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*Research Article*

# A Note on Fractional Equations of Volterra Type with Nonlocal Boundary Condition

**Zhenhai Liu[1,2] and Rui Wang[2]**

[1] *Guangxi Key Laboratory of Hybrid Computation and IC Design Analysis, Guangxi University for Nationalities, China*
[2] *College of Sciences, Guangxi University for Nationalities, Nanning, Guangxi 530006, China*

Correspondence should be addressed to Zhenhai Liu; zhhliu@hotmail.com

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We deal with nonlocal boundary value problems of fractional equations of Volterra type involving Riemann-Liouville derivative. Firstly, by defining a weighted norm and using the Banach fixed point theorem, we show the existence and uniqueness of solutions. Then, we obtain the existence of extremal solutions by use of the monotone iterative technique. Finally, an example illustrates the results.

## 1. Introduction

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, and so forth. There has been a significant theoretical development in fractional differential equations in recent years (see [1–18]). Monotone iterative technique is a useful tool for analyzing fractional differential equations.

In [3], Jankowski considered the existence of the solutions of the following problem:

$$D^q x(t) = f\left(t, x(t), \int_0^t k(t,s)\, x(s)\, ds\right), \quad 0 < q < 1, \quad t \in (0,T], \qquad \widetilde{x}(0) = r, \tag{1}$$

where $f \in C([0,T] \times R^2, R)$, $\widetilde{x}(0) = t^{1-q}x(t)|_{t=0}$ by using the Banach fixed point theorem and monotone iterative technique.

Motivated by [3], in this paper we investigate the following nonlocal boundary value problem:

$$D^\alpha x(t) = f\left(t, x(t), \int_0^t k(t,s)\, x(s)\, ds\right) \equiv Fx(t), \quad 0 < \alpha < 1,\ t \in (0,T], \qquad \widetilde{x}(0) = g(x), \tag{2}$$

where $f \in C([0,T] \times R^2, R)$, $g : C_{1-\alpha}([0,T]) \to R$ is a continuous functional, $J = [0,T]$, $\widetilde{x}(0) = t^{1-\alpha}x(t)|_{t=0}$, and $k(t,s) \in C(\Delta, R)$; here $\Delta = \{(t,s) \in J \times J : 0 \le s \le t \le T\}$.

Firstly, the nonlocal condition can be more useful than the standard initial condition to describe many physical and chemical phenomena. In contrast to the case for initial value problems, not much attention has been paid to the nonlocal fractional boundary value problems. Some recent results on the existence and uniqueness of nonlocal fractional boundary value problems can be found in [1, 2, 12, 14, 18]. However, discussion on nonlocal boundary value problems of fractional equations of Volterra type involving Riemann-Liouville derivative is rare. Secondly, in [3], in order to discuss the existence and uniqueness of problem (1), Jankowski divided $q \in (0,1)$ into two situations to discuss; one is $0 < q \le 1/2$ with an additional condition and the other is $1/2 < q < 1$. In this paper, we unify the two situations without using the additional condition. Thirdly, for the study of differential equation, monotone iterative technique is a useful tool (see [9, 10, 16, 17]). We know that it is important to build a comparison result when we use the monotone iterative technique. We transform the differential equation into integral equation and use the integral equation to build the comparison result which is different from [3]. It makes the calculation easier and is suitable for the more complicated forms of equations.

The paper is organized as follows. In Section 2, we present some useful definitions and fundamental facts of fractional calculus theory. In Section 3, by applying Banach fixed point theorem, we prove the existence and uniqueness of solution for problem (2). In Section 4, by the utility of the monotone iterative technique, we prove that (2) has extremal solutions. At last, we give an example to illustrate our main results.

## 2. Preliminaries

Let $C_{1-\alpha}(J,R)=\{x\in C((0,T],R):t^{1-\alpha}x(t)\in C(J,R)\}$ with the norm $\|x\|_{C_{1-\alpha}}=\max_{t\in J}|t^{1-\alpha}e^{-\lambda t}x(t)|$, where $\lambda$ is a fixed positive constant which will be fixed in Section 3. Obviously, the space $C_{1-\alpha}(J,R)$ is a Banach space. Now, let us recall the following definitions from fractional calculus. For more details, one can see [5, 11].

*Definition 1.* For $\alpha>0$, the integral

$$I^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)\,ds \tag{3}$$

is called the Riemann-Liouville fractional integral of order $\alpha$.

*Definition 2.* The Riemann-Liouville derivative of order $\alpha(n-1<\alpha\le n)$ can be written as

$$\begin{aligned} D^{\alpha}f(t)&=\left(\frac{d}{dt}\right)^n\left(I^{n-\alpha}f(t)\right)\\ &=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t(t-s)^{n-\alpha-1}f(s)\,ds,\quad t>0. \end{aligned} \tag{4}$$

**Lemma 3** (see [5]). *Let $n-1<\alpha\le n$. If $f(t)\in L(0,T)$ and $D_{0+}^{\alpha-n}f(t)\in AC^n[0,T]$, then one has the following equality:*

$$I^{\alpha}D^{\alpha}f(t)=f(t)-\sum_{i=1}^{n}\left[D^{\alpha-i}f(t)\right]_{t=0}\frac{t^{\alpha-i}}{\Gamma(\alpha-i+1)}. \tag{5}$$

## 3. Existence and Uniqueness of Solutions

In what follows, to discuss the existence and uniqueness of solutions of nonlocal boundary value problems for fractional equations of Volterra type involving Riemann-Liouville derivative, we suppose the following.

(H1) There exist nonnegative constants $L_1$, $L_2$, and $W$ such that $|k(t,s)|\le W$, for all $(t,s)\in\Delta$, and

$$\begin{aligned} &\left|f(t,v_1,v_2)-f(t,u_1,u_2)\right|\le L_1\left|v_1-u_1\right|+L_2\left|v_2-u_2\right|,\\ &\qquad\forall t\in J,\ \forall v_1,v_2,u_1,u_2\in R. \end{aligned} \tag{6}$$

(H2) There exists a nonnegative constant $L_3\in(0,1)$ such that

$$\begin{aligned} &\left|g(u_1)-g(u_2)\right|\le L_3\|u_1-u_2\|_{C_{1-\alpha}},\quad\forall t\in J,\\ &\qquad\forall u_1,u_2\in C_{1-\alpha}(J). \end{aligned} \tag{7}$$

**Lemma 4.** *Let (H1) hold. $x\in C_{1-\alpha}(J)$ and $x$ is a solution of the following problem:*

$$\begin{aligned} D^{\alpha}x(t)&=f\left(t,x(t),\int_0^t k(t,s)x(s)\,ds\right)\equiv Fx(t),\\ \tilde{x}(0)&=g(x), \end{aligned} \tag{8}$$

*if and only if $x(t)$ is a solution of the following integral equation:*

$$x(t)=g(x)t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}Fx(s)\,ds. \tag{9}$$

*Proof.* Assume that $x(t)$ satisfies (8). From the first equation of (8) and Lemma 3, we have

$$\begin{aligned} x(t)&=\frac{I_{0+}^{1-\alpha}x(t)\big|_{t=0}t^{\alpha-1}}{\Gamma(\alpha)}+I_{0+}^{\alpha}Fx(t)\\ &=g(x)t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}Fx(s)\,ds. \end{aligned} \tag{10}$$

Conversely, assume that $x(t)$ satisfies (9). Applying the operator $D^{\alpha}$ to both sides of (9), we have

$$D^{\alpha}x(t)=Fx(t). \tag{11}$$

In addition, by calculation, we can conclude $\tilde{x}(0)=t^{1-\alpha}x(t)|_{t=0}=g(x)$. The proof is completed. □

**Theorem 5.** *Let (H1), (H2) hold, $f\in C(J\times R^2,R)$, and $k\in C(\Delta,R)$. Then problem (2) has a unique solution.*

*Proof.* Define the operator $N:C_{1-\alpha}(J)\to C_{1-\alpha}(J)$ by

$$Nx(t)=g(x)t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}Fx(s)\,ds. \tag{12}$$

It is easy to check that the operator $N$ is well defined on $C_{1-\alpha}(J)$. Next we show that $N$ is a contradiction operator on $C_{1-\alpha}(J)$. For convenience, let

$$\begin{aligned} \rho\equiv&\frac{1}{q^{1/q}(1-L_3)}\\ &\times\left\{\frac{L_1}{\Gamma(\alpha)}\left[T^{p\alpha-p+1}\frac{\Gamma(p\alpha-p+1)^2}{\Gamma(2p\alpha-2p+2)}\right]^{1/p}\right.\\ &\left.+\frac{L_2W}{\alpha\Gamma(\alpha)}\left[T^{p\alpha+1}\frac{\Gamma(p\alpha-p+1)\Gamma(p\alpha+1)}{\Gamma(2p\alpha-p+2)}\right]^{1/p}\right\}, \end{aligned} \tag{13}$$

and choose

$$1<p<\frac{1}{1-\alpha},\qquad\frac{1}{p}+\frac{1}{q}=1,\qquad\lambda>\rho^q, \tag{14}$$

where $\lambda$ is a positive constant defined in the norm of the space $C_{1-\alpha}(J)$.

Then, for any $x, y \in C_{1-\alpha}(J)$, we have from (H1), (H2), and the Hölder inequality

$$
\begin{aligned}
&\left\|(Nx)(t)-(Ny)(t)\right\|_{C_{1-\alpha}} \\
&\quad= \max_{t\in[0,T]}\left|t^{1-\alpha}e^{-\lambda t}\left[(Nx)(t)-(Ny)(t)\right]\right| \\
&\quad\le \max_{t\in[0,T]} e^{-\lambda t}\left|g(x)-g(y)\right| \\
&\qquad+\max_{t\in[0,T]}\frac{1}{\Gamma(\alpha)}t^{1-\alpha}e^{-\lambda t}\int_0^t (t-s)^{\alpha-1}\left|Fx(s)-Fy(s)\right|ds \\
&\quad\le L_3\|x-y\|_{C_{1-\alpha}} \\
&\qquad+\max_{t\in[0,T]}\frac{L_1}{\Gamma(\alpha)}t^{1-\alpha}e^{-\lambda t}\int_0^t (t-s)^{\alpha-1}\left|x(s)-y(s)\right|ds \\
&\qquad+\max_{t\in[0,T]}\frac{L_2W}{\Gamma(\alpha)}t^{1-\alpha}e^{-\lambda t} \\
&\qquad\times\int_0^t (t-s)^{\alpha-1}\int_0^s\left|x(\tau)-y(\tau)\right|d\tau\, ds \\
&\quad\le L_3\|x-y\|_{C_{1-\alpha}} \\
&\qquad+\max_{t\in[0,T]}\frac{L_1}{\Gamma(\alpha)}t^{1-\alpha}e^{-\lambda t}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}e^{\lambda s}ds\|x-y\|_{C_{1-\alpha}} \\
&\qquad+\max_{t\in[0,T]}\frac{L_2W}{\alpha\Gamma(\alpha)}t^{1-\alpha}e^{-\lambda t}\int_0^t (t-s)^{\alpha-1}s^{\alpha}e^{\lambda s}ds\|x-y\|_{C_{1-\alpha}} \\
&\quad\le L_3\|x-y\|_{C_{1-\alpha}} \\
&\qquad+\max_{t\in[0,T]}\frac{L_1}{\Gamma(\alpha)}t^{1-\alpha}e^{-\lambda t}\left(\int_0^t\left((t-s)^{\alpha-1}s^{\alpha-1}\right)^p ds\right)^{1/p} \\
&\qquad\times\left(\int_0^t e^{\lambda sq}ds\right)^{1/q}\|x-y\|_{C_{1-\alpha}} \\
&\qquad+\max_{t\in[0,T]}\frac{L_2W}{\alpha\Gamma(\alpha)}t^{1-\alpha}e^{-\lambda t}\left(\int_0^t\left((t-s)^{\alpha-1}s^{\alpha}\right)^p ds\right)^{1/p} \\
&\qquad\times\left(\int_0^t e^{\lambda sq}ds\right)^{1/q}\|x-y\|_{C_{1-\alpha}} \\
&\quad\le L_3\|x-y\|_{C_{1-\alpha}} \\
&\qquad+\max_{t\in[0,T]}\frac{L_1}{\Gamma(\alpha)}e^{-\lambda t}\left(t^{p\alpha-p+1}\int_0^1(1-\eta)^{p\alpha-p}\eta^{p\alpha-p}d\eta\right)^{1/p} \\
&\qquad\times\frac{e^{\lambda t}}{(\lambda q)^{1/q}}\|x-y\|_{C_{1-\alpha}} \\
&\qquad+\max_{t\in[0,T]}\frac{L_2W}{\alpha\Gamma(\alpha)}e^{-\lambda t}\left(t^{p\alpha+1}\int_0^1(1-\eta)^{p\alpha-p}\eta^{p\alpha}d\eta\right)^{1/p} \\
&\qquad\times\frac{e^{\lambda t}}{(\lambda q)^{1/q}}\|x-y\|_{C_{1-\alpha}} \\
&\quad\le L_3\|x-y\|_{C_{1-\alpha}}+\frac{L_1}{\Gamma(\alpha)} \\
&\qquad\times\left[T^{p\alpha-p+1}\frac{\Gamma(p\alpha-p+1)^2}{\Gamma(2p\alpha-2p+2)}\right]^{1/p}\frac{1}{(\lambda q)^{1/q}}\|x-y\|_{C_{1-\alpha}} \\
&\qquad+\frac{L_2W}{\alpha\Gamma(\alpha)}\left[T^{p\alpha+1}\frac{\Gamma(p\alpha-p+1)\Gamma(p\alpha+1)}{\Gamma(2p\alpha-p+2)}\right]^{1/p} \\
&\qquad\times\frac{1}{(\lambda q)^{1/q}}\|x-y\|_{C_{1-\alpha}} \\
&\quad\le\left\{L_3+\frac{L_1}{\Gamma(\alpha)}\left[T^{p\alpha-p+1}\frac{\Gamma(p\alpha-p+1)^2}{\Gamma(2p\alpha-2p+2)}\right]^{1/p}\frac{1}{(\lambda q)^{1/q}}\right. \\
&\qquad+\frac{L_2W}{\alpha\Gamma(\alpha)}\left[T^{p\alpha+1}\frac{\Gamma(p\alpha-p+1)\Gamma(p\alpha+1)}{\Gamma(2p\alpha-p+2)}\right]^{1/p} \\
&\qquad\left.\times\frac{1}{(\lambda q)^{1/q}}\right\}\|x-y\|_{C_{1-\alpha}}.
\end{aligned}
\tag{$*$}
$$

According to $\lambda > \rho^q$ and the Banach fixed point theorem, the problem (2) has a unique solution. The proof is completed. □

*Remark 6.* Theorem 5 is an essential improvement of [3, Theorem 1].

## 4. The Monotone Iterative Technique for Problem (2)

In this section, the monotone iterative technique is presented and constructed for problem (2). This method leads to a simple and yet efficient linear iterative algorithm. It yields two sequences of iterations that converge monotonically from above and below, respectively, to a solution of the problem.

Let $M, N \in C(J)$. We may assume $|M(t)| \le M_1$, $|N(t)| \le N_1$, for all $t \in J$, $\sigma \in C_{1-\alpha}(J)$. Then, according to Lemma 4 and Theorem 5, the following linear problem

$$
\begin{gathered}
D^{\alpha}x(t)-M(t)x(t)-N(t)\int_0^t k(t,s)x(s)ds=\sigma(t), \\
t\in(0,T],\ 0<\alpha<1, \\
\tilde{x}(0)=t^{1-\alpha}x(t)\big|_{t=0}=g(x)
\end{gathered}
\tag{15}
$$

has a unique solution which satisfies

$$
\begin{aligned}
x(t) &= g(x)t^{\alpha-1} \\
&\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}
\end{aligned}
$$

$$\times\left(M(s)x(s)+N(s)\int_0^s k(s,\tau)x(\tau)\,d\tau+\sigma(s)\right)ds. \quad (16)$$

**Lemma 7.** *Let $0<\alpha<1$, $M,N\in C(J)$, $|M(t)|\le M_1$, $|N(t)|\le N_1$. Suppose that*

$$\frac{M_1T^{\alpha}\Gamma(\alpha)}{\Gamma(2\alpha)}+\frac{N_1WT^{\alpha+1}\Gamma(\alpha)}{\Gamma(2\alpha+1)}<1 \quad (17)$$

*and $p\in C_{1-\alpha}(J)$ satisfies the problem*

$$\begin{aligned}p(t)&\le\tilde{p}(0)t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\\&\quad\times\left(M(s)p(s)+N(s)\int_0^s k(s,\tau)p(\tau)\,d\tau\right)ds.\\&\tilde{p}(0)\le 0,\end{aligned} \quad (18)$$

*Then $p(t)\le 0$ for all $t\in(0,T]$.*

*Proof.* Suppose that the inequality $p(t)\le 0$, for all $t\in(0,T]$, is not true. Therefore, there exists at least a $t_*\in(0,T]$ such that $e^{-\lambda t_*}t_*^{1-\alpha}p(t_*)>0$. Without loss of generality, we assume $e^{-\lambda t_*}t_*^{1-\alpha}p(t_*)=\max\{e^{-\lambda t}t^{1-\alpha}p(t):t\in(0,T]\}=\rho_1>0$.

We obtain that

$$\begin{aligned}e^{-\lambda t}t^{1-\alpha}p(t)&\le e^{-\lambda t}\tilde{p}(0)+\frac{e^{-\lambda t}t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\\&\quad\times\left(M(s)p(s)+N(s)\int_0^s k(s,\tau)p(\tau)\,d\tau\right)ds\\&\le\frac{e^{-\lambda t}t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\\&\quad\times\left(M(s)p(s)+N(s)\int_0^s k(s,\tau)p(\tau)\,d\tau\right)ds\\&\le\frac{e^{-\lambda t}M_1t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\\&\quad\times s^{\alpha-1}e^{\lambda s}e^{-\lambda s}s^{1-\alpha}|p(s)|\,ds\\&\quad+\frac{e^{-\lambda t}N_1Wt^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\\&\quad\times\left(\int_0^s\tau^{\alpha-1}e^{\lambda\tau}e^{-\lambda\tau}\tau^{1-\alpha}|p(\tau)|\,d\tau\right)ds.\end{aligned} \quad (19)$$

Let $t=t_*$; we have

$$\begin{aligned}e^{-\lambda t_*}t_*^{1-\alpha}p(t_*)&\le\frac{M_1e^{-\lambda t_*}t_*^{1-\alpha}}{\Gamma(\alpha)}\\&\quad\times\int_0^{t_*}(t_*-s)^{\alpha-1}s^{\alpha-1}e^{\lambda s}e^{-\lambda s}s^{1-\alpha}|p(s)|\,ds\\&\quad+\frac{N_1We^{-\lambda t_*}t_*^{1-\alpha}}{\Gamma(\alpha)}\\&\quad\times\int_0^{t_*}(t_*-s)^{\alpha-1}\\&\quad\times\left(\int_0^s\tau^{\alpha-1}e^{\lambda\tau}e^{-\lambda\tau}\tau^{1-\alpha}|p(\tau)|\,d\tau\right)ds\\\rho_1&\le\frac{M_1t_*^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_*}(t_*-s)^{\alpha-1}s^{\alpha-1}ds\rho_1\\&\quad+\frac{N_1Wt_*^{1-\alpha}}{\alpha\Gamma(\alpha)}\int_0^{t_*}(t_*-s)^{\alpha-1}s^{\alpha}ds\rho_1,\\\rho_1&\le\left(\frac{M_1T^{\alpha}\Gamma(\alpha)}{\Gamma(2\alpha)}+\frac{N_1WT^{\alpha+1}\Gamma(\alpha)}{\Gamma(2\alpha+1)}\right)\rho_1.\end{aligned} \quad (20)$$

So

$$\frac{M_1T^{\alpha}\Gamma(\alpha)}{\Gamma(2\alpha)}+\frac{N_1WT^{\alpha+1}\Gamma(\alpha)}{\Gamma(2\alpha+1)}\ge 1. \quad (21)$$

This is a contradiction. Hence $p(t)\le 0$ for all $t\in(0,T]$. The proof is completed. □

*Definition 8.* We say that $x_0\in C_{1-\alpha}(J)$ is called a lower solution of problem (2) if

$$\begin{aligned}x_0(t)&\le\tilde{x}_0(0)t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}Fx_0(s)\,ds,\quad t\in(0,T],\\&\tilde{x}_0(0)\le g(x_0).\end{aligned} \quad (22)$$

We say that $y_0 \in C_{1-\alpha}(J)$ is called an upper solution of problem (2) if

$$y_0(t) \geq \tilde{y}_0(0)\, t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F y_0(s)\, ds, \quad t \in (0,T], \qquad \tilde{y}_0(0) \geq g(y_0). \tag{23}$$

In the following discussion, we need the following assumptions.

(H3) Assume that $g : C_{1-\alpha}(J) \to R$ is a nondecreasing continuous function, $f(t,\beta_1,\beta_2) \in C_{1-\alpha}(J)$ for all $t \in J$, $x_0 \leq \beta_1 \leq y_0$, $\int_0^t k(t,s)x_0(s)ds \leq \beta_2 \leq \int_0^t k(t,s)y_0(s)ds$. $x_0$ and $y_0$ are lower and upper solutions of problem (2), respectively, and $x_0 \leq y_0$.

(H4) Consider

$$f(t,v_1,v_2) - f(t,u_1,u_2) \geq M(t)(v_1-u_1) + N(t)(v_2-u_2), \tag{24}$$

where $x_0 \leq u_1 \leq v_1 \leq y_0$, $\int_0^t k(t,s)x_0(s)ds \leq u_2 \leq v_2 \leq \int_0^t k(t,s)y_0(s)ds$. $M, N \in C(J)$.

Let $[x_0, y_0] = \{z \in C_{1-\alpha}(J) : x_0(t) \leq z(t) \leq y_0(t), \tilde{x}_0(0) \leq \tilde{z}(0) \leq \tilde{y}_0(0)\}$.

**Theorem 9.** *Let inequality* (17), *(H2)–(H4) hold. Then there exist monotone sequences* $\{x_n\},\{y_n\} \subset [x_0, y_0]$ *which converge uniformly to the extremal solutions of* (2) *in* $[x_0, y_0]$, *respectively.*

*Proof.* This proof consists of the following three steps.

*Step 1.* Construct the sequences $\{x_n\}, \{y_n\}$.

For any $\eta \in [x_0, y_0]$, we consider the following linear problem:

$$\begin{aligned} D^\alpha x(t) - M(t)x(t) - N(t)\int_0^t k(t,s)x(s)\,ds &= f\left(t,\eta(t), \int_0^t k(t,s)\eta(s)\,ds\right) - M(t)\eta(t) - N(t)\int_0^t k(t,s)\eta(s)\,ds, \quad t \in (0,T], \\ \tilde{x}(0) &= g(\eta). \end{aligned} \tag{25}$$

By Theorem 5, (25) has a unique solution which satisfies

$$\begin{aligned} x(t) = {} & \tilde{x}(0)\,t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \times \left[ f\left(s,\eta(s),\int_0^s k(s,\tau)\eta(\tau)\,d\tau\right) - M(s)(\eta(s)-x(s)) - N(s)\int_0^s k(s,\tau) \times (\eta(\tau)-x(\tau))\,d\tau\right] ds, \\ \tilde{x}(0) = {} & g(\eta). \end{aligned} \tag{26}$$

Define an operator $A : [x_0, y_0] \to [x_0, y_0]$ by $x = A\eta$. It is easy to check that the operator $A$ is well defined on $[x_0, y_0]$. Let $\eta_1, \eta_2 \in [x_0, y_0]$ with $\eta_1 \leq \eta_2$.

Setting $p(t) = z_1(t) - z_2(t)$, $z_1(t) = A\eta_1(t)$, and $z_2(t) = A\eta_2(t)$, by (26), we obtain

$$\begin{aligned} p(t) = {} & \tilde{p}(0)\,t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \times \Bigg[ f\left(s,\eta_1(s),\int_0^s k(s,\tau)\eta_1(\tau)\,d\tau\right) - f\left(s,\eta_2(s),\int_0^s k(s,\tau)\eta_2(\tau)\,d\tau\right) \\ & - M(s)(\eta_1(s)-z_1(s)) + M(s)(\eta_2(s)-z_2(s)) - N(s)\int_0^s k(s,\tau)(\eta_1(\tau)-z_1(\tau))\,d\tau + N(s)\int_0^s k(s,\tau)(\eta_2(\tau)-z_2(\tau))\,d\tau\Bigg] ds \\ \leq {} & \tilde{p}(0)\,t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \times \left( M(s)p(s) + N(s)\int_0^s k(s,\tau)p(\tau)\,d\tau\right) ds. \end{aligned} \tag{27}$$

Besides,

$$\widetilde{p}(0) = \widetilde{z}_1(0) - \widetilde{z}_2(0) = g(\eta_1) - g(\eta_2) \le 0. \tag{28}$$

By Lemma 7, we know $p(t) \le 0$, $t \in (0, T]$. It means that $A$ is nondecreasing. Obviously, we can easily get that $A$ is a continuous map. Let $x_n = Ax_{n-1}$, $y_n = Ay_{n-1}$, $n = 1, 2, \ldots$.

*Step 2.* The sequences $\{t^{1-\alpha}x_n\}$, $\{t^{1-\alpha}y_n\}$ converge uniformly to $t^{1-\alpha}x^*$, $t^{1-\alpha}y^*$, respectively.

In fact, $x_n$, $y_n$ satisfy the following relation:

$$x_0 \le x_1 \le \cdots \le x_n \le \cdots \le y_n \le \cdots \le y_1 \le y_0. \tag{29}$$

Setting $p(t) = x_0(t) - x_1(t)$ and $x_0(t)$ is a lower solution of problem (2):

$$\begin{aligned} p(t) \le{} & \widetilde{x}_0(0) t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \times f\left(s, x_0(s), \int_0^s k(s,\tau) x_0(\tau) d\tau\right) ds \\ & - \widetilde{x}_1(0) t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \times \left[ f\left(s, x_0(s), \int_0^s k(s,\tau) x_0(\tau) d\tau\right) - M(s)(x_0(s) - x_1(s)) - N(s) \int_0^s k(s,\tau) \times (x_0(\tau) - x_1(\tau)) d\tau \right] ds \\ ={} & \widetilde{p}(0) t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \times \left( M(s) p(s) + N(s) \int_0^s k(s,\tau) p(\tau) d\tau \right) ds. \end{aligned} \tag{30}$$

Besides,

$$\widetilde{p}(0) = \widetilde{x}_0(0) - \widetilde{x}_1(0) \le g(x_0) - g(x_0) = 0. \tag{31}$$

By Lemma 7, we can obtain that $x_0 \le x_1$ for all $t \in (0, T]$. Similarly, we can show that $y_1 \le y_0$ for all $t \in (0, T]$. Applying the operator $A$ to both sides of $x_0 \le x_1$, $y_1 \le y_0$, and $x_0 \le y_0$, we can easily get (29). Obviously, the sequences $\{t^{1-\alpha}x_n\}$, $\{t^{1-\alpha}y_n\}$ are uniformly bounded and equicontinuous. Then by using the Ascoli-Arzela criterion, we can conclude that the sequences $\{t^{1-\alpha}x_n\}$, $\{t^{1-\alpha}y_n\}$ converge uniformly on $(0, T]$ with $\lim_{n\to\infty} t^{1-\alpha}x_n = t^{1-\alpha}x^*$, $\lim_{n\to\infty} t^{1-\alpha}y_n = t^{1-\alpha}y^*$ uniformly on $(0, T]$.

*Step 3.* $x^*$, $y^*$ are extremal solutions of (1).

$x^*$, $y^*$ are solutions of (1) on $[x_0, y_0]$, because of the continuity of operator $A$. Let $z \in [x_0, y_0]$ be any solution of (1). That is,

$$z(t) = \widetilde{z}(0) t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Fz(s) ds, \quad \widetilde{z}(0) = g(z). \tag{32}$$

Suppose that there exists a positive integer $n$ such that $x_n(t) \le z(t) \le y_n(t)$ on $(0, T]$. Let $p(t) = x_{n+1}(t) - z(t)$; we have

$$\begin{aligned} p(t) ={} & \widetilde{x}_{n+1}(0) t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \times \left[ f\left(s, x_n(s), \int_0^s k(s,\tau) x_n(\tau) d\tau\right) - M(s)(x_n(s) - x_{n+1}(s)) - N(s) \int_0^s k(s,\tau) \times (x_n(\tau) - x_{n+1}(\tau)) d\tau \right] ds \\ & - \widetilde{z}(0) t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \times f\left(s, z(s), \int_0^s k(s,\tau) z(\tau) d\tau\right) ds \\ \le{} & \widetilde{p}(0) t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \times \left( M(s) p(s) + N(s) \int_0^s k(s,\tau) p(\tau) d\tau \right) ds, \\ \widetilde{p}(0) ={} & \widetilde{x}_{n+1}(0) - \widetilde{z}(0) = g(x_n) - g(z) \le 0. \end{aligned} \tag{33}$$

By Lemma 7, we know that $p(t) \le 0$ on $(0, T]$, which implies $x_{n+1}(t) \le z(t)$ on $(0, T]$. Similarly, we obtain that $z(t) \le y_{n+1}(t)$ on $(0, T]$. Since $x_0(t) \le z(t) \le y_0(t)$ on $(0, T]$, by induction we get that $x_n(t) \le z(t) \le y_n(t)$ on $(0,T]$ for every $n$. Therefore, $x^*(t) \le z(t) \le y^*(t)$ on $(0,T]$ by taking $n \to \infty$. Thus, we completed this proof. □

## 5. An Example

*Example 1.* Consider the following problem:

$$D^{1/2}x(t) = t + \frac{1}{60}x(t) + \frac{1}{30}\int_0^t tsx(s)\,ds, \quad t \in (0,1],$$
$$\tilde{x}(0) = g(x) = \frac{\eta}{12}x(\eta), \quad 0 < \eta < 1. \tag{34}$$

Obviously, $T = 1$, $\alpha = 1/2$, $k(t,s) = ts$, and $f(t,v_1,v_2) = t + (1/60)v_1 + (1/30)v_2$.

Let $w = 1$, $L_1 = 1/60$, $L_2 = 1/30$, and $L_3 = 1/12$.

It is easy to check that

$$|k(t,s)| \le 1,$$
$$|f(t,v_1,v_2) - f(t,u_1,u_2)| \le \frac{1}{60}|v_1 - u_1| + \frac{1}{30}|v_2 - u_2|,$$
$$|g(x_1) - g(x_2)| \le \frac{1}{12}\|x_1 - x_2\|_{C_{1-\alpha}}. \tag{35}$$

So, (H1) and (H2) are satisfied. By the choice of $p = 3/2$, $q = 3$, we can get that $\lambda > \rho^3$ and $\rho \equiv (4/(11\times 3^{1/3}))\{(1/20\Gamma(1/2))[\Gamma(1/4)^2/\Gamma(1/2)]^{2/3} + (1/15\Gamma(1/2))[\Gamma(1/4)\Gamma(7/4)]^{2/3}\}$. According to Theorem 5, the problem (34) has a unique solution.

Consider the same equation as (34), taking $x_0(t) = 0$, $y_0(t) = t^{-1/2} + 6$, and then we have $\tilde{y}_0(0) = 1$.

Moreover,

$$\begin{aligned} y_0(t) &= t^{-1/2} + 6 \\ &\ge t^{-1/2} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\ &\quad \times \left[s + \frac{1}{60}\left(s^{-1/2} + 6\right)\right. \\ &\quad \left. + \frac{1}{30}\int_0^s s\tau\left(\tau^{-1/2} + 6\right)d\tau\right] ds, \\ \tilde{y}_0(0) &= 1 \ge \frac{\eta^{1/2}}{12} + \frac{\eta}{2}, \quad 0 < \eta < 1. \end{aligned} \tag{36}$$

On the other hand, it is easy to check that (H3) holds. And let $M(t) = 1/(t-1)$, $N(t) = \cos t/30$, and then we have

$$\begin{aligned} f(t,v_1,v_2) - f(t,u_1,u_2) &\ge \frac{1}{t-1}(v_1 - u_1) \\ &\quad + \frac{\cos t}{30}(v_2 - u_2), \end{aligned} \tag{37}$$

where $x_0 \le u_1 \le v_1 \le y_0$, $\int_0^t k(t,s)x_0(s)ds \le u_2 \le v_2 \le \int_0^t k(t,s)y_0(s)ds$. So (H4) is satisfied. Obviously, $M_1 = 1/30$, $N_1 = 1$, and then we can get

$$\frac{M_1 T^{\alpha}\Gamma(\alpha)}{\Gamma(2\alpha)} + \frac{N_1 W T^{\alpha+1}\Gamma(\alpha)}{\Gamma(2\alpha+1)} = \frac{31\pi^{1/2}}{60} < 1. \tag{38}$$

Inequality (17) holds. All conditions of Theorem 9 are satisfied, so problem (34) has extremal solutions.


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