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Research Article

Revised Variational Iteration Method for Solving Systems of Nonlinear Fractional-Order Differential Equations

C. Ünlü,[1] H. Jafari,[2,3] and D. Baleanu[4]

[1] *Department of Mathematics, Faculty of Science, Istanbul University, Vezneciler, 34134 Istanbul, Turkey*
[2] *International Institute for Symmetry Analysis and Mathematical Modelling, Department of Mathematical Sciences, North-West University, Mmabatho 2735, South Africa*
[3] *Department of Mathematics, University of Mazandaran, P.O. Box 47416-95447, Babolsar, Iran*
[4] *Department of Mathematics and Computer Science, Faculty of Arts and Sciences, Çankaya University, 06530 Ankara, Turkey*

Correspondence should be addressed to C. Ünlü; unlucanan@hotmail.com

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A modification of the variational iteration method (VIM) for solving systems of nonlinear fractional-order differential equations is proposed. The fractional derivatives are described in the Caputo sense. The solutions of fractional differential equations (FDE) obtained using the traditional variational iteration method give good approximations in the neighborhood of the initial position. The main advantage of the present method is that it can accelerate the convergence of the iterative approximate solutions relative to the approximate solutions obtained using the traditional variational iteration method. Illustrative examples are presented to show the validity of this modification.

1. Introduction

Recently, fractional-order calculus has been studied as an alternative calculus in mathematics. Numerous problems in physics, chemistry, biology, and engineering can be modeled with fractional derivatives [1–12]. On the other hand, in control society, fractional-order dynamic systems and controls have gained an increasing attention [13–17], and also motion of an elastic column fixed at one end loaded at the other can be formulated in terms of a system of fractional differential equations [18]. Since most fractional differential equations do not have exact analytic solutions, approximate and numerical techniques, therefore, are used extensively.

The variational iteration method is relatively new approaches to provide approximate solutions to linear and nonlinear problems. The variational iteration method, which is proposed by He [19], was successfully applied to find the solutions of several classes of variational problems. Some research works in this field are [20–25]. Recently, the application of the method is extended for fractional differential equations [26, 27].

Daftardar-Gejji and Jafari [28] have explored the Adomian decomposition method to obtain solution of a system of linear and nonlinear fractional differential equations. Further in [29], they have suggested a modification (termed as "revised ADM") of this method and applied revised method for solving systems of linear/nonlinear ordinary and fractional differential equations [30].

The objective of this paper is the use of revised variational iteration method (RVIM) for solving systems of nonlinear fractional-order differential equations. We demonstrate that the approximate solution thus obtained converges faster relative to the approximate solutions by standard variational iteration method. Several illustrative examples have been presented.

2. Definitions and Preliminaries

In this section, we give some definitions and properties of the fractional calculus [9] which are used further in this paper.

Definition 1. A real function $f(t)$, $t > 0$ is said to be in the space C_α, $\alpha \in \mathfrak{R}$ if there exists a real number p $(> \alpha)$, such that $f(t) = t^p f_1(t)$, where $f_1 \in C[0, \infty]$.

Definition 2. A function $f(t)$, $t > 0$ is said to be in the space C_{α}^{m}, $m \in N \bigcup\{0\}$ if $f^{(m)} \in C_{\alpha}$.

Definition 3. The left-sided Riemann-Liouville fractional integral of order $\mu \geq 0$, of a function $f \in C_{\alpha}$, $\alpha \geq -1$ is defined as

$$I_t^{\mu} f(t) = \begin{cases} \dfrac{1}{\Gamma(\mu)} \displaystyle\int_0^t \dfrac{f(\tau)}{(t-\tau)^{1-\mu}} d\tau, & \mu > 0,\ t > 0, \\ f(t), & \mu = 0. \end{cases} \quad (1)$$

Definition 4. The left-sided Caputo fractional derivative of f, $f \in C_{-1}^{m}$, $m \in N \cup \{0\}$, is defined as

$$\begin{aligned} D_t^{\mu} f(t) &= \frac{d^{\mu} f(t)}{dt^{\mu}} \\ &= \begin{cases} I^{m-\mu}\left[\dfrac{d^m f(t)}{dt^m}\right], & m-1 < \mu < m,\ m \in N, \\ \dfrac{d^m f(t)}{dt^m}, & \mu = m. \end{cases} \end{aligned} \quad (2)$$

3. The VIM for FDE

The principles of the variational iteration method and its applicability for various kinds of differential equations are given in [20, 31]. In [26], He showed that the variational iteration method is also valid for fractional differential equations. In this section, following the discussion presented in [26], we extend the application of the variational iteration method to solve the fractional differential equation as follows:

$$D_t^{\alpha} y(t) + N(y(t)) = f(t), \quad 0 < \alpha \leq 1, \quad (3)$$

where N is an operator with respect to $y(t)$ and $f(t)$ is a known function. According to the variational iteration method, we can construct the correction functional for (3) as follows:

$$\begin{aligned} y_{n+1}(t) &= y_n(t) + I_t^{\alpha}\left[\lambda\left(D_t^{\alpha} y_n(t) + N(y_n(t)) - f(t)\right)\right] \\ &= y_n(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lambda(s) \\ &\qquad \times \left[D_s^{\alpha} y_n(s) - N(y_n(s)) \right. \\ &\qquad \left. -f(s)\right] ds, \end{aligned} \quad (4)$$

where λ is the general Lagrange multiplier, which can be identified optimally via variational theory [32].

To identify approximately Lagrange multiplier, some approximations must be made. The correction functional equation (4) can be approximately expressed as follows:

$$\begin{aligned} y_{n+1}(t) &= y_n(t) \\ &\quad + \int_0^t \lambda(s)\left[y_n'(s) + N(\tilde{y}_n(s)) - f(s)\right] ds. \end{aligned} \quad (5)$$

Here we apply restricted variations $\tilde{y}_n$ to the term $N(y)$, and in this case, we can easily determine the multiplier. Making the aforementioned functional stationary, noticing that $\delta \tilde{y}_n = 0$,

$$\delta y_{n+1}(t) = \delta y_n(t) + \delta \int_0^t \lambda(s)\left[y_n'(s) - f(s)\right] ds \quad (6)$$

yields the Lagrange multiplier $\lambda = -1$, and substituting into the functional equation (4), we obtain the following iteration formula:

$$y_{n+1}(t) = y_n(t) - I_t^{\alpha}\left[D_t^{\alpha} y_n(t) + N(y_n(t)) - f(t)\right]. \quad (7)$$

The initial approximation $y_0(t)$ can be freely chosen if it satisfies the initial conditions of the problem. Finally, we approximate the solution $y(t) = \lim_{n \to 1} y_n(t)$ by the Nth term $y_N(t)$.

4. The System of FDE and Revised VIM

Let us consider the following system of fractional differential equations:

$$\begin{aligned} D_t^{\alpha_i} y_i(t) + N_i(y_1(t), \ldots, y_m(t)) &= f_i(t), \\ 0 < \alpha_i \leq 1,\ i &= 1, 2, \ldots, m, \end{aligned} \quad (8)$$

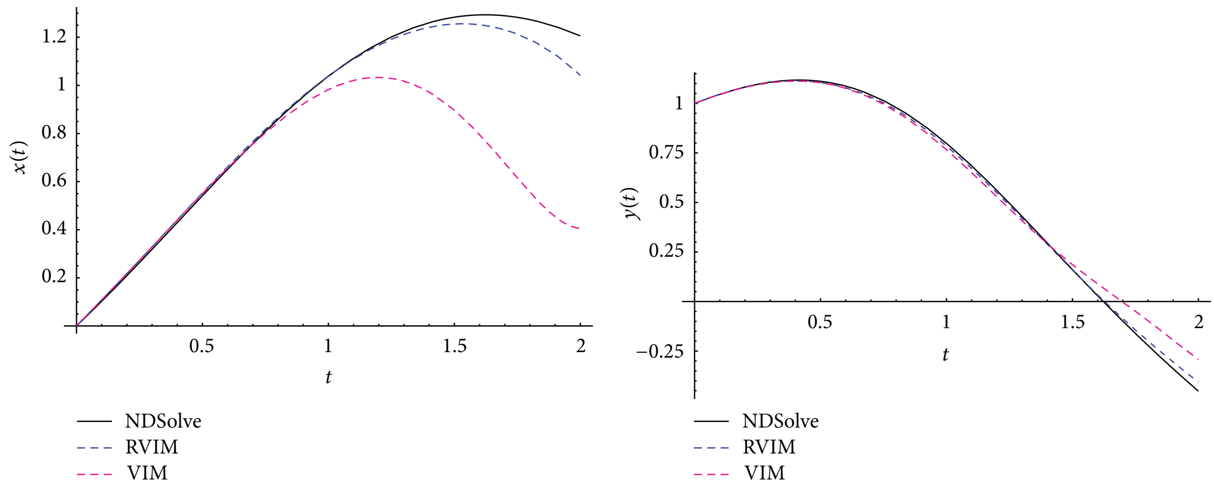
where N_i are operators with respect to $y_i(t)$ and $f_i(t)$ are known functions. In this case, the correction functionals are obtained as follows:

$$\begin{aligned} y_{i(n+1)}(t) = y_{in}(t) - I_t^{\alpha_i}\left[D_t^{\alpha_i} y_{in}(t)\right. \\ \left. + N_i(y_{1n}(t), \ldots, y_{mn}(t)) - f_i(t)\right] \\ i = 1, 2, \ldots, m. \end{aligned} \quad (9)$$

Here we construct the following iteration formula instead of the iteration formula obtained with the standard variational iteration method equation (9):

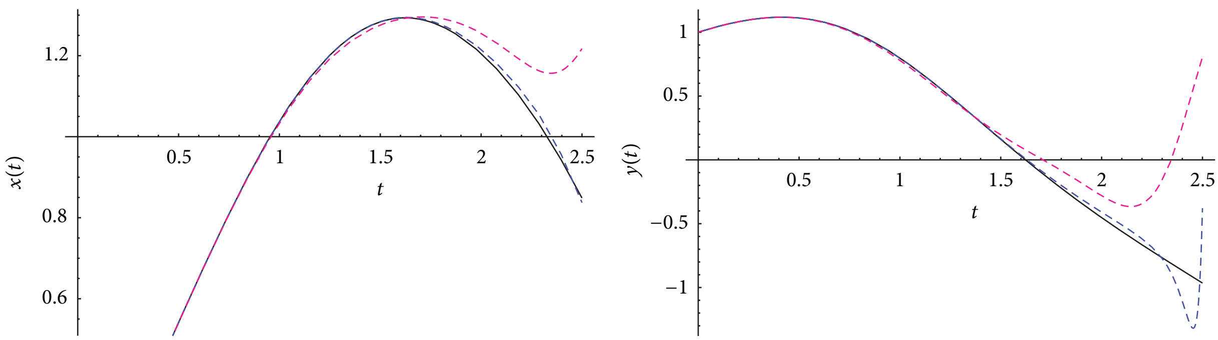
$$\begin{aligned} &y_{i(n+1)}(t) \\ &\quad = y_{in}(t) - I_t^{\alpha_i}\left[D_t^{\alpha_i} y_{in}(t)\right. \\ &\qquad + N_i\left(y_{1(n+1)}(t), \ldots, y_{(i-1)(n+1)}(t),\right. \\ &\qquad \left.\left. y_{(i)n}(t), \ldots, y_{mn}(t)\right) - f_i(t)\right]. \end{aligned} \quad (10)$$

In fact, the updated values $y_{1(n+1)}(t), \ldots, y_{(i-1)(n+1)}(t)$ are used for finding $y_{i(n+1)}(t)$. We called it the revised variational iteration method (RVIM). This technique can accelerate the convergence of iterative approximate solutions relative to the approximate solutions obtained using the traditional variational iteration method. The effect of this correction is clear in $y_{i(n+1)}(t)$ because the updated values are used



(a) (b)

Figure 1: Simulation of the approximate results for $\alpha = .98$ and exact solution for $\alpha = 1$ of the Van der pol system.



(a) (b)

Figure 2: Simulation of the approximate results for $\alpha = 1$ and exact solution of the Van der pol system.

to compute it. Tatari and Dehghan have employed this technique for systems of ordinary differential equations [33].

5. Test Examples

In this section, we illustrate the applicability of revised variational iteration method to systems of nonlinear fractional-order differential equations.

Example 1. Consider the following system of nonlinear fractional-order Van der pol:

$$\begin{aligned} D_t^{\alpha} x(t) &= y(t), \\ \frac{dy(t)}{dt} &= -x(t) - .5x(t)^2 y(t) + .5y(t), \\ & \qquad (0 < \alpha \le 1) \end{aligned} \tag{11}$$

with the initial condition

$$x(0) = 0, \quad y(0) = 1. \tag{12}$$

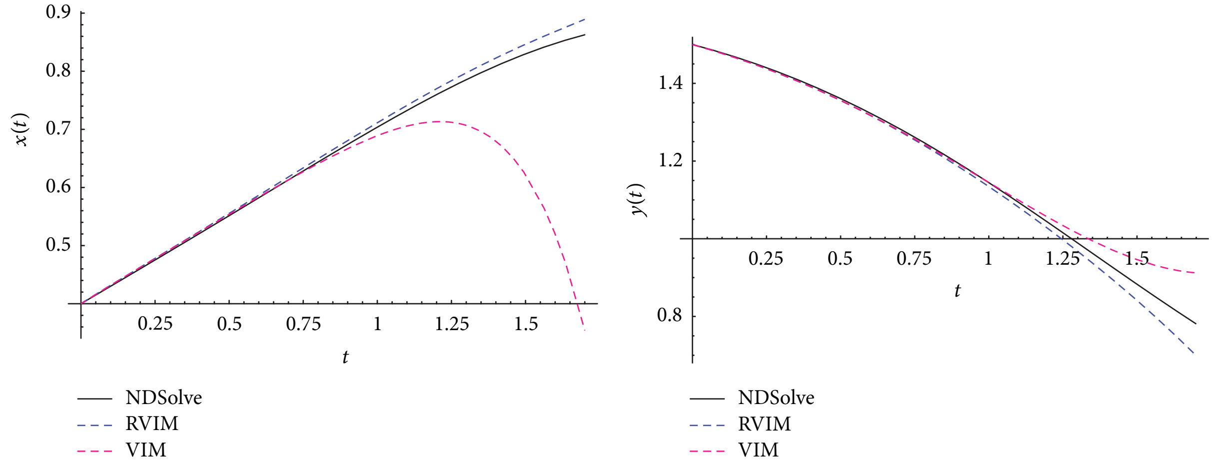
The standard VIM for (11) leads to the following iteration formula:

$$\begin{aligned} x_{(n+1)}(t) &= x_n(t) - I_t^{\alpha}\left[D_t^{\alpha} x_n(t) - y_n(t)\right], \\ y_{(n+1)}(t) &= y_n(t) - \int_0^t \left[\frac{dy_n(s)}{ds} + x_n(s) \right. \\ & \quad \left. + .5x_n(s)^2 y_n(s) - .5y_n(s)\right] ds. \end{aligned} \tag{13}$$

The use of the revised VIM for (11) results in the following formula:

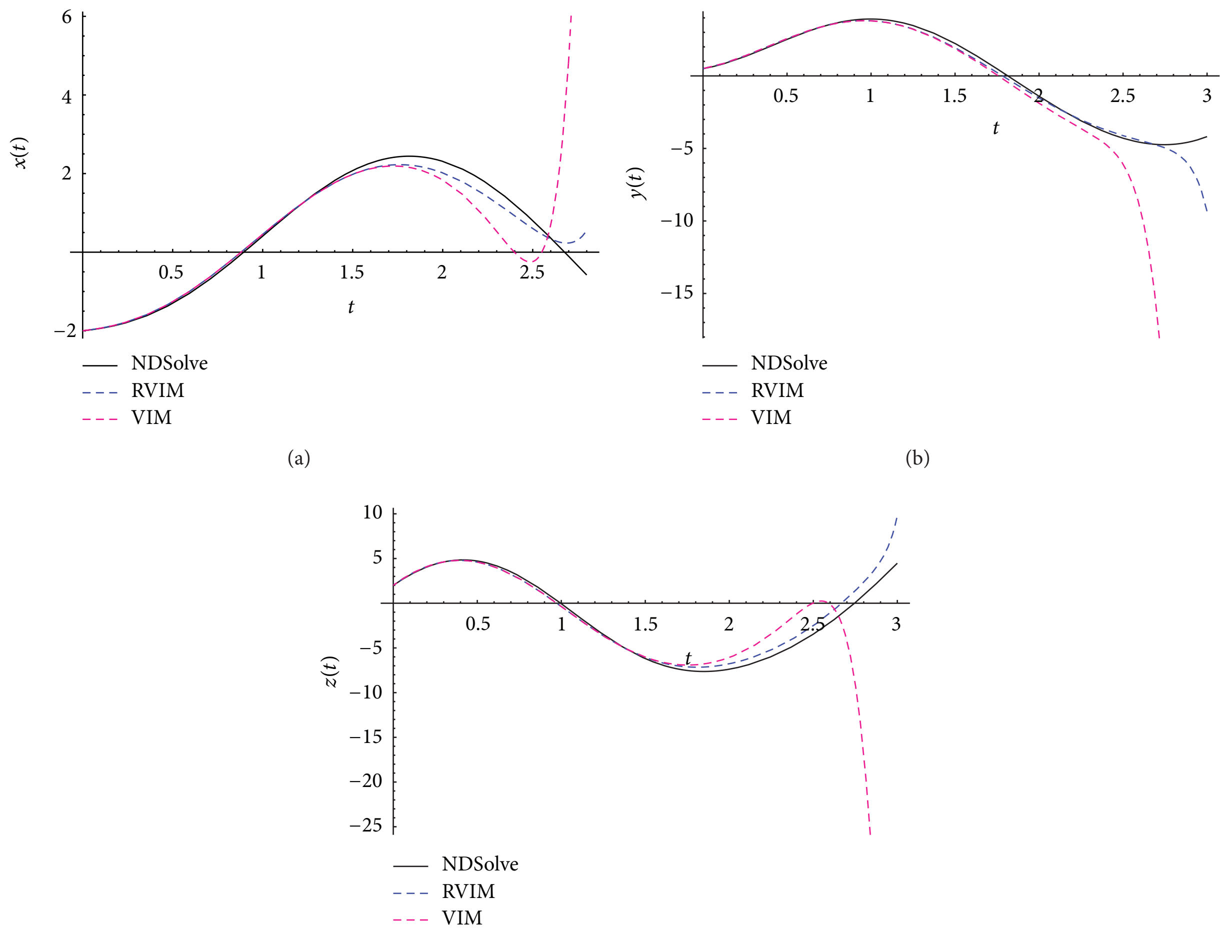
$$\begin{aligned} x_{(n+1)}(t) &= x_n(t) - I_t^{\alpha}\left[D_t^{\alpha} x_n(t) - y_n(t)\right], \\ y_{(n+1)}(t) &= y_n(t) - \int_0^t \left[\frac{dy_n(s)}{ds} + x_{(n+1)}(s) \right. \\ & \quad \left. + .5x_{(n+1)}(s)^2 y_n(s) - .5y_n(s)\right] ds. \end{aligned} \tag{14}$$

Starting with the initial approximations $x_0(t) = 0$ and $y_0(t) = 1$, we can easily obtain the results using (13) and (14).



(a) (b)

FIGURE 3: Simulation of the approximate results for $\alpha_1 = \alpha_2 = .98$ and exact solution for $\alpha_1 = \alpha_2 = 1$ of the Brusselator system.



(c)

FIGURE 4: Simulation of the approximate results for $\alpha_1 = \alpha_2 = \alpha_3 = .98$ and exact results for $\alpha_1 = \alpha_2 = \alpha_3 = 1$ of the Genesio-Tesi system for parameters $\beta_1 = 6.5$, $\beta_2 = 2.92$, $\beta_3 = 1.2$, and $\beta_4 = 1.0$.

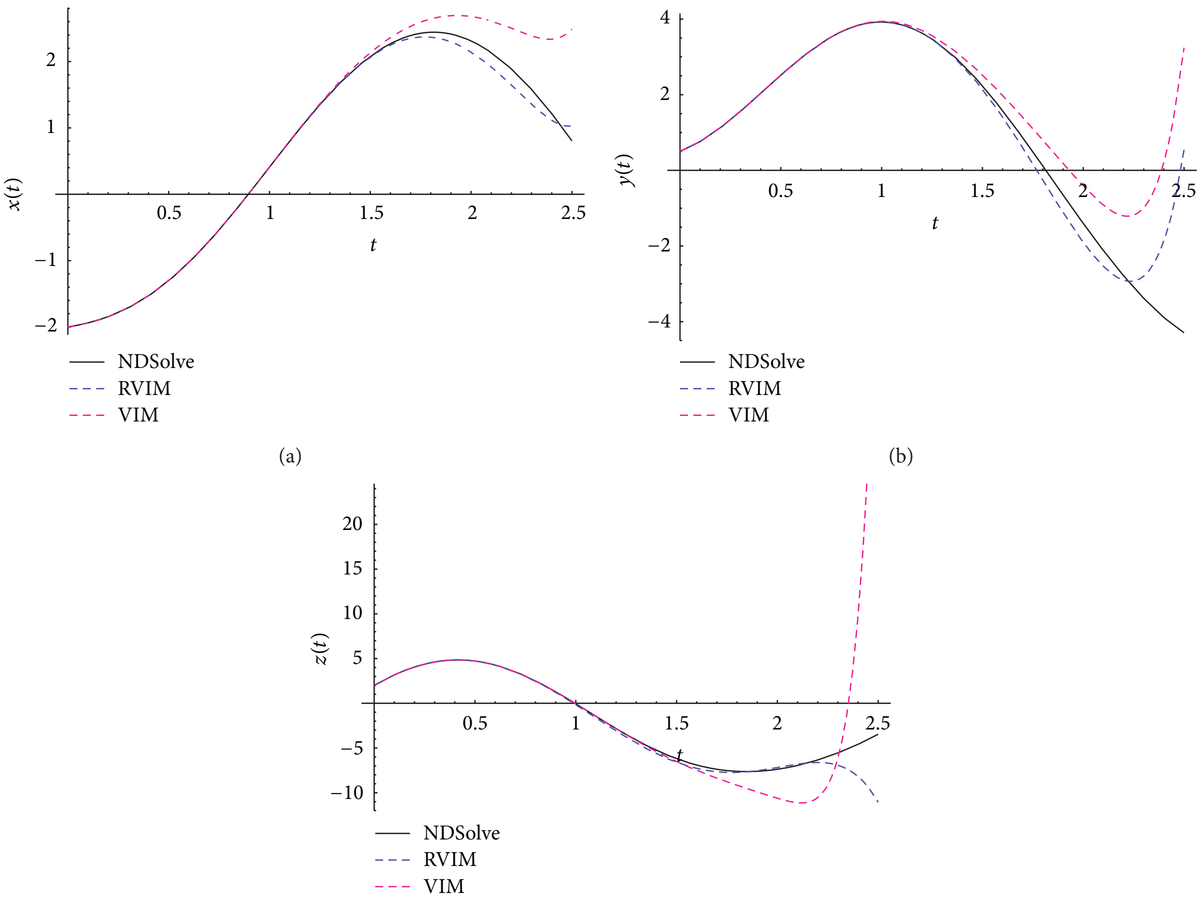


FIGURE 5: Simulation of the approximate results for $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and exact results of the Genesio-Tesi system for parameters $\beta_1 = 6.5$, $\beta_2 = 2.92$, $\beta_3 = 1.2$, and $\beta_4 = 1.0$.

In Figures 1 and 2, the results from VIM and RVIM are shown. Figures 1(a) and 1(b) show comparison between the approximate solutions $(x(t) \cong x_4(t))$, $((y(t) \cong y_4(t))$ of (11) obtained using VIM and RVIM for the special case $\alpha = .98$ and the numerical solutions for the special case $\alpha = 1$, respectively. Figures 2(a) and 2(b), show approximate solutions $(x(t) \cong x_5(t))$, $((y(t) \cong y_4(t))$ of (11) using VIM and RVIM for the special case $\alpha = 1$ and the numerical solutions, respectively.

Example 2. Consider the system of nonlinear fractional-order Brusselator:

$$D^{\alpha_1} y(t) = .1x(t) - x(t)^2 y(t),$$
$$D^{\alpha_2} x(t) = .5 - 1.1x(t) + x(t)^2 y(t) \qquad (15)$$
$$(0 < \alpha_1, \alpha_2 \leq 1)$$

with the initial conditions

$$x(0) = .4, \quad y(0) = 1.5. \qquad (16)$$

In this example, the use of the variational iteration method leads to

$$y_{n+1}(t) = y_n(t) - I_t^{\alpha_1}\left[D_t^{\alpha_1} y_n(t) - .1x_n(t) + x_n^2(t)\, y_n(t)\right],$$
$$x_{n+1}(t) = x_n(t) - I_t^{\alpha_2}\left[D_t^{\alpha_2} x_n(t) - .5 + 1.1x_n(t) - x_n^2(t)\, y_n(t)\right]. \qquad (17)$$

Using the RVIM, we obtain

$$y_{n+1}(t) = y_n(t) - I_t^{\alpha_1}\left[D_t^{\alpha_1} y_n(t) - .1x_n(t) + x_n^2(t)\, y_n(t)\right],$$
$$x_{n+1}(t) = x_n(t) - I_t^{\alpha_2}\left[D_t^{\alpha_2} x_n(t) - .5 + 1.1x_n(t) - x_n^2(t)\, y_{(n+1)}(t)\right]. \qquad (18)$$

We consider the initial approximations $x_0(t) = .4 + (.5 \times (t^{\alpha_2}/\Gamma[\alpha_2 + 1]))$ and $y_0(t) = 1.5$. Results are shown in Figure 3. Figures 3(a) and 3(b) show comparison between the approximate solutions $(x(t) \cong x_2(t))$, $(y(t) \cong y_3(t))$ of (15) obtained using VIM and RVIM for the special case

$\alpha_1 = \alpha_2 = .98$ and the numerical solutions for the special case $\alpha_1 = \alpha_2 = 1$, respectively.

Example 3. Consider the system of nonlinear fractional-order Genesio-Tesi

$$\begin{aligned} D^{\alpha_1} x(t) &= y(t), \\ D^{\alpha_2} y(t) &= z(t), \\ D^{\alpha_3} z(t) &= -\beta_1 x(t) - \beta_2 y(t) - \beta_3 z(t) + \beta_4 x(t)^2, \\ &\qquad (0 < \alpha_1, \alpha_2, \alpha_3 \le 1), \end{aligned} \tag{19}$$

where β_1, β_2, β_3, and β_4 are system parameters. With the initial conditions,

$$x(0) = -2, \quad y(0) = .5, \quad z(0) = 2. \tag{20}$$

In view of the variational iteration method, we set

$$\begin{aligned} x_{n+1}(t) &= x_n(t) - I_t^{\alpha_1}\left[D_t^{\alpha_1} x_n(t) - y_n(t)\right], \\ y_{n+1}(t) &= y_n(t) - I_t^{\alpha_2}\left[D_t^{\alpha_2} y_n(t) - z_n(t)\right], \\ z_{n+1}(t) &= z_n(t) - I_t^{\alpha_3}\left[D_t^{\alpha_3} z_n(t) + \beta_1 x_n(t) + \beta_2 y_n(t) \right. \\ &\qquad \left. + \beta_3 z_n(t) - \beta_4 x_n(t)^2\right]. \end{aligned} \tag{21}$$

The revised variational iteration method would lead to

$$\begin{aligned} x_{n+1}(t) &= x_n(t) - I_t^{\alpha_1}\left[D_t^{\alpha_1} x_n(t) - y_n(t)\right], \\ y_{n+1}(t) &= y_n(t) - I_t^{\alpha_2}\left[D_t^{\alpha_2} y_n(t) - z_n(t)\right], \\ z_{n+1}(t) &= z_n(t) - I_t^{\alpha_3}\left[D_t^{\alpha_3} z_n(t) + \beta_1 x_{(n+1)}(t) + \beta_2 y_{(n+1)}(t) \right. \\ &\qquad \left. + \beta_3 z_n(t) - \beta_4 x_{(n+1)}(t)^2\right]. \end{aligned} \tag{22}$$

Beginning with the initial approximations $x_0(t) = -2$ and $y_0(t) = .5, z_0(t) = 2$, we can easily obtain the results. Results are shown in Figures 4 and 5. Figures 4(a), 4(b), and 4(c) show comparison between the approximate solutions $(x(t) \cong x_9(t))$, $(y(t) \cong y_9(t))$ and $(z(t) \cong z_9(t))$ of (19) obtained using VIM and RVIM for the special case $\alpha_1 = \alpha_2 = \alpha_3 = .98$ and the numerical solutions for the special case $\alpha_1 = \alpha_2 = \alpha_3 = 1$, respectively.

Figures 5(a), 5(b), and 5(c), show approximate solutions $(x(t) \cong x_8(t))$, $(y(t) \cong y_7(t))$, and $(z(t) \cong z_7(t))$ of (19) using VIM and RVIM for the special case $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and the numerical solutions, respectively.

6. Conclusion

The variational iteration method is an efficient method for solving various kinds of problems. In this paper, we have suggested a modification of this method which is called "revised variational iteration method." We employ the revised VIM for solving a systems of nonlinear fractional-order differential equations. The revised method yields a series solution which converges faster than the series obtained by standard VIM. Illustrative examples presented clear support for this claim.

Mathematica has been used for computation and graphs presented in this paper.

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