

Research Article

Existence and Exact Asymptotic Behavior of Positive Solutions for a Fractional Boundary Value Problem

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Received 4 November 2012; Accepted 25 December 2012

Academic Editor: Chuanzhi Bai

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We establish the existence and uniqueness of a positive solution u for the fractional boundary value problem $D^\alpha u(x) = -a(x)u^\sigma(x)$, $x \in (0, 1)$ with the condition $\lim_{x \rightarrow 0^+} D^{\alpha-1}u(x) = 0$, $u(1) = 0$, where $1 < \alpha \leq 2$, $\sigma \in (-1, 1)$, and a is a nonnegative continuous function on $(0, 1)$ that may be singular at $x = 0$ or $x = 1$.

1. Introduction

Fractional differential equations arise in various fields of science and engineering such as control, porous media, electrochemistry, viscoelasticity, and electromagnetism. They also serve as an excellent tool for the description of hereditary properties of various materials and processes (see [1–3]). In consequence, the subject of fractional differential equations is gaining much importance and attention. Motivated by the surge in the development of this subject, we consider the following problem:

$$\begin{aligned} D^\alpha u(x) &= -a(x)u^\sigma(x), \quad x \in (0, 1), \\ \lim_{x \rightarrow 0^+} D^{\alpha-1}u(x) &= 0, \quad u(1) = 0, \end{aligned} \quad (1)$$

where $1 < \alpha \leq 2$, $-1 < \sigma < 1$, a is a nonnegative continuous function on $(0, 1)$ that may be singular at $x = 0$ or $x = 1$ and D^α is the Riemann-Liouville fractional derivative. Then we study the existence and exact asymptotic behavior of positive solutions for this problem.

We recall that for a measurable function v , the Riemann-Liouville fractional integral $I_\beta v$ and the Riemann-Liouville

derivative $D^\beta v$ of order $\beta > 0$ are, respectively, defined by

$$\begin{aligned} I_\beta v(x) &= \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} v(t) dt, \\ D^\beta v(x) &= \frac{1}{\Gamma(n-\beta)} \left(\frac{d}{dx} \right)^n \int_0^x (x-t)^{n-\beta-1} v(t) dt \quad (2) \\ &= \left(\frac{d}{dx} \right)^n I_{n-\beta} v(x), \end{aligned}$$

provided that the right-hand sides are pointwise defined on $(0, 1]$. Here $n = [\beta] + 1$ and $[\beta]$ means the integral part of the number β and Γ is the Euler Gamma function.

Moreover, we have the following well-known properties (see [2, 4]):

- (i) $I_\beta I_\gamma v(x) = I_{\beta+\gamma} v(x)$ for $x \in [0, 1]$, $v \in L^1((0, 1])$, $\beta + \gamma \geq 1$,
- (ii) $D^\beta I_\beta v(x) = v(x)$ for a.e. $x \in [0, 1]$, where $v \in L^1((0, 1])$, $\beta > 0$,
- (iii) $D^\beta v(x) = 0$ if and only if $v(x) = \sum_{j=1}^n c_j t^{\beta-j}$, where $n = [\beta] + 1$ and $(c_1, c_2, \dots, c_n) \in \mathbb{R}^n$.

Several results are obtained for fractional differential equation with different boundary conditions (see [5–15]) and the

references therein), but none of them deals with the existence of a positive solution for problem (1).

Our aim in this paper is to establish the existence and uniqueness of a positive solution $u \in C_{2-\alpha}([0, 1])$ for problem (1) with a precise asymptotic behavior, where $C_{2-\alpha}([0, 1])$ is the set of all functions f such that $t \rightarrow t^{2-\alpha} f(t)$ is continuous on $[0, 1]$. Note that for $0 < \alpha < 2$, the solution u for problem (1) blows up at $x = 0$.

To state our result, we need some notations. We will use \mathcal{K} to denote the set of Karamata functions L defined on $(0, \eta]$ by

$$L(t) := c \exp \left(\int_t^\eta \frac{z(s)}{s} ds \right), \tag{3}$$

for some $\eta > 1$, where $c > 0$ and $z \in C([0, \eta])$ such that $z(0) = 0$. It is clear that a function L is in \mathcal{K} if and only if L is a positive function in $C^1((0, \eta])$ such that

$$\lim_{t \rightarrow 0^+} \frac{tL'(t)}{L(t)} = 0. \tag{4}$$

For two nonnegative functions f and g defined on a set S , the notation $f(x) \approx g(x)$, $x \in S$, means that there exists $c > 0$ such that $(1/c)f(x) \leq g(x) \leq cf(x)$, for all $x \in S$. We denote also $x^+ = \max(x, 0)$ for $x \in \mathbb{R}$.

Throughout this paper we assume that a is nonnegative on $(0, 1)$ and satisfies the following condition.

(H_0) $a \in C((0, 1))$ such that

$$a(t) \approx t^{-\lambda} L_1(t) (1-t)^{-\mu} L_2(1-t), \quad t \in (0, 1), \tag{5}$$

where $\lambda + (2 - \alpha)\sigma \leq 1$, $\mu \leq \alpha$, $L_1, L_2 \in \mathcal{K}$ satisfying

$$\int_0^\eta \frac{L_1(t)}{t^{\lambda+(2-\alpha)\sigma}} dt < \infty, \quad \int_0^\eta \frac{L_2(t)}{t^{\mu-\alpha+1}} dt < \infty. \tag{6}$$

In the sequel, we introduce the function θ defined on $(0, 1)$ by

$$\theta(x) = \begin{cases} 1-x & \text{if } \mu < \sigma + \alpha - 1, \\ (1-x) \left(\int_{1-x}^\eta \frac{L_2(s)}{s} ds \right)^{1/(1-\sigma)} & \text{if } \mu = \sigma + \alpha - 1, \\ (1-x)^{(\alpha-\mu)/(1-\sigma)} (L_2(1-x))^{1/(1-\sigma)} & \text{if } \sigma + \alpha - 1 < \mu < \alpha, \\ \left(\int_0^{1-x} \frac{L_2(s)}{s} ds \right)^{1/(1-\sigma)} & \text{if } \mu = \alpha. \end{cases} \tag{7}$$

Our main result is the following.

Theorem 1. *Let $\sigma \in (-1, 1)$ and assume that a satisfies (H_0) . Then problem (1) has a unique positive solution $u \in C_{2-\alpha}([0, 1])$ satisfying for $x \in (0, 1)$*

$$u(x) \approx x^{\alpha-2} \theta(x). \tag{8}$$

This paper is organized as follows. Some preliminary lemmas are stated and proved in the next section, involving some already known results on Karamata functions. In Section 3, we give the proof of Theorem 1.

2. Technical Lemmas

To keep the paper self-contained, we begin this section by recapitulating some properties of Karamata regular variation theory. The following is due to [16, 17].

Lemma 2. *The following hold.*

(i) *Let $L \in \mathcal{K}$ and $\varepsilon > 0$, then one has*

$$\lim_{t \rightarrow 0^+} t^\varepsilon L(t) = 0. \tag{9}$$

(ii) *Let $L_1, L_2 \in \mathcal{K}$ and $p \in \mathbb{R}$. Then one has $L_1 + L_2 \in \mathcal{K}$, $L_1 L_2 \in \mathcal{K}$, and $L_1^p \in \mathcal{K}$.*

Example 3. Let $m \in \mathbb{N}^*$. Let $c > 0$, $(\mu_1, \mu_2, \dots, \mu_m) \in \mathbb{R}^m$ and let d be a sufficiently large positive real number such that the function

$$L(t) = c \prod_{k=1}^m \left(\log_k \left(\frac{d}{t} \right) \right)^{-\mu_k} \tag{10}$$

is defined and positive on $(0, \eta]$, for some $\eta > 1$, where $\log_k x = \log \circ \log \circ \dots \circ \log x$ (k times). Then $L \in \mathcal{K}$.

Applying Karamata's theorem (see [16, 17]), we get the following.

Lemma 4. *Let $\mu \in \mathbb{R}$ and let L be a function in \mathcal{K} defined on $(0, \eta]$. One has the following.*

(i) *If $\mu < -1$, then $\int_0^\eta s^\mu L(s) ds$ diverges and $\int_t^\eta s^\mu L(s) ds \sim_{t \rightarrow 0^+} (t^{1+\mu} L(t))/(\mu + 1)$.*

(ii) *If $\mu > -1$, then $\int_0^\eta s^\mu L(s) ds$ converges and $\int_0^t s^\mu L(s) ds \sim_{t \rightarrow 0^+} (t^{1+\mu} L(t))/(\mu + 1)$.*

Lemma 5. *Let $L \in \mathcal{K}$ be defined on $(0, \eta]$. Then one has*

$$\lim_{t \rightarrow 0^+} \frac{L(t)}{\int_t^\eta (L(s)/s) ds} = 0. \tag{11}$$

If further $\int_0^\eta (L(s)/s) ds$ converges, then one has

$$\lim_{t \rightarrow 0^+} \frac{L(t)}{\int_0^t (L(s)/s) ds} = 0. \tag{12}$$

Proof. We distinguish two cases.

Case 1. We suppose that $\int_0^\eta (L(s)/s) ds$ converges. Since the function $t \rightarrow L(t)/t$ is nonincreasing in $(0, \omega]$, for some $\omega < \eta$, it follows that, for each $t \leq \omega$, we have

$$L(t) \leq \int_0^t \frac{L(s)}{s} ds. \tag{13}$$

It follows that $\lim_{t \rightarrow 0^+} L(t) = 0$. So we deduce (11).

Now put

$$\varphi(t) = \frac{L(t)}{t}, \quad \text{for } t \in (0, \eta). \tag{14}$$

Using that $\lim_{t \rightarrow 0^+} (t\varphi'(t)/\varphi(t)) = -1$, we obtain

$$\int_0^t \varphi(s) ds \underset{t \rightarrow 0^+}{\sim} - \int_0^t s\varphi'(s) ds = -t\varphi(t) + \int_0^t \varphi(s) ds. \tag{15}$$

This implies that

$$\int_0^t \frac{L(s)}{s} ds \underset{t \rightarrow 0^+}{\sim} L(t) + \int_0^t \frac{L(s)}{s} ds. \tag{16}$$

So (12) holds.

Case 2. We suppose that $\int_0^\eta (L(s)/s) ds$ diverges. We have, for some $\omega < \eta$,

$$\int_t^\omega \varphi(s) ds \underset{t \rightarrow 0^+}{\sim} t\varphi(t) - \omega\varphi(\omega) + \int_t^\omega \varphi(s) ds. \tag{17}$$

This implies that

$$\int_t^\omega \frac{L(s)}{s} ds \underset{t \rightarrow 0^+}{\sim} L(t) - \omega\varphi(\omega) + \int_t^\omega \frac{L(s)}{s} ds. \tag{18}$$

This proves (11) and completes the proof. \square

Remark 6. Let $L \in \mathcal{K}$ defined on $(0, \eta]$; then, using (4) and (11), we deduce that

$$t \longrightarrow \int_t^\eta \frac{L(s)}{s} ds \in \mathcal{K}. \tag{19}$$

If further $\int_0^\eta (L(s)/s) ds$ converges, we have by (11) that

$$t \longrightarrow \int_0^t \frac{L(s)}{s} ds \in \mathcal{K}. \tag{20}$$

Lemma 7. Given $1 < \alpha \leq 2$ and f is such that the function $t \rightarrow (1-t)^{\alpha-1} f(t)$ is continuous and integrable on $(0, 1)$, then the boundary value problem

$$\begin{aligned} D^\alpha u(t) &= -f(t), \quad t \in (0, 1), \\ \lim_{x \rightarrow 0} D^{\alpha-1} u(x) &= 0, \quad u(1) = 0, \end{aligned} \tag{21}$$

has a unique solution given by

$$u(x) = G_\alpha f(x) := \int_0^1 G_\alpha(x, t) f(t) dt, \tag{22}$$

where

$$G_\alpha(x, t) = \frac{1}{\Gamma(\alpha)} \left[x^{\alpha-2} (1-t)^{\alpha-1} - ((x-t)^+)^{\alpha-1} \right], \tag{23}$$

is the Green function for the boundary value problem (21).

Proof. Since $u_0 = -I_\alpha f$ is a solution of the equation $D^\alpha u = -f$, then $D^\alpha(u + I_\alpha f) = 0$. Consequently there exist two constants $c_1, c_2 \in \mathbb{R}$ such that $u(x) + I_\alpha f(x) = c_1 x^{\alpha-1} + c_2 x^{\alpha-2}$.

Using the fact that $\lim_{x \rightarrow 0} D^{\alpha-1} u(x) = 0$ and $u(1) = 0$, we obtain $c_1 = 0$ and $c_2 = I_\alpha f(1)$. So

$$\begin{aligned} u(x) &= \frac{1}{\Gamma(\alpha)} x^{\alpha-2} \int_0^1 (1-t)^{\alpha-1} f(t) dt \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt \\ &= \int_0^1 G_\alpha(x, t) f(t) dt. \end{aligned} \tag{24}$$

In the following, we give some estimates on the function G_α . So, we need the following lemma. \square

Lemma 8. For $\lambda, \mu \in (0, \infty)$, and $a, t \in [0, 1]$, one has

$$\min\left(1, \frac{\mu}{\lambda}\right) (1-at^\lambda) \leq 1-at^\mu \leq \max\left(1, \frac{\mu}{\lambda}\right) (1-at^\lambda). \tag{25}$$

Proposition 9. On $(0, 1) \times (0, 1)$, one has

$$G_\alpha(x, t) \approx x^{\alpha-2} (1-t)^{\alpha-2} (1 - \max(x, t)). \tag{26}$$

Proof. For $x, t \in (0, 1) \times (0, 1)$, we have

$$G_\alpha(x, t) = \frac{(1-t)^{\alpha-1} x^{\alpha-2}}{\Gamma(\alpha)} \left[1 - x \left(\frac{(x-t)^+}{x(1-t)} \right)^{\alpha-1} \right]. \tag{27}$$

Since $(x-t)^+/x(1-t) \in (0, 1)$ for $t \in (0, 1)$, then, by applying Lemma 8 with $\mu = \alpha - 1$ and $\lambda = 1$, we obtain

$$\begin{aligned} G_\alpha(x, t) &\approx x^{\alpha-2} (1-t)^{\alpha-1} \left(1 - \frac{(x-t)^+}{1-t} \right) \\ &= x^{\alpha-2} (1-t)^{\alpha-2} (1 - \max(x, t)), \end{aligned} \tag{28}$$

which completes the proof. \square

In the sequel we put

$$b(t) = t^{-\beta} L_3(t) (1-t)^{-\gamma} L_4(1-t), \tag{29}$$

where $L_3, L_4 \in \mathcal{K}$ and we aim to give some estimates on $x^{2-\alpha} G_\alpha b(x)$.

Proposition 10. Assume that $L_3, L_4 \in \mathcal{K}, \beta \leq 1, \gamma \leq \alpha$ with

$$\int_0^\eta t^{-\beta} L_3(t) dt < \infty, \quad \int_0^\eta t^{\alpha-1-\gamma} L_4(t) dt < \infty. \tag{30}$$

Then for $x \in (0, 1)$,

$$x^{2-\alpha} G_\alpha b(x) \approx \begin{cases} 1-x & \text{if } \gamma < \alpha-1, \\ (1-x) \int_{1-x}^\eta \frac{L_4(t)}{t} dt & \text{if } \gamma = \alpha-1, \\ (1-x)^{\alpha-\gamma} L_4(1-x) & \text{if } \alpha-1 < \gamma < \alpha, \\ \int_0^{1-x} \frac{L_4(t)}{t} dt & \text{if } \gamma = \alpha. \end{cases} \tag{31}$$

Proof. Using Proposition 9, we have

$$\begin{aligned} x^{2-\alpha}G_\alpha b(x) &\approx (1-x) \int_0^x (1-t)^{\alpha-2-\gamma} t^{-\beta} L_3(t) L_4(1-t) dt \\ &\quad + \int_x^1 (1-t)^{\alpha-1-\gamma} t^{-\beta} L_3(t) L_4(1-t) dt \\ &= (1-x)I(x) + J(x). \end{aligned} \tag{32}$$

For $0 < x \leq 1/2$, we use Lemma 4 and hypotheses (30) to deduce that

$$\begin{aligned} I(x) &\approx \begin{cases} x^{1-\beta} L_3(x) & \text{if } \beta < 1, \\ \int_0^x \frac{L_3(t)}{t} dt & \text{if } \beta = 1, \end{cases} \\ J(x) &\approx \int_{1/2}^1 (1-t)^{\alpha-1-\gamma} L_4(1-t) dt + \int_x^{1/2} t^{-\beta} L_3(t) dt \\ &\approx 1 + \int_x^{1/2} t^{-\beta} L_3(t) dt \\ &\approx 1. \end{aligned} \tag{33}$$

Hence, it follows from Lemma 2 and hypothesis (30) that, for $0 < x \leq 1/2$, we have

$$x^{2-\alpha}G_\alpha b(x) \approx 1. \tag{34}$$

Now, for $1/2 \leq x < 1$, we use again Lemma 4 and hypothesis (30) to deduce that

$$\begin{aligned} I(x) &\approx \int_0^{1/2} t^{-\beta} L_3(t) dt + \int_{1/2}^x (1-t)^{\alpha-2-\gamma} L_4(1-t) dt \\ &\approx 1 + \int_{1-x}^{1/2} t^{\alpha-2-\gamma} L_4(t) dt \\ &\approx \begin{cases} 1 & \text{if } \gamma < \alpha - 1, \\ \int_{1-x}^{\eta} \frac{L_4(t)}{t} dt & \text{if } \gamma = \alpha - 1, \\ (1-x)^{\alpha-1-\gamma} L_4(1-x) & \text{if } \gamma > \alpha - 1, \end{cases} \end{aligned} \tag{35}$$

$$\begin{aligned} J(x) &\approx \int_0^{1-x} t^{\alpha-1-\gamma} L_4(t) dt \\ &\approx \begin{cases} (1-x)^{\alpha-\gamma} L_4(1-x) & \text{if } \gamma < \alpha, \\ \int_0^{1-x} \frac{L_4(t)}{t} dt & \text{if } \gamma = \alpha. \end{cases} \end{aligned}$$

Hence, it follows from Lemmas 2 and 5 that, for $x \in [1/2, 1)$, we have

$$x^{2-\alpha}G_\alpha b(x) \approx \begin{cases} 1-x & \text{if } \gamma < \alpha - 1, \\ (1-x) \int_{1-x}^{\eta} \frac{L_4(t)}{t} dt & \text{if } \gamma = \alpha - 1, \\ (1-x)^{\alpha-\gamma} L_4(1-x) & \text{if } \alpha - 1 < \gamma < \alpha, \\ \int_0^{1-x} \frac{L_4(t)}{t} dt & \text{if } \gamma = \alpha. \end{cases} \tag{36}$$

This together with (34) implies that (36) holds on $(0, 1)$. \square

3. Proof of Theorem 1

We begin this section by giving a preliminary result that will play a crucial role in the proof of Theorem 1.

Proposition 11. *Assume that the function a satisfies (H_0) and put $\omega(t) = a(t)t^{(\alpha-2)\sigma} \theta^\sigma(t)$ for $t \in (0, 1)$. Then one has, for $x \in (0, 1)$,*

$$x^{2-\alpha} G_\alpha \omega(x) \approx \theta(x). \tag{37}$$

Proof. For $t \in (0, 1)$, we have

$$\begin{aligned} \omega(t) &= a(t) t^{(\alpha-2)\sigma} \theta^\sigma(t) \\ &= \begin{cases} t^{-\lambda-(2-\alpha)\sigma} (1-t)^{-\mu+\sigma} L_1(t) L_2(1-t) & \text{if } \mu < \sigma + \alpha - 1, \\ t^{-\lambda-(2-\alpha)\sigma} (1-t)^{-\mu+\sigma} L_1(t) L_2(1-t) \\ \quad \times \left(\int_{1-t}^{\eta} \frac{L_2(s)}{s} ds \right)^{\sigma/(1-\sigma)} & \text{if } \mu = \sigma + \alpha - 1, \\ t^{-\lambda-(2-\alpha)\sigma} (1-t)^{-\mu} L_1(t) L_2(1-t) \\ \quad \times (L_2(1-t))^{\sigma/(1-\sigma)} & \text{if } \sigma + \alpha - 1 < \mu < \alpha, \\ t^{-\lambda-(2-\alpha)\sigma} (1-t)^{-\mu} L_1(t) L_2(1-t) \\ \quad \times \left(\int_0^{1-t} \frac{L_2(s)}{s} ds \right)^{\sigma/(1-\sigma)} & \text{if } \mu = \alpha. \end{cases} \end{aligned} \tag{38}$$

So, we can see that

$$\omega(t) = t^{-\beta} (1-t)^{-\gamma} \tilde{L}_1(t) \tilde{L}_2(1-t), \tag{39}$$

where $\beta \leq 1, \gamma \leq \alpha$ and, according to Lemma 2, the functions $t \rightarrow \tilde{L}_1(t)$ and $t \rightarrow \tilde{L}_2(t)$ are in \mathcal{X} . Moreover, using Lemma 4, we have $\int_0^\eta t^{-\beta} \tilde{L}_1(t) dt < \infty$ and $\int_0^\eta t^{-\gamma} \tilde{L}_2(t) dt < \infty$. So the result follows from Proposition 10. \square

Proof of Theorem 1. From Proposition 11, there exists $M > 1$ such that for each $x \in (0, 1)$

$$\frac{1}{M} \theta(x) \leq x^{2-\alpha} G_\alpha \omega(x) \leq M \theta(x), \tag{40}$$

where $\omega(t) = a(t)t^{(\alpha-2)\sigma} \theta^\sigma(t)$. Put $c_0 = M^{1/(1-|\sigma|)}$ and let

$$\Lambda = \left\{ \nu \in C([0, 1]) : \frac{1}{c_0} \theta \leq \nu \leq c_0 \theta \right\}. \tag{41}$$

In order to use a fixed point theorem, we denote $\tilde{a}(t) = a(t)t^{(\alpha-2)\sigma}$ and we define the operator T on Λ by

$$T\nu(x) = x^{2-\alpha} G_\alpha (\tilde{a}\nu^\sigma)(x). \tag{42}$$

For this choice of c_0 , we can easily prove that for $\nu \in \Lambda$, we have $T\nu \leq c_0 \theta$ and $T\nu \geq (1/c_0) \theta$.

Now, we have

$$\begin{aligned}
 Tv(x) &= \frac{x^{2-\alpha}}{\Gamma(\alpha)} \int_0^1 G_\alpha(x, t) \bar{a}(t) v^\sigma(t) dt \\
 &= \frac{1}{\Gamma(\alpha)} \int_0^1 \left[(1-t)^{\alpha-1} - x^{2-\alpha} ((x-t)^+)^{\alpha-1} \right] \\
 &\quad \times \bar{a}(t) v^\sigma(t) dt.
 \end{aligned} \tag{43}$$

Since the function $(x, t) \rightarrow (1-t)^{\alpha-1} - x^{2-\alpha}((x-t)^+)^{\alpha-1}$ is continuous on $[0, 1] \times [0, 1]$ and the function $t \rightarrow (1-t)^{\alpha-1} \bar{a}(t) \theta^\sigma(t)$ is integrable on $(0, 1)$, we deduce that the operator T is compact from Λ to itself. It follows by the Schauder fixed point theorem that there exists $v \in \Lambda$ such that $Tv = v$. Put $u(x) = x^{\alpha-2} v(x)$. Then $u \in C_{2-\alpha}([0, 1])$ and u satisfies the equation

$$u(x) = G_\alpha(au^\sigma)(x). \tag{44}$$

Since the function $t \rightarrow (1-t)^{\alpha-1} a(t) u^\sigma(t)$ is continuous and integrable on $(0, 1)$, then by Lemma 7 the function u is a positive continuous solution of problem (1).

Finally, let us prove that u is the unique positive continuous solution satisfying (8). To this aim, we assume that (1) has two positive solutions $u, v \in C_{2-\alpha}([0, 1])$ satisfying (8) and consider the nonempty set $J = \{m \geq 1 : 1/m \leq u/v \leq m\}$ and put $c = \inf J$. Then $c \geq 1$ and we have $(1/c)v \leq u \leq cv$. It follows that $u^\sigma \leq c^{|\sigma|} v^\sigma$ and consequently

$$\begin{aligned}
 -D^\alpha(c^{|\sigma|}v - u) &= a(c^{|\sigma|}v^\sigma - u^\sigma) \geq 0, \\
 \lim_{t \rightarrow 0^+} D^{\alpha-1}(c^{|\sigma|}v - u)(t) &= 0, \quad (c^{|\sigma|}v - u)(1) = 0.
 \end{aligned} \tag{45}$$

Which implies by Lemma 7 that $c^{|\sigma|}v - u = G_\alpha(a(c^{|\sigma|}v^\sigma - u^\sigma)) \geq 0$. By symmetry, we obtain also that $v \leq c^{|\sigma|}u$. Hence $c^{|\sigma|} \in J$ and $c \leq c^{|\sigma|}$. Since $|\sigma| < 1$, then $c = 1$ and consequently $u = v$. \square

Example 12. Let $\sigma \in (-1, 1)$ and a be a positive continuous function on $(0, 1)$ such that

$$a(t) \approx (1-t)^{-\mu} \log\left(\frac{3}{1-t}\right)^{-\beta}, \tag{46}$$

where $\mu < \alpha$ and $\beta \in \mathbb{R}$ or $\mu = \alpha$ and $\beta > 1$. Then, using Theorem 1, problem (1) has a unique positive continuous solution u satisfying the following estimates.

(i) If $\mu < \sigma + \alpha - 1$ or $\mu = \sigma + \alpha - 1$ and $\beta > 1$, then for $x \in (0, 1)$,

$$u(x) \approx x^{\alpha-2} (1-x). \tag{47}$$

(ii) If $\mu = \sigma + \alpha - 1$ and $\beta = 1$, then for $x \in (0, 1)$,

$$u(x) \approx x^{\alpha-2} (1-x) \left[\log\left(\log\left(\frac{3}{1-x}\right)\right) \right]^{1/(1-\sigma)}. \tag{48}$$

(iii) If $\mu = \sigma + \alpha - 1$ and $\beta < 1$, then for $x \in (0, 1)$,

$$u(x) \approx x^{\alpha-2} (1-x) \left[\log\left(\frac{3}{1-x}\right) \right]^{(1-\beta)/(1-\sigma)}. \tag{49}$$

(iv) If $\sigma + \alpha - 1 < \mu < \alpha$, then for $x \in (0, 1)$,

$$u(x) \approx x^{\alpha-2} (1-x)^{(\alpha-\mu)/(1-\sigma)} \left[\log\left(\frac{3}{1-x}\right) \right]^{-\beta/(1-\sigma)}. \tag{50}$$

(v) If $\mu = \alpha$ and $\beta > 1$, then for $x \in (0, 1)$,

$$u(x) \approx x^{\alpha-2} \left[\log\left(\frac{3}{1-x}\right) \right]^{(1-\beta)/(1-\sigma)}. \tag{51}$$

Acknowledgment

The authors thank the anonymous referees for a careful reading of the paper and for their helpful suggestions.

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