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Research Article

Product and Commutativity of *k*th-Order Slant Toeplitz Operators

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The commutativity of kth-order slant Toeplitz operators with harmonic polynomial symbols, analytic symbols, and coanalytic symbols is discussed. We show that, on the Lebesgue space and Bergman space, necessary and sufficient conditions for the commutativity of kth-order slant Toeplitz operators are that their symbol functions are linearly dependent. Also, we study the product of two kth-order slant Toeplitz operators and give some necessary and sufficient conditions.

1. Preliminaries

Throughout this paper, k is a fixed positive integer, and $k \geq 2$. Let $\varphi(z) = \sum_{i=-\infty}^{\infty} a_i z^i$ be a bounded measurable function on the unit circle $\mathbb{T}$, where $a_i = \langle \varphi, z^i \rangle$ is the ith Fourier coefficient of φ and $\{z^i : i \in \mathbb{Z}\}$ is the usual basis of $L^2(\mathbb{T})$, with $\mathbb{Z}$ being the set of integers. The kth-order slant Toeplitz operator U_φ with symbol φ in $L^\infty(\mathbb{T})$ is defined on $L^2(\mathbb{T})$ as follows:

$$U_\varphi\left(z^l\right) = \sum_{i=-\infty}^{\infty} a_{ki-l} z^i. \quad (1)$$

In the past several decades, slant Toeplitz operators have played outstanding roles in wavelet analysis, curve and surface modelling, and dynamical systems (e.g., see [1–10]). For instance, Villemoes [7] has associated the Besov regularity of solution of the refinement equation with the spectral radius of an associated slant Toeplitz operators and has used the spectral radius of the slant Toeplitz operators to characterize the L_p $(1 \leq p \leq \infty)$ regularity of refinable functions; Goodman et al. [6] have shown the connection between the spectral radii and conditions for the solutions of certain differential equations that in the Lipschitz classes. However, these mathematicians concentrated mainly on the applications, but these considerations serve as a source of motivation to introduce and study the properties of slant Toeplitz operators.

In 1995, Ho [11–14] began a systematic study of the slant Toeplitz operators on the Hardy space. In [15–17], the authors discussed some properties of kth-order slant Toeplitz operator. In [18, 19], the authors defined the slant Toeplitz operator and kth-order slant Toeplitz operator on the Bergman spaces, respectively, and studied some properties of these operators.

In this paper, properties of kth-order slant Toeplitz operators with harmonic polynomial symbols, analytic symbols, and coanalytic symbols are discussed. We show that, on the Lebesgue space and Bergman space, the necessary and sufficient conditions for the commutativity of kth-order slant Toeplitz operators are that their symbol functions are linearly dependent. Meanwhile, we study the product of two kth-order slant Toeplitz operators and give some necessary and sufficient conditions.

2. Commutativity of kth-Order Slant Toeplitz Operators on $L^2(\mathbb{T})$

In [17], we investigated the properties of kth-order slant Toeplitz operators on $L^2(\mathbb{T})$ and have obtained that for $\varphi, \psi \in L^\infty(\mathbb{T})$, U_φ and U_ψ commute (essentially commute) if and only if $\varphi(z^k)\psi - \varphi\psi(z^k) = 0$.

Immediately we come up with the following problem.

Could the commutativity of two kth-order slant Toeplitz operators be fully characterized by their symbols?

The partial answer to the pervious problem has been obtained in [17]: for $\psi \in L^{\infty}(\mathbb{T})$ and $\varphi(z)=\sum_{p=0}^{k-1} a_p z^p$ or $\varphi(z) = \sum_{p=-k+1}^{0} a_p z^p$, U_φ and U_ψ commute (essentially commute) if and only if there exist scalars α and β, not both zero, such that $\alpha\varphi + \beta\psi = 0$.

In this section the commutativity of kth-order slant Toeplitz operators with analytic symbols and harmonic symbols will be studied. First we discuss the commutativity of two kth-order slant Toeplitz operators with analytic symbols.

Proposition 1. *Let $\varphi, \psi \in H^{\infty}(\mathbb{T})$, then the following statements are equivalent:*

(1.1) $\varphi(z^k)\psi(z) = \varphi(z)\psi(z^k)$;

(1.2) *there exist scalars α and β, not both zero, such that $\alpha\varphi + \beta\psi = 0$.*

Proof. Begin with the easy direction. First, suppose that (1.2) holds. Without loss of generality, let $\alpha \neq 0$, so that $\varphi = -(\beta/\alpha)\psi$. Thus, $\varphi(z^k)\psi(z) = \varphi(z)\psi(z^k)$.

To prove the other direction of the proposition, suppose that (1.1) holds; that is, $\varphi(z^k)\psi(z) = \varphi(z)\psi(z^k)$. Let $\varphi(z) = \sum_{p=0}^{\infty} a_p z^p$ and $\psi(z) = \sum_{p=0}^{\infty} b_p z^p$, then

$$\sum_{p=0}^{\infty} a_p z^{kp} \cdot \sum_{p=0}^{\infty} b_p z^p = \sum_{p=0}^{\infty} a_p z^p \cdot \sum_{p=0}^{\infty} b_p z^{kp}, \tag{2}$$

that is,

$$\sum_{p=0}^{\infty} \sum_{ki+j=p} a_i b_j z^p = \sum_{p=0}^{\infty} \sum_{i+kj=p} a_i b_j z^p, \tag{3}$$

where i and j are both nonnegative integers. Hence, for all nonnegative integers i, j, and p,

$$\sum_{ki+j=p} a_i b_j = \sum_{i+kj=p} a_i b_j. \tag{4}$$

Now we would give the proof in four cases.

Case I. Suppose that $a_0 b_0 \neq 0$. Let $\lambda = b_0/a_0$ and we continue the proof by the induction.

When $p = 1$, from (4) we get that $a_0 b_1 = a_1 b_0$, which means that $b_1 = \lambda a_1$.

When $p = 2$, from (4) we get that

$$\begin{aligned} a_0 b_2 + a_1 b_0 = a_2 b_0 + a_0 b_1, \quad &\text{if } k = 2, \\ a_0 b_2 = a_2 b_0, \quad &\text{if } k > 2, \end{aligned} \tag{5}$$

which means that $b_2 = \lambda a_2$, since $b_1 = \lambda a_1$.

Now suppose that $b_i = \lambda a_i$ for all integers i with $0 \leq i \leq m$. Then, observe the connection between a_{m+1} and b_{m+1}. Let $m+1 = kl + r$, where l and r are nonnegative integers with $0 \leq r \leq k-1$.

When $p = m+1$, from (4) we get

$$\begin{aligned} &a_0 b_{m+1} + a_1 b_{m+1-k} + a_2 b_{m+1-2k} + \cdots + a_l b_{m+1-lk} \\ &\quad = b_0 a_{m+1} + b_1 a_{m+1-k} + b_2 a_{m+1-2k} + \cdots + b_l a_{m+1-lk}. \end{aligned} \tag{6}$$

From the assumption we have $a_0 b_{m+1} = b_0 a_{m+1}$; that is; $b_{m+1} = \lambda a_{m+1}$.

Hence, from the above discussion we get that $b_i = \lambda a_i$ for all integers i with $i \geq 0$ by the induction, which means that $\psi(z) = \lambda\varphi(z)$. So, the required result holds.

Case II. Suppose that $a_0 = 0$ and $b_0 \neq 0$. We want to show that $a_i = 0$ for all integers i with $i \geq 0$; that is, $\varphi \equiv 0$. Suppose that there exists some a_i which is not zero. Without loss of generality, let $a_i = 0$ for all integers i with $0 \leq i \leq l-1$ and $a_l \neq 0$, where $l \geq 1$ is an integer, then $\varphi(z) = z^l \varphi_1(z)$ and $\varphi_1(z) = \sum_{i=0}^{\infty} a_{i+l} z^i \doteq \sum_{i=0}^{\infty} a'_i z^i$.

Because $\varphi(z^k)\psi(z) = \varphi(z)\psi(z^k)$, we can get that $z^{(k-1)l}\varphi_1(z^k)\psi(z) = \varphi_1(z)\psi(z^k)$, which means that

$$\sum_{p=0}^{\infty} \sum_{ki+j=p} a'_i b_j z^{p+(k-1)l} = \sum_{p=0}^{\infty} \sum_{i+kj=p} a'_i b_j z^p. \tag{7}$$

Thus, we have $a'_0 b_0 = 0$, and, so, since $a'_0 = a_l \neq 0$, we have $b_0 = 0$. This leads to a contradiction.

Hence, $\varphi \equiv 0$ and the required result holds.

Case III. Suppose that $a_0 \neq 0$ and $b_0 = 0$. Similar to Case II, we can get that $\psi \equiv 0$. So, the required result holds.

Case IV. Suppose that $a_0 = 0$ and $b_0 = 0$. If $\varphi \equiv 0$ or $\psi \equiv 0$, then the required result holds. Otherwise, without loss of generality, let $a_i = 0$ for all integers i with $0 \leq i \leq l-1$ and $a_l \neq 0$, and let $b_j = 0$ for all integers j with $0 \leq j \leq m-1$ and $b_m \neq 0$, where l and m are positive integers. Then, $\varphi(z) = z^l \varphi_1(z)$ and $\psi(z) = z^m \psi_1(z)$.

If $l = m$, then $\varphi(z^k)\psi(z) = \varphi(z)\psi(z^k)$ is equal to $\varphi_1(z^k)\psi_1(z) = \varphi_1(z)\psi_1(z^k)$. The proof is similar to Case I.

If $l > m$, then $\varphi(z^k)\psi(z) = \varphi(z)\psi(z^k)$ is equal to $z^{k(l-m)}\varphi_1(z^k)\psi_1(z) = \varphi_1(z)\psi_1(z^k)$. The proof is similar to Case II.

If $l < m$, then $\varphi(z^k)\psi(z) = \varphi(z)\psi(z^k)$ is equal to $\varphi_1(z^k)\psi_1(z) = z^{k(m-l)}\varphi_1(z)\psi_1(z^k)$. The proof is similar to Case III. □

From the preceding Proposition 1, it is evident that Corollary 2 holds.

Corollary 2. *Let $\overline{\varphi}, \overline{\psi} \in H^{\infty}(\mathbb{T})$, then the following statements are equivalent:*

(1.1) $\varphi(z^k)\psi(z) = \varphi(z)\psi(z^k)$;

(1.2) *there exist scalars α and β, not both zero, such that $\alpha\varphi + \beta\psi = 0$.*

From Theorem 2.8 in [17], Proposition 1, and Corollary 2, we can obtain the following Theorem 3.

Theorem 3. *Let φ, $\psi \in H^\infty(\mathbb{T})$ or $\overline{\varphi}, \overline{\psi} \in H^\infty(\mathbb{T})$, the following statements are equivalent:*

(1.1) *U_φ and U_ψ commute;*

(1.2) *U_φ and U_ψ essentially commute;*

(1.3) $\varphi(z^k)\psi(z) = \varphi(z)\psi(z^k)$;

(1.4) *there exist scalars α and β, not both zero, such that $\alpha\varphi + \beta\psi = 0$.*

Now we start to study the commutativity of two kth-order slant Toeplitz operators with harmonic symbols.

Proposition 4. *Let $\varphi(z) = \sum_{p=-n}^{n} a_p z^p$ and $\psi(z) = \sum_{p=-n}^{n} b_p z^p$, where $a_{-n}^2 + b_{-n}^2 \neq 0$ and n is a positive integer, then the following statements are equivalent:*

(1.1) $\varphi(z^k)\psi(z) = \varphi(z)\psi(z^k)$;

(1.2) *there exist scalars α and β, not both zero, such that $\alpha\varphi + \beta\psi = 0$.*

Proof. We begin with the easy direction. First, suppose that (1.2) holds and let $\alpha \neq 0$ without lost of generality, so that $\varphi = -(\beta/\alpha)\psi$. Thus, $\varphi(z^k)\psi(z) = \varphi(z)\psi(z^k)$.

To prove the other direction of the proposition, suppose that (1.1) holds. Since $\varphi(z) = \sum_{p=-n}^{n} a_p z^p$ and $\psi(z) = \sum_{p=-n}^{n} b_p z^p$, then $\varphi(z^k) = \sum_{p=-n}^{n} a_p z^{kp}$, $\psi(z^k) = \sum_{p=-n}^{n} b_p z^{kp}$ and

$$\begin{aligned} \varphi\left(z^k\right)\psi(z) &= \sum_{p=-n}^{n} a_p z^{kp} \cdot \sum_{p=-n}^{n} b_p z^p \\ &= \sum_{p=-(k+1)n}^{(k+1)n} \sum_{ki+j=p}^{n} a_i b_j z^p, \\ \varphi(z)\psi\left(z^k\right) &= \sum_{p=-n}^{n} a_p z^p \cdot \sum_{p=-n}^{n} b_p z^{kp} \\ &= \sum_{p=-(k+1)n}^{(k+1)n} \sum_{i+kj=p}^{n} a_i b_j z^p, \end{aligned} \tag{8}$$

where both i and j are integers with $-n \le i,\ j \le n$. Because $\varphi(z^k)\psi(z) = \varphi(z)\psi(z^k)$, we can get that

$$\sum_{ki+j=p} a_i b_j = \sum_{i+kj=p} a_i b_j, \tag{9}$$

for any integers p with $-(k+1)n \le p \le (k+1)n$.

Now we start to investigate the connection between a_j and b_j ($j = -n, \ldots, n$) by induction. Since $a_{-n}^2 + b_{-n}^2 \neq 0$, without loss of generality, let $a_{-n} \neq 0$ and let $\lambda = b_{-n}/a_{-n}$, which also means that $b_{-n} = \lambda a_{-n}$.

When $p = -(k+1)n + 1$, then by (9), we can get that $a_{-n}b_{-n+1} = a_{-n+1}b_{-n}$, which means that $b_{-n+1} = \lambda a_{-n+1}$, since $a_{-n} \neq 0$.

Suppose that $b_{-n+j} = \lambda a_{-n+j}$ for any integers j with $0 \le j \le l-1$, where $1 \le l \le 2n$. Now we consider the connection between b_{-n+l} and a_{-n+l}. Let $l = mk + r$, where m and r are both nonnegative integers with $0 \le r \le k-1$.

When $p = -(k+1)n + l$, by (9) we get that

$$\begin{aligned} &a_{-n}b_{-n+l} + a_{-n+1}b_{-n+l-k} + \cdots + a_{-n+m}b_{-n+l-mk} \\ &\quad = b_{-n}a_{-n+l} + b_{-n+1}a_{-n+l-k} + \cdots + b_{-n+m}a_{-n+l-mk}. \end{aligned} \tag{10}$$

From the assumption we get that $a_{-n}b_{-n+l} = a_{-n+l}b_{-n}$, which means that $b_{-n+l} = \lambda a_{-n+l}$.

Hence, by the induction we obtain that $b_j = \lambda a_j$ ($j = -n, \ldots, n$); that is, $\psi(z) = \sum_{j=-n}^{n} \lambda a_j z^j = \lambda \sum_{j=-n}^{n} a_j z^j = \lambda\varphi(z)$. So, the required result holds. □

Lemma 5. *Let $\varphi(z) = \sum_{p=-n}^{n} a_p z^p$, $\psi(z) = \sum_{p=-m}^{m} b_p z^p$, and $b_m b_{-m} \neq 0$, where n and m are integers and $n > m \ge 1$. If $\varphi(z^k)\psi(z) = \varphi(z)\psi(z^k)$, then $a_j = 0$ for any integers j with $m+1 \le |j| \le n$.*

Proof. Since $\varphi(z) = \sum_{p=-n}^{n} a_p z^p$, $\psi(z) = \sum_{p=-m}^{m} b_p z^p$, and $\varphi(z^k)\psi(z) = \varphi(z)\psi(z^k)$, then

$$\sum_{p=-(kn+m)}^{kn+m} \sum_{ki+j=p} a_i b_j z^p = \sum_{p=-(n+km)}^{n+km} \sum_{i+kj=p} a_i b_j z^p. \tag{11}$$

Since $n > m \ge 1$, we can get that for any integers p with $n + km + 1 \le |p| \le kn + m$,

$$\sum_{ki+j=p} a_i b_j = 0, \tag{12}$$

for any integers p with $0 \le |p| \le n + km$,

$$\sum_{ki+j=p} a_i b_j = \sum_{i+kj=p} a_i b_j. \tag{13}$$

We want to show that $a_j = 0$ for any integers j with $m + 1 \le j \le n$. Here are two cases: $n < m + k$ and $n \ge m + k$. Let $m = lk + r$, where l and r are nonnegative integers with $0 \le r \le k-1$.

First Case. If $n < m + k$, then $k(m+1) + m = km + m + k > km + n$. Now we, continue the discussion by induction.

When $p = kn + m$, by (12), we can get that $a_n b_m = 0$. So, $a_n = 0$, since $b_{-m}b_m \neq 0$.

When $p = kn + m - k$, by (12) we can get that $a_{n-1}b_m + (m-k)^+ a_n b_{m-k} = 0$, where $(m-k)^+ = \max\{\operatorname{sgn}(2m - k + 1), 0\}$ and sgn is a sign function. So, $a_{n-1} = 0$, since $b_{-m}b_m \neq 0$ and $a_n = 0$.

Suppose that $a_{n-j} = 0$ for any integers j with $0 \le j \le t < n - m - 1$. Now, consider the value of a_{n-t-1}.

When $p = kn + m - k(t+1)$, by (12) we get that

$$a_{n-t-1}b_m + \cdots + a_{n-t-1+\lambda}b_{m-\lambda k} = 0, \tag{14}$$

where

$$\begin{aligned} \lambda &= t + 1, \quad \text{if } 2m \ge (t+1)k, \\ \lambda &= \left[\frac{2m}{k}\right], \quad \text{if } 2m < (t+1)k, \end{aligned} \tag{15}$$

and $[2m/k]$ is the biggest integer which is not bigger than $2m/k$. Then, by the assumption and (14), we get that $a_{n-t-1} = 0$, since $b_m b_{-m} \neq 0$.

Hence, from the above discussion we obtain that $a_j = 0$ for all integers j with $m+1 \leq j \leq n$ by the induction.

Second Case. If $n \geq m+k$, then $k(m+1)+m = km+m+k \leq km+n$. Now we continue the discussion by induction.

When $p = kn+m$, by (12), we can get that $a_n b_m = 0$. So, $a_n = 0$, since $b_{-m} b_m \neq 0$.

Suppose that a_{n-j} for any integers j with $0 \leq j \leq t < n-m-1$. Now, consider the value of a_{n-t-1}.

If $p = kn+m-k(t+1) > km+n$, by (12) we get that

$$a_{n-t-1} b_m + \cdots + a_{n-t-1+\lambda} b_{m-\lambda k} = 0, \tag{16}$$

where

$$\begin{aligned} \lambda &= t+1, \quad \text{if } 2m \geq (t+1)k, \\ \lambda &= \left[\frac{2m}{k}\right], \quad \text{if } 2m < (t+1)k, \end{aligned} \tag{17}$$

and $[x]$ is the biggest integer which is not bigger than x. Then, by the assumption and (16), we get that $a_{n-t-1} = 0$, since $b_m b_{-m} \neq 0$.

If $p = kn+m-k(t+1) \leq km+n$, by (13), we get that

$$\begin{aligned} a_{n-t-1} b_m &+ \cdots + a_{n-t-1+\lambda_1} b_{m-\lambda_1 k} \\ &= b_m a_{kn-km-kt-k+m} + \cdots + b_{m-\lambda_2} a_{kn-km-kt-k+m+\lambda_2 k}, \end{aligned} \tag{18}$$

where

$$\begin{aligned} \lambda_1 &= t+1, \quad \text{if } 2m \geq (t+1)k, \\ \lambda_1 &= \left[\frac{2m}{k}\right], \quad \text{if } 2m < (t+1)k, \\ \lambda_2 &= 2m, \quad \text{if } kn+km-kt-k+m \leq n, \\ \lambda_2 &= \left[\frac{n+km+kt+k-kn-m}{k}\right], \\ &\quad \text{if } kn+km-kt-k+m > n, \end{aligned} \tag{19}$$

and $[x]$ is the biggest integer which is not bigger than x. Then, by the assumption and (18), we get that $a_{n-t-1} = 0$, since $b_m b_{-m} \neq 0$.

Hence, from the above discussion we obtain that $a_j = 0$ for all integers j with $m+1 \leq j \leq n$ by the induction.

Similarly, we could get that $a_j = 0$ for all integers j with $-n \leq j \leq -(m+1)$. □

From Proposition 4 and Lemma 5, it is evident that Proposition 6 holds.

Proposition 6. *Let $\varphi(z) = \sum_{p=-n}^{n} a_p z^p$, $\psi(z) = \sum_{p=-m}^{m} b_p z^p$ and $b_m b_{-m} \neq 0$, where n, m are integers and $n > m \geq 1$, then the following statements are equivalent:*

(1.1) *$\varphi(z^k)\psi(z) = \varphi(z)\psi(z^k)$;*

(1.2) *there exist scalars α and β, not both zero, such that $\alpha\varphi + \beta\psi = 0$.*

Theorem 7 is obvious from Theorem 2.8 in [17] and Proposition 6.

Theorem 7. *Let $\varphi(z) = \sum_{p=-n}^{n} a_p z^p$, $\psi(z) = \sum_{p=-m}^{m} b_p z^p$, and $b_m b_{-m} \neq 0$, where n and m are integers and $n > m \geq 1$, and the following statements are equivalent:*

(1.1) *U_φ and U_ψ commute;*

(1.2) *U_φ and U_ψ essentially commute;*

(1.3) *$\varphi(z^k)\psi(z) = \varphi(z)\psi(z^k)$;*

(1.4) *there exist scalars α and β, not both zero, such that $\alpha\varphi + \beta\psi = 0$.*

3. Product of Two kth-Order Slant Toeplitz Operators on $L^2(\mathbb{T})$

In [15, 17], the authors have investigated properties of the product of two kth-order slant Toeplitz operators on $L^2(\mathbb{T})$ and have obtained the following result.

Theorem 8 (see [15, 17]). *Let $\varphi, \psi \in L^\infty(\mathbb{T})$, then the following statements are equivalent:*

(1.1) *$U_\varphi U_\psi$ is a kth-order slant Toeplitz operator;*

(1.2) *$U_\varphi U_\psi$ is a zero operator;*

(1.3) *$U_\varphi U_\psi$ is compact;*

(1.4) *$\varphi(z^k)\psi(z) = 0$.*

In this section, we will describe properties of the product of two kth-order slant Toeplitz operators with analytic symbols and harmonic symbols on $L^2(\mathbb{T})$ by their symbols. First, we start to discuss properties of two kth-order slant Toeplitz product with analytic symbols.

Proposition 9. *Let $\varphi, \psi \in H^\infty(\mathbb{T})$, then $\varphi(z^k)\psi(z) = 0$ if and only if $\varphi = 0$ or $\psi = 0$.*

Proof. As we know, the "if" direction of the proposition is trivial.

Now suppose that $\varphi(z^k)\psi(z) = 0$. Since $\varphi, \psi \in H^\infty(\mathbb{T})$, we have $\widehat{\varphi}(z^k)\widehat{\psi}(z) = 0$, where $\widehat{\varphi}$ and $\widehat{\psi}$ are the Poisson extensions of φ and ψ, respectively, and they are analytic on the unit disk $\mathbb{D}$. Hence, we get that $\widehat{\varphi}(z^k)$ is identically 0 or $\widehat{\psi}$ is identically 0; that is, φ is identically 0 or ψ is identically 0. □

Similarly, we could obtain Corollary 10.

Corollary 10. *Let $\overline{\varphi}, \overline{\psi} \in H^\infty(\mathbb{T})$, then $\varphi(z^k)\psi(z) = 0$ if and only if $\varphi = 0$ or $\psi = 0$.*

It is obvious that Theorem 11 holds from the preceding analysis.

Theorem 11. *Let $\varphi, \psi \in H^\infty(\mathbb{T})$ or $\overline{\varphi}, \overline{\psi} \in H^\infty(\mathbb{T})$, then the following statements are equivalent:*

(1.1) $U_\varphi U_\psi$ *is a kth-order slant Toeplitz operator;*

(1.2) $U_\varphi U_\psi$ *is a zero operator;*

(1.3) $U_\varphi U_\psi$ *is compact;*

(1.4) $\varphi(z^k)\psi(z) = 0$*;*

(1.5) $\varphi = 0$ *or* $\psi = 0$.

Now, we start to discuss the properties of two kth-order slant Toeplitz product with harmonic symbols.

Proposition 12. *Let $\varphi(z) = \sum_{p=0}^{n} a_p z^p$ and $\psi(z) = \sum_{p=0}^{m} b_{-p} z^{-p}$, where n and m are both positive integers, then $\varphi(z^k)\psi(z) = 0$ if and only if $\varphi = 0$ or $\psi = 0$.*

Proof. We begin with the easy direction. First, suppose that $\varphi = 0$ or $\psi = 0$; then it is clear that $\varphi(z^k)\psi(z) = 0$.

Without loss of generality, let $a_n^2 + b_{-m}^2 \neq 0$. Otherwise, $a_n^2 + b_{-m}^2 = 0$; then we can consider the value of a_i and b_j, where i and j are both integers with $0 \le i \le n-1$ and $0 \le j \le m-1$, since $\varphi(z)$ and ψ are both polynomial functions. There are four cases: (1) $a_i = 0$ for all integers i with $0 \le i \le n$; (2) $b_{-j} = 0$ for all integers j with $0 \le j \le m$; (3) $a_s \neq 0$ and $a_i = 0$ for all integers i with $s+1 \le i \le n$; (4) $b_{-t} \neq 0$ and $b_{-j} = 0$ for all integers j with $t+1 \le j \le m$, where s and t are both nonnegative integers with $s \le n-1$ and $t \le m-1$. If the first two cases hold, then the required result holds; if the latter two cases holds, then we have $\varphi(z) = \sum_{p=0}^{s} a_p z^p$ and $a_s^2 + b_{-m}^2 \neq 0$ or $\psi(z) = \sum_{p=0}^{t} b_{-p} z^{-p}$ and $a_n^2 + b_{-t}^2 \neq 0$.

Now suppose that $\varphi(z^k)\psi(z) = 0$ and $a_n^2 + b_{-m}^2 \neq 0$. Since $\varphi(z) = \sum_{p=0}^{n} a_p z^p$, $\psi(z) = \sum_{p=0}^{m} b_{-p} z^{-p}$, then

$$\varphi\left(z^k\right)\psi(z) = \sum_{p=0}^{n} a_p z^{kp} \cdot \sum_{p=0}^{m} b_{-p} z^{-p} = \sum_{p=-m}^{kn} \sum_{ki-j=p} a_i b_{-j} z^p, \tag{20}$$

where i and j are both integers with $0 \le i \le n$ and $0 \le j \le m$. Because $\varphi(z^k)\psi(z) = 0$, we get, for any integers p with $-m \le p \le kn$,

$$\sum_{ki-j=p} a_i b_{-j} = 0, \tag{21}$$

where i and j are both integers with $0 \le i \le n$ and $0 \le j \le m$. Since $a_n^2 + b_{-m}^2 \neq 0$, yet we only obtain either $a_n \neq 0$ or $b_{-m} \neq 0$.

First Case. If $a_n \neq 0$. Now we want to show that $\psi \equiv 0$ by the induction.

When $p = kn$, by (21), we get that $a_n b_0 = 0$, which means that $b_0 = 0$, since $a_n \neq 0$.

When $p = kn-1$, by (21), we get that $a_n b_{-1} = 0$, which means that $b_{-1} = 0$, since $a_n \neq 0$.

Now suppose that $b_{-l} = 0$ for any integers l with $0 \le l \le t$, where t is an integer with $0 \le t \le m-1$. Considering the value of b_{-t-1}, when $p = kn-t-1$, by (21), we get that

$$a_n b_{-t-1} + a_{n-1} b_{-t-1+k} + a_{n-2} b_{-t-1+2k} + \cdots + a_{n-\lambda} b_{-t-1+k\lambda} = 0, \tag{22}$$

where $\lambda = \min\{n, [(t+1)/k]\}$ and $[x]$ is the biggest integer which is not bigger than x. By the assumption and the above equation, we get that $a_n b_{-t-1} = 0$; that is, $b_{-t-1} = 0$, since $a_n \neq 0$.

From the preceding analysis, by the induction, we can obtain that $b_{-l} = 0$ for any integers l with $0 \le l \le m$. Hence, $\psi \equiv 0$.

Second Case. If $b_{-m} \neq 0$. Arguing as in the First case, we obtain that $\varphi \equiv 0$. □

Proposition 13. *Let $\varphi(z) = \sum_{p=0}^{n} a_p z^p$ and $\psi(z) = \sum_{p=-m}^{m} b_p z^p$, where n and m are positive integers and $a_n^2 + b_m^2 + b_{-m}^2 \neq 0$, then the following statements are equivalent:*

(1.1) $\varphi(z^k)\psi(z) = 0$*;*

(1.2) $\varphi(z)\psi(z^k) = 0$*;*

(1.3) $\varphi = 0$ *or* $\psi = 0$.

Proof. We begin with the easy direction. First, suppose that $\varphi = 0$ or $\psi = 0$; then it is clear that (1.1) and (1.2) hold.

Now suppose that (1.1) holds. Since $\varphi(z) = \sum_{p=0}^{n} a_p z^p$, $\psi(z) = \sum_{p=-m}^{m} b_p z^p$, then

$$\varphi\left(z^k\right)\psi(z) = \sum_{p=0}^{n} a_p z^{kp} \cdot \sum_{p=-m}^{m} b_p z^p = \sum_{p=-m}^{kn+m} \sum_{ki+j=p} a_i b_j z^p, \tag{23}$$

where i and j are both integers with $0 \le i \le n$ and $-m \le j \le m$. Because $\varphi(z^k)\psi(z) = 0$, we get that for any integers p with $-m \le p \le kn+m$,

$$\sum_{ki+j=p} a_i b_j = 0, \tag{24}$$

where i and j are both integers with $0 \le i \le n$ and $-m \le j \le m$. Since $a_n^2 + b_m^2 + b_{-m}^2 \neq 0$, yet we can obtain that $a_n \neq 0$, $b_{-m} \neq 0$, or $b_m \neq 0$.

First Case. If $a_n \neq 0$. Now we start to continue the proof by induction.

When $p = kn+m$, by (24), we get that $a_n b_m = 0$, which means that $b_m = 0$, since $a_n \neq 0$.

When $p = kn+m-1$, by (24), we get that $a_n b_{m-1} = 0$, which means that $b_{m-1} = 0$, since $a_n \neq 0$.

Now suppose that $b_l = 0$ for any integers l with $t \le l \le m$, where t is an integer with $0 < t \le m$. Considering the value of b_{t-1}, when $p = kn+t-1$, by (24), we get that

$$a_n b_{t-1} + \cdots + a_{n-\lambda} b_{t-1+k\lambda} = 0, \tag{25}$$

where $\lambda = \min\{n, [(m-t+1)/2]\}$ and $[x]$ is the biggest integer which is not bigger than x. By the assumption and the above equation, we get that $a_n b_{t-1} = 0$; that is, $b_{t-1} = 0$, since $a_n \neq 0$.

From the preceding analysis, by the induction we can obtain that $b_l = 0$ for any integers l with $0 \leq l \leq m$. Then, by Proposition 12 we get that $\psi \equiv 0$, since φ is not 0 identically.

Second Case. If $b_m \neq 0$. In the following, we will continue the proof by induction.

When $p = kn + m$, by (24), we get that $a_n b_m = 0$, which means that $a_n = 0$, since $b_m \neq 0$.

When $p = kn + m - k$, by (24), we get that $a_{n-1} b_m + (m-k)^+ a_n b_{m-k} = 0$, where $(m-k)^+ = \max\{0, \operatorname{sgn}(2m-k+1)\}$ and sgn is a sign function. So, $a_{n-1} = 0$, since $a_n = 0$ and $b_m \neq 0$.

Now, suppose that $a_l = 0$ for any integers l with $t \leq l \leq n$, where t is an integer with $0 < t \leq n$. Considering the value of a_{t-1}, when $p = m + kt - k$, by (24), we get that

$$b_m a_{t-1} + \cdots + b_{m-k\lambda} a_{t-1+\lambda} = 0, \tag{26}$$

where $\lambda = \min\{n-t+1, [2m/k]\}$ and $[x]$ is the biggest integer which is not bigger than x. By the assumption and the above equation, we get that $b_m a_{t-1} = 0$; that is, $a_{t-1} = 0$, since $b_m \neq 0$.

From the preceding analysis, by the induction we can obtain that $a_l = 0$ for any integers l with $0 \leq l \leq n$, which means that $\varphi \equiv 0$.

Third Case. If $b_{-m} \neq 0$. A computation analogous to the one done in the second case from which we can get that $\varphi \equiv 0$.

From the above analysis, we have that (1.3) holds.

Arguing as in the pervious discussion, we obtain that if (1.2) holds, then (1.3) is true. □

Proposition 14. *Let $\varphi(z) = \sum_{p=-n}^{n} a_p z^p$, $\psi(z) = \sum_{p=-m}^{m} b_p z^p$, where n and m are both positive integers and $a_n^2 + b_m^2 + a_{-n}^2 + b_{-m}^2 \neq 0$, then $\varphi(z^k)\psi(z) = 0$ if and only if $\varphi = 0$ or $\psi = 0$.*

Proof. We begin with the easy direction. First, suppose that $\varphi = 0$ or $\psi = 0$; then it is clear that $\varphi(z^k)\psi(z) = 0$.

Now suppose that $\varphi(z^k)\psi(z) = 0$. Since $\varphi(z) = \sum_{p=-n}^{n} a_p z^p$, $\psi(z) = \sum_{p=-m}^{m} b_p z^p$, then

$$\begin{aligned} \varphi\left(z^k\right)\psi(z) &= \sum_{p=-n}^{n} a_p z^{kp} \cdot \sum_{p=-n}^{n} b_p z^p \\ &= \sum_{p=-(kn+m)}^{kn+m} \sum_{ki+j=p} a_i b_j z^p, \end{aligned} \tag{27}$$

where i and j are both integers with $-n \leq i \leq n$ and $-m \leq j \leq m$. Because $\varphi(z^k)\psi(z) = 0$, we get that, for any integers p with $-(kn+m) \leq p \leq kn+m$,

$$\sum_{ki+j=p} a_i b_j = 0, \tag{28}$$

where i and j are both integers with $-n \leq i \leq n$ and $-m \leq j \leq m$.

Since $a_n^2 + b_m^2 + a_{-n}^2 + b_{-m}^2 \neq 0$, without loss of generality, suppose that $a_n \neq 0$. We want to continue the proof by induction.

When $p = kn + m$, by (28), we get that $a_n b_m = 0$, which means that $b_m = 0$, since $a_n \neq 0$, when $p = kn + m - 1$, by (28), we get that $a_n b_{m-1} = 0$, which means that $b_{m-1} = 0$, since $a_n \neq 0$.

Now suppose that $b_l = 0$ for any integers l with $t \leq l \leq m$, where t is an integer with $0 < t \leq m$. Considering the value of b_{t-1}. When $p = kn + t - 1$, by (28) we get that

$$a_n b_{t-1} + \cdots + a_{n-\lambda} b_{t-1+k\lambda} = 0, \tag{29}$$

where $\lambda = \min\{2n, [(m-t+1)/k]\}$ and $[x]$ is the biggest integer which is not bigger than x. By the assumption and the above equation, we get that $a_n b_{t-1} = 0$; that is, $b_{t-1} = 0$, since $a_n \neq 0$.

From the preceding analysis, by the induction we can obtain that $b_l = 0$ for any integers l with $0 \leq l \leq m$. Then, by Proposition 13 we get that $\psi \equiv 0$, since φ is not 0 identically and $\varphi(z^k)\psi(z) = 0$ is equivalent to $\overline{\varphi(z^k)\psi(z)} = \overline{\varphi(z^k)} \cdot \overline{\psi(z)} = 0$. □

Form Proposition 3.4 and Theorem 3.1 in [15, 17], we will obtain the following theorem which describes the product of two slant Toeplitz operators with harmonic symbols.

Theorem 15. *Let $\varphi(z) = \sum_{p=-n}^{n} a_p z^p$, $\psi(z) = \sum_{p=-m}^{m} b_p z^p$, where n and m are both positive integers and $a_n^2 + b_m^2 + a_{-n}^2 + b_{-m}^2 \neq 0$, then the following statements are equivalent:*

(1.1) $U_\varphi U_\psi$ *is a kth-order slant Toeplitz operator;*

(1.2) $U_\varphi U_\psi$ *is a zero operator;*

(1.3) $U_\varphi U_\psi$ *is compact;*

(1.4) $\varphi(z^k)\psi(z) = 0$;

(1.5) $\varphi = 0$ *or* $\psi = 0$.

4. Commutativity of kth-Order Slant Toeplitz Operators on Bergman Spaces

Let $\mathbb{C}$ be the complex plane and let $\mathbb{D}$ be the unit disk in $\mathbb{C}$. Let dA be the area measure on $\mathbb{D}$ normalized so that $\int_{\mathbb{D}} 1 dA = 1$. Let $A^2(\mathbb{D})$ be the space of analytic functions in $L^2(\mathbb{D}, dA)$ which consists of Lebesgue measurable functions f on $\mathbb{D}$ with

$$\int_{\mathbb{D}} |f(z)|^2 dA(z) < +\infty. \tag{30}$$

It is well known that $A^2(\mathbb{D})$ is a closed subspace of the Hilbert space $L^2(\mathbb{D}, dA)$ with the inner product $\langle \cdot, \cdot \rangle$ and $\{z^i : i \in \mathbb{Z}^+\}$ are the orthogonal basis in $A^2(\mathbb{D})$, where $\mathbb{Z}^+$ is the set of nonnegative integers. Let $L^\infty(\mathbb{D})$ be the Banach space of Lebesgue measurable functions f on $\mathbb{D}$ with

$$\|f\|_\infty = \operatorname{ess\,sup}\{|f(z)| : z \in \mathbb{D}\} < +\infty. \tag{31}$$

Let P denote the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $A^2(\mathbb{D})$. For $f \in L^\infty(\mathbb{D})$, the Toeplitz operator T_f on $A^2(\mathbb{D})$ is defined by

$$T_f(g) = P(fg), \quad g \in A^2(\mathbb{D}), \tag{32}$$

and the kth-order slant Toeplitz operator B_f on $A^2(\mathbb{D})$ is defined by

$$B_f = W_k T_f, \tag{33}$$

where W_k is a bounded linear operator on $A^2(\mathbb{D})$ which is defined as

$$W_k\left(z^i\right) = \begin{cases} z^{i/k}, & \text{if } i \text{ is divisible by } k, \\ 0, & \text{otherwise.} \end{cases} \tag{34}$$

In this section we will investigate the commutativity of kth-order slant Toeplitz operators with coanalytic symbols and harmonic symbols on Bergman space $A^2(\mathbb{D})$. First, we study the commutativity of kth-order slant Toeplitz operators with coanalytic symbols.

Lemma 16. *Let $q \geq 0$ be an integer, let $f, g \in H^\infty(\mathbb{D})$, both of which are not 0 identically. If $fW_k^* g = gW_k^* f$, the following statements are equivalent:*

(1.1) *$f^{(i)}(0) = 0$ for any integers i with $0 \leq i \leq q$ and $f^{(q+1)}(0) \neq 0$;*

(1.2) *$g^{(i)}(0) = 0$ for any integers i with $0 \leq i \leq q$ and $g^{(q+1)}(0) \neq 0$.*

Proof. First, suppose that (1.1) holds. Since $f \in H^\infty(\mathbb{D})$, we get that $f(z) = z^{q+1} f_1(z)$ and $(W_k^* f)(z) = z^{kq+k} f_2(z)$, where $f_1(0) \neq 0$, $f_2(0) \neq 0$, and $f_1, f_2 \in H^\infty(\mathbb{D})$. Because $fW_k^* g = gW_k^* f$ and $g \in H^\infty(\mathbb{D})$, we get that

$$f_1(z)\left(W_k^* g\right)(z) = z^{(k-1)(q+1)} f_2(z)\, g(z), \tag{35}$$

so $g(0) = 0$. Since g is not 0 identically, without loss of generality, take $g^{(j)}(0) = 0$ for any integers j with $0 \leq j \leq p$ and $g^{(p+1)}(0) \neq 0$, then $g(z) = z^{p+1} g_1(z)$ and $(W_k^* g)(z) = z^{k(p+1)} g_2(z)$, where $g_1(0) \neq 0$, $g_2(0) \neq 0$, and $g_1, g_2 \in H^\infty(\mathbb{D})$; so, by (35) we get that

$$f_1(z)\, g_2(z)\, z^{(k-1)(p+1)} = z^{(k-1)(q+1)} f_2(z)\, g_1(z), \tag{36}$$

so $p = q$. Otherwise, if $p > q$, then by the above equation we get that $f_1(z)g_2(z)z^{(k-1)(p-q)} = f_2(z)g_1(z)$, which means that $f_2(0)g_1(0) = 0$, that leads a contradiction; if $p < q$, then by the above equation we get that $f_1(z)g_2(z) = z^{(k-1)(q-p)} f_2(z)g_1(z)$, which means that $f_1(0)g_2(0) = 0$, that leads a contradiction. Hence (1.2) holds.

Similarly, we can obtain the other direction of the Lemma. □

Theorem 17. *Let $\overline{\varphi}, \overline{\psi} \in H^\infty(\mathbb{D})$, then the following statements are equivalent:*

(1.1) *B_φ and B_ψ commute;*

(1.2) *there exist scalars α and β, not both zero, such that $\alpha\varphi + \beta\psi = 0$.*

Proof. First suppose that (1.2) holds; then it is obvious that B_φ and B_ψ commute.

Now suppose that (1.1) holds. So, we get that $B_\varphi^* B_\psi^*(1) = B_\psi^* B_\varphi^*(1)$; that is, $\overline{\varphi} W_k^* \overline{\psi} = \overline{\psi} W_k^* \overline{\varphi}$.

Now we continue the discussion in three cases.

First Case. If $\varphi \equiv 0$ or $\psi \equiv 0$. It is obvious that the required result holds.

Second Case. If $\overline{\varphi}(0) \neq 0$ and $\overline{\psi}(0) \neq 0$. Since $\overline{\varphi}, \overline{\psi} \in H^\infty(\mathbb{D})$, let $\overline{\varphi}(z) = \sum_{p=0}^\infty a_p z^p$ and $\overline{\psi}(z) = \sum_{p=0}^\infty b_p z^p$, then $a_0 \neq 0$, $b_0 \neq 0$, and $W_k^* \overline{\varphi}(z) = \sum_{p=0}^\infty a_p((kp+1)/(p+1)) z^{kp}$ and $W_k^* \overline{\psi}(z) = \sum_{p=0}^\infty b_p((kp+1)/(p+1)) z^{kp}$, so

$$\begin{aligned} \overline{\psi} W_k^* \overline{\varphi}(z) &= \sum_{p=0}^\infty b_p z^p \cdot \sum_{p=0}^\infty a_p \frac{kp+1}{p+1} z^{kp} \\ &= \sum_{p=0}^\infty \sum_{ki+j=p} \frac{ki+1}{i+1} a_i b_j z^p, \\ \overline{\varphi} W_k^* \overline{\psi}(z) &= \sum_{p=0}^\infty a_p z^p \cdot \sum_{p=0}^\infty b_p \frac{kp+1}{p+1} z^{kp} \\ &= \sum_{p=0}^\infty \sum_{i+kj=p} \frac{kj+1}{j+1} a_i b_j z^p. \end{aligned} \tag{37}$$

Because $\overline{\varphi} W_k^* \overline{\psi} = \overline{\psi} W_k^* \overline{\varphi}$, we get that

$$\sum_{p=0}^\infty \sum_{ki+j=p} \frac{ki+1}{i+1} a_i b_j z^p = \sum_{p=0}^\infty \sum_{i+kj=p} \frac{kj+1}{j+1} a_i b_j z^p, \tag{38}$$

that is, for any integers $p \geq 0$,

$$\sum_{ki+j=p} \frac{ki+1}{i+1} a_i b_j = \sum_{i+kj=p} \frac{kj+1}{j+1} a_i b_j, \tag{39}$$

where i and j are both nonnegative integers. In the following, we want to continue the proof by the induction.

When $p = 0$, by (39), we get that $a_0 b_0 = a_0 b_0$, so $b_0 = (b_0/a_0)a_0$, since $a_0 \neq 0$. Let $\lambda = b_0/a_0$, then $b_0 = \lambda a_0$.

When $p = 1$, by (39), we get that $a_0 b_1 = a_1 b_0$, so $b_1 = \lambda a_1$.

Suppose that $b_j = \lambda a_j$ for any integers j with $0 \leq j \leq l$, where l is a nonnegative integer. Now, consider the connection between a_{l+1} and b_{l+1}.

When $p = l + 1$, by (39), we get that

$$\sum_{ki+j=l+1} \frac{ki+1}{i+1} a_i b_j = \sum_{i+kj=l+1} \frac{kj+1}{j+1} a_i b_j, \tag{40}$$

that is,

$$\begin{aligned} & a_0 b_{l+1} + \cdots + \frac{k\lambda+1}{\lambda+1} a_\lambda b_{l+1-\lambda k} \\ &\quad = b_0 a_{l+1} + \cdots + \frac{k\lambda+1}{\lambda+1} b_\lambda a_{l+1-\lambda k}, \end{aligned} \tag{41}$$

where $\lambda = [(l+1)/k]$ and $[x]$ is the biggest integer which is not bigger than x. By this assumption, we can obtain that $a_0 b_{l+1} = b_0 a_{l+1}$; that is, $b_{l+1} = \lambda a_{l+1}$, since $a_0 \neq 0$.

Hence, by the induction, we obtain that $b_j = \lambda a_j$ for any nonnegative integers j from the pervoius discussion; that is, $\overline{\psi(z)} = \sum_{p=0}^{\infty} \lambda a_p z^p = \lambda\overline{\varphi(z)}$. So, the required result holds.

Third Case. If φ and ψ are both not 0 identically, and $\overline{\varphi}(0) = 0$ or $\overline{\psi}(0) = 0$. Without loss of generality, take $\overline{\varphi}^{(i)}(0) = 0$ for any integers $0 \le i \le i_1$ and $\overline{\varphi}^{(i_1+1)}(0) \neq 0$. By Lemma 16, we get that $\overline{\psi}^{(i)}(0) = 0$ for any integers $0 \le i \le i_1$ and $\overline{\psi}^{(i_1+1)}(0) \neq 0$, and

$$\overline{\varphi}(z) = z^{i_1+1}\varphi_1(z), \qquad \overline{\psi}(z) = z^{i_1+1}\psi_1(z), \tag{42}$$

where $\varphi_1, \psi_1 \in H^{\infty}(\mathbb{D})$, and $\varphi_1(0) \neq 0$, $\psi_1(0) \neq 0$, so $\overline{\psi}W_k^*(\overline{\varphi}) = z^{(k+1)(i_1+1)}\psi_1 W_k^*(\varphi_1)$ and $\overline{\varphi}W_k^*(\overline{\psi}) = z^{(k+1)(i_1+1)}\varphi_1 W_k^*(\psi_1)$. Since $\overline{\psi}W_k^*(\overline{\varphi}) = \overline{\varphi}W_k^*(\overline{\psi})$, yet we can get that $\psi_1 W_k^*(\varphi_1) = \varphi_1 W_k^*(\psi_1)$. Since $\varphi_1(0) \neq 0$ and $\psi_1(0) \neq 0$, yet by the second case we get that $\psi_1 = \lambda_1\varphi_1$, where $\lambda_1 = \psi_1(0)/\varphi_1(0)$. So, $\overline{\psi}(z) = z^{i_1+1}\psi_1(z) = \lambda_1 z^{i_1+1}\varphi_1(z) = \lambda_1\varphi_1(z)$. The required result holds. □

Now we are in a position to discuss the commutativity of slant Toeplitz operators with harmonic symbols.

Theorem 18. *Let $\varphi(z) = \sum_{p=0}^{n} a_p z^p + \sum_{k=1}^{n} a_{-p}\overline{z}^p$ and $\psi(z) = \sum_{p=0}^{n} b_p z^p + \sum_{k=1}^{n} b_{-p}\overline{z}^p$, where $a_{-n}^2 + b_{-n}^2 \neq 0$ and $n \ge 1$ is an integer, then the following statements are equivalent:*

(1.1) *B_φ and B_ψ commute;*

(1.2) *there exist scalars α and β, not both zero, such that $\alpha\varphi + \beta\psi = 0$.*

Proof. First suppose that (1.2) holds. It is obvious that B_φ and B_ψ commute.

Now suppose that (1.1) holds. Let $\varphi_1(z) = \sum_{p=0}^{n} a_p z^p$, $\overline{\varphi_2}(z) = \sum_{p=1}^{n} a_{-p}\overline{z}^p$, $\psi_1(z) = \sum_{p=0}^{n} b_p z^p$, and $\overline{\psi_2}(z) = \sum_{p=1}^{n} b_{-p}\overline{z}^p$, then $\varphi = \varphi_1 + \overline{\varphi_2}$ and $\psi = \psi_1 + \overline{\psi_2}$. Since B_φ and B_ψ commute, we have $T_{\overline{\varphi}}W_k^* T_{\overline{\psi}}W_k^* 1 = T_{\overline{\psi}}W_k^* T_{\overline{\varphi}}W_k^* 1$, that is,

$$\begin{aligned}&\varphi_2 W_k^*\psi_2 + \overline{\psi_1(0)}\varphi_2 + P\left(\overline{\varphi_1}W_k^*\psi_2\right)\\&\quad = \psi_2 W_k^*\varphi_2 + \overline{\varphi_1(0)}\psi_2 + P\left(\overline{\psi_1}W_k^*\varphi_2\right).\end{aligned} \tag{43}$$

Then, by (43), we get that for any integers p with $kn+1 \le p \le kn+n$,

$$\sum_{i+kj=p} \overline{a_{-i}}\,\overline{b_{-j}}\frac{kj+1}{j+1} = \sum_{ki+j=p} \overline{a_{-i}}\,\overline{b_{-j}}\frac{ki+1}{i+1}, \tag{44}$$

where i and j are positive integers which are not bigger than n. Since $a_{-n}^2 + b_{-n}^2 \neq 0$, without loss of generality, take $b_{-n} \neq 0$. Now we continue the proof by the induction.

When $p = kn+n$, by (44), we get that $\overline{a_{-n}}\,\overline{b_{-n}}((kn+1)/(n+1)) = \overline{a_{-n}}\,\overline{b_{-n}}((kn+1)/(n+1))$, so $\overline{a_{-n}} = (\overline{a_{-n}}/\overline{b_{-n}})\overline{b_{-n}}$, since $b_{-n} \neq 0$. Let $\lambda = \overline{a_{-n}}/\overline{b_{-n}}$, then $\overline{a_{-n}} = \lambda\overline{b_{-n}}$.

When $p = kn+n-1$, by (44), we get that $\overline{a_{-n+1}}\,\overline{b_{-n}}((kn+1)/(n+1)) = \overline{a_{-n}}\,\overline{b_{-n+1}}((kn+1)/(n+1))$, so $\overline{a_{-n+1}} = \lambda\overline{b_{-n+1}}$, since $b_{-n} \neq 0$.

Suppose that $\overline{a_{-n+i}} = \lambda\overline{b_{-n+i}}$ for any integers i with $0 \le i \le l < n-1$. Now, consider the connection between a_{-n+l+1} and b_{-n+l+1}.

When $p = kn+n-l-1$, by (44), we get that

$$\begin{aligned}&\overline{a_{-n+l+1}}\,\overline{b_{-n}}\frac{2n+1}{n+1} + \cdots + \overline{a_{-n+l+1-k\gamma}}\,\overline{b_{-n+\gamma}}\frac{k\gamma+1}{k+1}\\&\quad = \overline{b_{-n+l+1}}\,\overline{a_{-n}}\frac{2n+1}{n+1} + \cdots + \overline{b_{-n+l+1-k\gamma}}\,\overline{a_{-n+\gamma}}\frac{k\gamma+1}{k+1},\end{aligned} \tag{45}$$

where $\gamma = [(l+1)/k]$ and $[x]$ is the biggest integer which is not bigger than x. From the assumption we obtain that $\overline{a_{-n+l+1}}\,\overline{b_{-n}}((2n+1)/(n+1)) = \overline{b_{-n+l+1}}\,\overline{a_{-n}}((2n+1)/(n+1))$, so $\overline{a_{-n+l+1}} = \lambda\overline{b_{-n+l+1}}$, since $b_{-n} \neq 0$.

From the pervous discussion, by the induction we obtain that $\overline{a_{-n+i}} = \lambda\overline{b_{-n+i}}$ for any integers i with $0 \le i \le n-1$. Hence, $\varphi_2(z) = \sum_{p=1}^{n} a_{-p}\overline{z}^p = \sum_{p=1}^{n} \overline{\lambda} b_{-p}\overline{z}^p = \overline{\lambda}\psi_2(z)$.

Since $\varphi_2(z) = \overline{\lambda}\psi_2(z)$, by (43), we get that

$$\overline{\psi_1(0)}\varphi_2 + P\left(\overline{\varphi_1}W_k^*\psi_2\right) = \overline{\varphi_1(0)}\psi_2 + P\left(\overline{\psi_1}W_k^*\varphi_2\right). \tag{46}$$

Then,

$$\begin{aligned}&\left\langle \overline{\psi_1(0)}\varphi_2 + P\left(\overline{\varphi_1}W_k^*\psi_2\right), z^{kn}\right\rangle\\&\quad = \left\langle W_k^*\psi_2, z^{kn}\varphi_1\right\rangle = b_{-n}\overline{a_0}\frac{1}{n+1}\\&\quad = \left\langle \overline{\varphi_1(0)}\psi_2 + P\left(\overline{\psi_1}W_k^*\varphi_2\right), z^{kn}\right\rangle\\&\quad = \left\langle W_k^*\varphi_2, z^{kn}\psi_1\right\rangle = a_{-n}\overline{b_0}\frac{1}{n+1},\end{aligned} \tag{47}$$

and we get that $b_{-n}\overline{a_0} = a_{-n}\overline{b_0}$; that is, $a_0 = \lambda b_0$. So, by (46) we have that $\overline{\psi_1(0)}\varphi_2 = \overline{\varphi_1(0)}\psi_2$ and $P(\overline{\varphi_1}W_k^*\psi_2) = P(\overline{\psi_1}W_k^*\varphi_2)$, that is, $P(\overline{\lambda\psi_1 - \varphi_1}W_k^*\psi_2) = 0$. Hence, for any integers l with $kn-n \le l \le kn-1$, we have

$$\begin{aligned}0 &= \left\langle z^l, 0\right\rangle = \left\langle z^l, P\left(\overline{\lambda\psi_1 - \varphi_1}W_k^*\psi_2\right)\right\rangle\\&= \left\langle z^l, \overline{\lambda\psi_1 - \varphi_1}W_k^*\psi_2\right\rangle = \left\langle\left(\lambda\psi_1 - \varphi_1\right)z^l, W_k^*\psi_2\right\rangle,\end{aligned} \tag{48}$$

that is,

$$\left\langle \sum_{p=0}^{n}\left(\lambda b_p - a_p\right)z^{p+l}, \sum_{p=1}^{n} b_{-p}\frac{kp+1}{p+1}z^{kp}\right\rangle = 0. \tag{49}$$

Since $a_0 = \lambda b_0$, we have

$$\left\langle \sum_{p=1}^{n}\left(\lambda b_p - a_p\right)z^{p+l}, \sum_{p=1}^{n} b_{-p}\frac{kp+1}{p+1}z^{kp}\right\rangle = 0. \tag{50}$$

When $l = kn-1$, by (50), we get that $(\lambda b_1 - a_1)\overline{b_{-n}}(1/(n+1)) = 0$, so $a_1 = \lambda b_1$, since $b_{-n} \neq 0$. Then we have

$$\left\langle \sum_{p=2}^{n}\left(\lambda b_p - a_p\right)z^{p+l}, \sum_{p=1}^{n} b_{-p}\frac{kp+1}{p+1}z^{kp}\right\rangle = 0. \tag{51}$$

Suppose that $a_j = \lambda b_j$ for any integers j with $0 \le j \le s$, where $0 \le s \le n-1$. By (50), we get that

$$\left\langle \sum_{p=s+1}^{n} \left(\lambda b_p - a_p\right) z^{p+l}, \sum_{p=1}^{n} b_{-p} \frac{kp+1}{p+1} z^{kp} \right\rangle = 0. \tag{52}$$

Now consider the connection between a_{s+1} and b_{s+1}.

When $l = kn - s - 1$, by (52), we get that $(\lambda b_{s+1} - a_{s+1})\overline{b_{-n}}(1/(n+1)) = 0$, so $a_{s+1} = \lambda b_{s+1}$, since $b_{-n} \neq 0$.

Form the above discussion, by the induction we can obtain that $a_j = \lambda b_j$ for any integers j with $0 \le j \le n$, so $\varphi_1(z) = \sum_{p=0}^{n} a_p z^p = \sum_{p=0}^{n} \lambda b_p z^p = \lambda \psi_1(z)$.

Since $\varphi_2 = \overline{\lambda}\psi_2$, we have $\varphi = \varphi_1 + \overline{\varphi_2} = \lambda\psi_1 + \lambda\overline{\psi_2} = \lambda\psi$. Hence, the required result holds. □

Lemma 19. *Let $\varphi(z) = \sum_{p=0}^{n} a_p z^p + \sum_{p=1}^{n} a_{-p}\overline{z}^p$ and $\psi(z) = \sum_{p=0}^{m} b_p z^p + \sum_{p=1}^{m} b_{-p}\overline{z}^p$, where n and m are integers with $n > m \ge 1$ and $b_{-m} \neq 0$. If B_φ and B_ψ commute, then $a_{-j} = 0$ for any integers j with $m+1 \le j \le n$.*

Proof. Let $\varphi_1(z) = \sum_{p=0}^{n} a_p z^p$, $\overline{\varphi_2}(z) = \sum_{p=1}^{n} a_{-p}\overline{z}^p$, $\psi_1(z) = \sum_{p=0}^{m} b_p z^p$ and $\overline{\psi_2}(z) = \sum_{p=1}^{m} b_{-p}\overline{z}^p$, then $\varphi = \varphi_1 + \overline{\varphi_2}$ and $\psi = \psi_1 + \overline{\psi_2}$. Since B_φ and B_ψ commute, we have $T_{\overline{\varphi}}W_k^* T_{\overline{\psi}} W_k^* 1 = T_{\overline{\psi}}W_k^* T_{\overline{\varphi}} W_k^* 1$; that is,

$$\begin{aligned}
&\varphi_2 W_k^* \psi_2 + \overline{\psi_1(0)}\varphi_2 + P\left(\overline{\varphi_1} W_k^* \psi_2\right) \\
&\quad = \psi_2 W_k^* \varphi_2 + \overline{\varphi_1(0)}\psi_2 + P\left(\overline{\psi_1} W_k^* \varphi_2\right),
\end{aligned}$$

$$\begin{aligned}
&\sum_{p=k+1}^{km+n} \sum_{i+kj=p} \overline{a_{-i}}\,\overline{b_{-j}} \frac{kj+1}{j+1} z^p + \overline{\psi_1(0)} \sum_{p=1}^{n} \overline{a_{-p}} z^p \\
&\qquad + \sum_{p=0}^{km} \sum_{kj-i=p} \overline{a_i}\,\overline{b_{-j}} \frac{kj-i+1}{kj+1} z^p \\
&\quad = \sum_{p=k+1}^{kn+m} \sum_{ki+j=p} \overline{a_{-i}}\,\overline{b_{-j}} \frac{ki+1}{i+1} z^p \\
&\qquad + \overline{\varphi_1(0)} \sum_{p=1}^{m} \overline{b_{-p}} z^p + \sum_{p=0}^{kn} \sum_{ki-j=p} \overline{b_j}\,\overline{a_{-i}} \frac{ki-j+1}{ki+1} z^p.
\end{aligned} \tag{53}$$

Since $n > m$, let $n = m + r$, where r is a positive integer. Then, by the pervoius equation, we get that

$$\sum_{ki+j=kn+m} \overline{a_{-i}}\,\overline{b_{-j}} \frac{ki+1}{i+1} = 0, \quad \text{that is,} \quad \overline{a_{-n}}\,\overline{b_{-m}} \frac{kn+1}{n+1} = 0, \tag{54}$$

so $a_{-n} = 0$, since $b_{-m} \neq 0$.

Suppose that $a_{-n+t} = 0$ for any integers t with $0 \le t \le r-2$. Now consider the value of a_{-n+t+1}. By the assumption and the above equation we get that

$$\begin{aligned}
&\sum_{p=k+1}^{km+n-t-1} \sum_{i+kj=p} \overline{a_{-i}}\,\overline{b_{-j}} \frac{kj+1}{j+1} z^p + \overline{\psi_1(0)} \sum_{p=1}^{n-t-1} \overline{a_{-p}} z^p \\
&\qquad + \sum_{p=0}^{km} \sum_{kj-i=p} \overline{a_i}\,\overline{b_{-j}} \frac{kj-i+1}{kj+1} z^p \\
&\quad = \sum_{p=k+1}^{kn+m-kt-k} \sum_{ki+j=p} \overline{a_{-i}}\,\overline{b_{-j}} \frac{ki+1}{i+1} z^p + \overline{\varphi_1(0)} \sum_{p=1}^{m} \overline{b_{-p}} z^p \\
&\qquad + \sum_{p=0}^{kn-kt-k} \sum_{ki-j=p} \overline{b_j}\,\overline{a_{-i}} \frac{ki-j+1}{ki+1} z^p.
\end{aligned} \tag{55}$$

Since $t+1 < r$, we can get that $\sum_{ki+j=kn+m-kt-k} \overline{a_{-i}}\,\overline{b_{-j}}((ki+1)/(i+1)) = 0$; that is,

$$\begin{aligned}
&\overline{a_{-n+t+1}}\,\overline{b_{-m}} \frac{k(n-t-1)+1}{n-t-1} \\
&\quad + \cdots + \overline{a_{-n+t+1-\lambda}}\,\overline{b_{-m+k\lambda}} \frac{k(n-t-1+\lambda)+1}{n-t-1+\lambda} = 0,
\end{aligned} \tag{56}$$

where $\lambda = \min\{t+1, [2m/k]\}$ and $[2m/k]$ is the biggest integer which is not bigger than $2m/k$. Since $b_{-m} \neq 0$, by assumption, we get that $a_{-n+t+1} = 0$.

From the preceding discussion, by the induction we can get that $a_{-j} = 0$ for any integers j with $m+1 \le j \le n$. □

Theorem 20. *Let $\varphi(z) = \sum_{p=0}^{n} a_p z^p + \sum_{p=1}^{n} a_{-p}\overline{z}^p$ and $\psi(z) = \sum_{p=0}^{m} b_p z^p + \sum_{p=1}^{m} b_{-p}\overline{z}^p$, where $a_{-n}^2 + b_{-m}^2 \neq 0$, n and m are integers with $n > m \ge 1$, then the following statements are equivalent:*

(1.1) *B_φ and B_ψ commute;*

(1.2) *there exist scalars α and β, not both zero, such that $\alpha\varphi + \beta\psi = 0$.*

Proof. First suppose that (1.2) holds. It is obvious that B_φ and B_ψ commute.

Now suppose that (1.1) holds. Since $a_{-n}^2 + b_{-m}^2 \neq 0$, we can get that $a_{-n} \neq 0$ or $b_{-m} \neq 0$.

If $a_{-n} \neq 0$, by Theorem 18, we can get the required result.

If $b_{-m} \neq 0$, by Lemma 19, we can get that $\varphi(z) = \sum_{p=0}^{n} a_p z^p + \sum_{k=1}^{m} a_{-p}\overline{z}^p$. Let $\varphi_1(z) = \sum_{p=0}^{n} a_p z^p$, $\overline{\varphi_2}(z) = \sum_{p=1}^{m} a_{-p}\overline{z}^p$, $\psi_1(z) = \sum_{p=0}^{m} b_p z^p$, and $\overline{\psi_2}(z) = \sum_{p=1}^{m} b_{-p}\overline{z}^p$, then $\varphi = \varphi_1 + \overline{\varphi_2}$ and $\psi = \psi_1 + \overline{\psi_2}$.

Since B_φ and B_ψ commute, we have $T_{\overline{\varphi}}W_k^* T_{\overline{\psi}} W_k^* 1 = T_{\overline{\psi}}W_k^* T_{\overline{\varphi}} W_k^* 1$; that is,

$$\begin{aligned}
&\varphi_2 W_k^* \psi_2 + \overline{\psi_1(0)}\varphi_2 + P\left(\overline{\varphi_1} W_k^* \psi_2\right) \\
&\quad = \psi_2 W_k^* \varphi_2 + \overline{\varphi_1(0)}\psi_2 + P\left(\overline{\psi_1} W_k^* \varphi_2\right).
\end{aligned} \tag{57}$$

By (57) we get that for any integers p with $km+1 \leq p \leq km+m$,

$$\sum_{i+kj=p} \overline{a_{-i}}\,\overline{b_{-j}}\,\frac{kj+1}{j+1} = \sum_{ki+j=p} \overline{a_{-i}}\,\overline{b_{-j}}\,\frac{ki+1}{i+1}, \tag{58}$$

where i and j are positive integers with $-m \leq i \leq n$ and $-m \leq j \leq m$. Now we continue the proof by the induction.

When $p = km+m$, by (58), we get that $\overline{a_{-m}}\,\overline{b_{-m}}((km+1)/(m+1)) = \overline{a_{-m}}\,\overline{b_{-m}}((km+1)/(m+1))$, so $\overline{a_{-m}} = (\overline{a_{-m}}/\overline{b_{-m}})\overline{b_{-m}}$, since $b_{-m} \neq 0$. Let $\lambda = (\overline{a_{-m}}/\overline{b_{-m}})$, then $\overline{a_{-m}} = \lambda\overline{b_{-m}}$.

When $p = km+m-1$, by (58), we get that $\overline{a_{-m+1}}\,\overline{b_{-m}}((km+1)/(m+1)) = \overline{a_{-m}}\,\overline{b_{-m+1}}((km+1)/(m+1))$, so $\overline{a_{-m+1}} = \lambda\overline{b_{-m+1}}$, since $b_{-m} \neq 0$.

Suppose that $\overline{a_{-m+i}} = \lambda\overline{b_{-m+i}}$ for any integers i with $0 \leq i \leq l < m-1$. Now consider the connection between a_{-m+l+1} and b_{-m+l+1}.

When $p = km+m-l-1$, by (58), we get that

$$\begin{aligned}
&\overline{a_{-m+l+1}}\,\overline{b_{-m}}\,\frac{km+1}{m+1} + \cdots \\
&\qquad + \overline{a_{-m+l+1-rk}}\,\overline{b_{-m+r}}\,\frac{k(m-r)+1}{m-r+1} \\
&\quad = \overline{b_{-m+l+1}}\,\overline{a_{-m}}\,\frac{km+1}{m+1} + \cdots \\
&\qquad + \overline{b_{-m+l+1-rk}}\,\overline{a_{-m+r}}\,\frac{k(m-r)+1}{m-r+1}.
\end{aligned} \tag{59}$$

From the assumption we obtain that $\overline{a_{-m+l+1}}\,\overline{b_{-m}}((km+1)/(m+1)) = \overline{b_{-m+l+1}}\,\overline{a_{-m}}((km+1)/(m+1))$, so $\overline{a_{-m+l+1}} = \lambda\overline{b_{-m+l+1}}$, since $b_{-m} \neq 0$.

From the pervious discussion, by the induction we obtain that $\overline{a_{-m+i}} = \lambda\overline{b_{-m+i}}$ for any integers i with $0 \leq i \leq m-1$. Hence, $\overline{\varphi_2}(z) = \sum_{p=1}^{m} a_{-p}\overline{z}^p = \sum_{p=1}^{m} \overline{\lambda} b_{-p}\overline{z}^p = \overline{\lambda}\overline{\psi_2}(z)$.

Since $\varphi_2(z) = \lambda\psi_2(z)$, by (57), we get that

$$\overline{\psi_1(0)}\varphi_2 + P(\overline{\varphi_1}W^*\psi_2) = \overline{\varphi_1(0)}\psi_2 + P(\overline{\psi_1}W^*\varphi_2). \tag{60}$$

Then,

$$\begin{aligned}
&\left\langle \overline{\psi_1(0)}\varphi_2 + P(\overline{\varphi_1}W^*\psi_2), z^{km} \right\rangle \\
&\quad = \left\langle W^*\psi_2, z^{km}\varphi_1 \right\rangle = \overline{b_{-m}}\,\overline{a_0}\,\frac{1}{m+1} \\
&\quad = \left\langle \overline{\varphi_1(0)}\psi_2 + P(\overline{\psi_1}W^*\varphi_2), z^{km} \right\rangle \\
&\quad = \left\langle W^*\varphi_2, z^{km}\psi_1 \right\rangle = \overline{a_{-m}}\,\overline{b_0}\,\frac{1}{m+1},
\end{aligned} \tag{61}$$

and we get that $\overline{b_{-m}}\,\overline{a_0} = \overline{a_{-m}}\,\overline{b_0}$; that is, $a_0 = \overline{\lambda}b_0$. So, by (60), we have that $\overline{\psi_1(0)}\varphi_2 = \overline{\varphi_1(0)}\psi_2$ and $P(\overline{\varphi_1}W^*\psi_2) = P(\overline{\psi_1}W_k^*\varphi_2)$; that is, $P((\lambda\overline{\psi_1} - \overline{\varphi_1})W_k^*\psi_2) = 0$. Hence, for any integers l with $km-m \leq l \leq km-1$, we have

$$\begin{aligned}
0 &= \left\langle z^l, 0 \right\rangle = \left\langle z^l, P\left((\lambda\overline{\psi_1} - \overline{\varphi_1})W_k^*\psi_2\right) \right\rangle \\
&= \left\langle z^l, (\lambda\overline{\psi_1} - \overline{\varphi_1})W_k^*\psi_2 \right\rangle = \left\langle \left(\overline{\lambda}\psi_1 - \varphi_1\right)z^l, W_k^*\psi_2 \right\rangle,
\end{aligned} \tag{62}$$

that is,

$$\left\langle \sum_{p=0}^{m}\left(\overline{\lambda}b_p - a_p\right)z^{p+l} - \sum_{p=m+1}^{n} a_p z^{p+l}, \sum_{p=1}^{m} b_{-p}\frac{kp+1}{p+1}z^{kp} \right\rangle = 0. \tag{63}$$

Since $a_0 = \overline{\lambda}b_0$, we have

$$\left\langle \sum_{p=1}^{m}\left(\overline{\lambda}b_p - a_p\right)z^{p+l} - \sum_{p=m+1}^{n} a_p z^{p+l}, \sum_{p=1}^{m} b_{-p}\frac{kp+1}{p+1}z^{kp} \right\rangle = 0. \tag{64}$$

When $l = km-1$, by (64), we get that $(\overline{\lambda}b_1 - a_1)\overline{b_{-m}}((km+1)/(m+1)) = 0$, so $a_1 = \overline{\lambda}b_1$, since $b_{-m} \neq 0$. Then, we have

$$\left\langle \sum_{p=2}^{m}\left(\overline{\lambda}b_p - a_p\right)z^{p+l} - \sum_{p=m+1}^{n} a_p z^{p+l}, \sum_{p=1}^{m} b_{-p}\frac{kp+1}{p+1}z^{kp} \right\rangle = 0. \tag{65}$$

Suppose that $a_j = \overline{\lambda}b_j$ for any integers j with $0 \leq j \leq s$, where $0 \leq s \leq m-1$. By (64), we get that

$$\left\langle \sum_{p=s+1}^{m}\left(\overline{\lambda}b_p - a_p\right)z^{p+l} - \sum_{p=m+1}^{n} a_p z^{p+l}, \sum_{p=1}^{m} b_{-p}\frac{kp+1}{p+1}z^{kp} \right\rangle = 0. \tag{66}$$

Now consider the connection between a_{s+1} and b_{s+1}.

When $l = km-s-1$, by (66), we get that $(\overline{\lambda}b_{s+1} - a_{s+1})\overline{b_{-m}}((km+1)/(m+1)) = 0$, so $a_{s+1} = \overline{\lambda}b_{s+1}$, since $b_{-m} \neq 0$.

From the pervious discussion, by the induction, we can obtain that $a_j = \overline{\lambda}b_j$ for any integers j with $0 \leq j \leq m$. Hence,

$$\begin{aligned}
\varphi_1(z) &= \sum_{p=0}^{n} a_p z^p = \sum_{p=0}^{m}\overline{\lambda}b_p z^p + \sum_{p=m+1}^{n} a_p z^p \\
&= \overline{\lambda}\psi_1(z) + \sum_{p=m+1}^{n} a_p z^p.
\end{aligned} \tag{67}$$

Let $\varphi_3(z) = \sum_{p=m+1}^{n} a_p z^p$, then $\varphi = \varphi_1 + \overline{\varphi_2} = \overline{\lambda}\psi_1 + \varphi_3 + \overline{\lambda}\,\overline{\psi_2} = \overline{\lambda}\psi + \varphi_3$, since $\varphi_2 = \lambda\psi_2$.

Now we want to show that $\varphi_3 \equiv 0$. Because $B_\varphi B_\psi = B_\psi B_\varphi$, yet we get that $B_{\varphi_3}B_\psi = B_\psi B_{\varphi_3}$, which means that

$$T_{\overline{\varphi_3}}W_k^* T_{\overline{\psi}}W_k^* = T_{\overline{\psi}}W_k^* T_{\overline{\varphi_3}}W_k^*. \tag{68}$$

Since $m \geq 1$, we have $T_{\overline{\varphi_3}}W_k^* T_{\overline{\psi}}W_k^* 1 = 0$; that is, $T_{\overline{\varphi_3}}W_k^*[\overline{\psi_2} + \overline{\psi_1(0)}] = 0$. Then, we get that for any integers l with $0 \leq l \leq km-m-1$, $\langle T_{\overline{\varphi_3}}W_k^*[\overline{\psi_2} + \overline{\psi_1(0)}],\ z^l \rangle = 0$; that is,

$$\left\langle \sum_{j=1}^{m}\overline{b_{-j}}\frac{kj+1}{j+1}z^{kj} + \overline{\psi_1(0)}, \sum_{i=m+1}^{n} a_i z^{i+l} \right\rangle = 0. \tag{69}$$

By (69), we can get that for any integers l with $\max\{0, km - n\} \leq l \leq km-m-1$, $\overline{b_{-m}}\,\overline{a_{km-l}}(1/(m+1)) = 0$; that is, $a_{km-l} = 0$, since $b_{-m} \neq 0$.

If $n \leq km$, then $\varphi_3 \equiv 0$, which means that $\varphi = \overline{\lambda}\psi$, so the required result holds.

If $n \geq km+1$, then $\varphi_3(z) = \sum_{p=km+1}^{n} a_p z^p$. So, by (68), we can get that $T_{\overline{\varphi_3}} W_k^* T_{\overline{\psi}} W_k^* (z^m) = 0$, that is, $T_{\overline{\varphi_3}} W_k^* [\overline{\psi_2} z^{km} + P(\overline{\psi_1} z^{km})] = 0$. So we get that for any integers l with $\max\{0, k^2m + km - n\} \leq l \leq k^2m - 1$,

$$\left\langle W_k^* \left[\overline{\psi_2} z^{km} + P\left(\overline{\psi_1} z^{km}\right)\right], \sum_{i=km+1}^{n} a_i z^{i+l} \right\rangle = 0. \tag{70}$$

By (70), we can get that for any integers l with $\max\{0, k^2m + km - n\} \leq l \leq k^2m - 1$, $\overline{b_{-m}}\,\overline{a_{k^2m+km-l}}(1/(k^2m + km + 1)) = 0$, that is, $a_{k^2m+km-l} = 0$, since $b_{-m} \neq 0$.

If $n \leq k^2m + km$, then $\varphi_3 \equiv 0$, which means that $\varphi = \overline{\lambda}\psi$, so, the required result holds.

If $n \geq k^2m+km+1$, then $\varphi_3(z) = \sum_{p=k^2m+km+1}^{n} a_p z^p$. Then successively by the pervoius method, we can get that $a_i = 0$ for all integers i with $m+1 \leq i \leq n$, which means that $\varphi = \overline{\lambda}\psi$. So, the required result holds. □

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