* (43) *hold, and* (52) *and* (53) *also hold with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$ *and* $\alpha = 1$, *respectively, hold with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$ *and* d_{nk} *instead of* a_{nk}.

(v) $A = (a_{nk}) \in (A_\lambda(c) : \hat{f})_\rho$ *if and only if the conditions of Corollary 28 hold with* d_{nk} *instead of* a_{nk}.

Corollary 34. *Let* $A = (a_{nk})$ *be an infinite matrix over the complex field. Then, the following statements hold.*

(i) $A = (a_{nk}) \in (A_\lambda(\ell_\infty) : \tilde{f})$ *if and only if* (42), (43), (52), *and* (53) *hold with* c_{nk} *instead of* a_{nk}, *where* $\tilde{f}$ *denotes the space of all sequences* $x = (x_k)$ *such that* $C_1 x \in f$ *and was introduced by Kayaduman and Şengönül [20].*

(ii) $A = (a_{nk}) \in (A_\lambda(\ell_\infty) : \tilde{f}_0)$ *if and only if* (42) *and* (43) *hold, and* (52) *and* (53) *hold with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$ *and* c_{nk} *instead of* a_{nk}, *where* $\tilde{f}_0$ *denotes the space of all sequences* $x = (x_k)$ *such that* $C_1 x \in f_0$ *and was introduced by Kayaduman and Şengönül [20].*

(iii) $A = (a_{nk}) \in (A_\lambda(c) : \tilde{f})$ *if and only if* (42), (43), (52), *and* (53) *hold with* c_{nk} *instead of* a_{nk}.

(iv) $A = (a_{nk}) \in (A_\lambda(c) : \tilde{f}_0)$ *if and only if* (42) *and* (43) *hold, and* (52) *and* (53) *also hold with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$ *and* $\alpha = 1$, *respectively, with* c_{nk} *instead of* a_{nk}.

(v) $A = (a_{nk}) \in (A_\lambda(c) : \tilde{f})_\rho$ *if and only if the conditions of Corollary 28 hold with* c_{nk} *instead of* a_{nk}.

Corollary 35. *Let* $A = (a_{nk})$ *be an infinite matrix over the complex field. Then, the following statements hold.*

(i) $A = (a_{nk}) \in (A_\lambda(\ell_\infty) : f(B))$ *if and only if* (42), (43), (52), *and* (53) *hold with* e_{nk} *instead of* a_{nk}, *where* $f(B)$ *denotes the space of all sequences* $x = (x_k)$ *such that* $B(r,s,t)x \in f$ *and was introduced by Sönmez [21].*

(ii) $A = (a_{nk}) \in (A_\lambda(\ell_\infty) : f_0(B))$ *if and only if* (42) *and* (43) *hold, and* (52) *and* (53) *hold with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$ *and* e_{nk} *instead of* a_{nk}, *where* $f_0(B)$ *denotes the space of all sequences* $x = (x_k)$ *such that* $B(r,s,t)x \in f_0$ *and was introduced by Sönmez [21].*

(iii) $A = (a_{nk}) \in (A_\lambda(c) : f(B))$ *if and only if the conditions* (42), (43), (52), *and* (53) *hold with* e_{nk} *instead of* a_{nk}.

(iv) $A = (a_{nk}) \in (A_\lambda(c) : f_0(B))$ *if and only if the conditions* (42) *and* (43) *hold, and the conditions* (52) *and* (53) *also hold with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$ *and* $\alpha = 1$, *respectively, hold with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$ *and* e_{nk} *instead of* a_{nk}.

(v) $A = (a_{nk}) \in (A_\lambda(c) : f(B))_\rho$ *if and only if the conditions of Corollary 28 hold with* e_{nk} *instead of* a_{nk}.

7. Conclusion

Mursaleen and Noman [2, 22, 23] have studied the domains ℓ_∞^λ, c^λ, c_0^λ, and ℓ_p^λ of the matrix Λ in the classical sequence spaces ℓ_∞, c, c_0, and ℓ_p, respectively. Malkowsky and Rakočević [24] characterized some classes of matrix transformations and investigated related compact operators involving the spaces of Λ-null, Λ-convergent, and Λ-bounded sequences. Quite recently, Sönmez and Başar [25] have introduced the spaces $c_0^\lambda(B)$ and $c^\lambda(B)$ of generalized difference sequences which generalize the paper due to Mursaleen and Noman [26]. Mursaleen and Noman [26] have derived some inclusion relations and determined the alpha-, beta-, and gamma-duals of those spaces and constructed their Schauder bases. Finally, Sönmez and Başar [25] have characterized some matrix classes from the spaces $c_0^\lambda(B)$ and $c^\lambda(B)$ to the spaces ℓ_p, c_0, and c. In the present paper, we emphasize the domains $A_\lambda(c_0)$, $A_\lambda(c)$, and $A_\lambda(\ell_\infty)$ of the matrix $A(\lambda)$ in the classical sequence spaces c_0, c, and ℓ_∞. Our results are more general and comprehensive than the corresponding results of Mursaleen and Noman [2, 22, 23] derived with the matrix Λ. We should note that, as a natural continuation of the present paper, one can study the domains $A_\lambda(\ell_p)$ and $A_\lambda(bv_p)$ of the matrix $A(\lambda)$ in the classical sequence space ℓ_p and in the sequence space bv_p with $0 < p < 1$ and $1 \leq p < \infty$, where bv_p denotes the space of all sequences $x = (x_k)$ such that $(x_k - x_{k-1}) \in \ell_p$ and introduced in the case $1 \leq p < \infty$ by Başar and Altay [11] and in the case $0 < p < 1$ by Altay and Başar [27].

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