

## Research Article

# Meir-Keeler Type Multidimensional Fixed Point Theorems in Partially Ordered Metric Spaces

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We study the existence and uniqueness of a fixed point of the multidimensional operators which satisfy Meir-Keeler type contraction condition. Our results extend, improve, and generalize the results mentioned above and the recent results on these topics in the literature.

## 1. Introduction

Fixed point theory plays a crucial role in nonlinear functional analysis. In particular, fixed point results are used to prove the existence (and also uniqueness) when solving various type of equations. On the other hand, fixed point theory has a wide application potential in almost all positive sciences, such as Economics, Computer Science, Biology, Chemistry, and Engineering. One of the initial results in this direction (given by S. Banach), which is known as *Banach fixed point theorem* or *Banach contraction mapping principle* [1] is as follows. Every contraction in a complete metric space has a unique fixed point. In fact, this principle not only guarantees the existence and uniqueness of a fixed point, but it also shows how to get the desired fixed point. Since then, this celebrated principle has attracted the attention of a number of authors (e.g., see [1–39]). Due to its importance in nonlinear functional analysis, Banach contraction mapping principle has been generalized in many ways with regards to different abstract spaces. One of the most interesting results on generalization was reported by Guo and Lakshmikantham [18] in 1987. In their paper, the authors introduced the notion of *coupled fixed point* and proved some related theorems for certain type mappings. After this pioneering work, Gnana Bhaskar and Lakshmikantham [10] reconsidered coupled fixed point in the context of partially ordered sets by defining

the notion of mixed monotone mapping. In this outstanding paper, the authors proved the existence and uniqueness of coupled fixed points for mixed monotone mappings and they also discussed the existence and uniqueness of solution for a periodic boundary value problem. Following these initial papers, a significant number of papers on coupled fixed point theorems have been reported (e.g., see [6, 11, 13, 19, 22, 23, 29, 31–33, 36, 38, 40]).

Following this trend, Berinde and Borcut [8] extended the notion of coupled fixed point to tripled fixed point. Inspired by this interesting paper, Karapınar [24] improved this idea by defining quadruple fixed point (see also [25–28]). Very recently, Roldán et al. [35] generalized this idea by introducing the notion of  $\Phi$ -fixed point, that is to say, the multidimensional fixed point.

Another remarkable generalization of Banach contraction mapping principle was given by Meir and Keeler [34]. In the literature of this topic, Meir-Keeler type contraction has been studied densely by many selected mathematicians (e.g., see [2–4, 9, 20, 21, 36, 39]).

In this paper, we prove the existence and uniqueness of fixed point of multidimensional Meir-Keeler contraction in a complete partially ordered metric space. Our results improve, extend, and generalize the existence results on the topic in fixed point theory.

## 2. Preliminaries

Preliminaries and notation about coincidence points can also be found in [35]. Let  $n$  be a positive integer. Henceforth,  $X$  will denote a nonempty set, and  $X^n$  will denote the product space  $X \times X \times \dots \times X$ . Throughout this paper,  $m$  and  $k$  will denote nonnegative integers and  $i, j, s \in \{1, 2, \dots, n\}$ . Unless otherwise stated, “for all  $m$ ” will mean “for all  $m \geq 0$ ” and “for all  $i$ ” will mean “for all  $i \in \{1, 2, \dots, n\}$ .”

A metric on  $X$  is a mapping  $d : X \times X \rightarrow \mathbb{R}$  satisfying, for all  $x, y, z \in X$ ,

- (i)  $d(x, y) = 0$ , iff  $x = y$ ;
- (ii)  $d(x, y) \leq d(z, x) + d(z, y)$ .

From these properties, we can easily deduce that  $d(x, y) \geq 0$  and  $d(y, x) = d(x, y)$  for all  $x, y \in X$ . The last requirement is called the *triangle inequality*. If  $d$  is a metric on  $X$ , we say that  $(X, d)$  is a *metric space* (for short, an *MS*).

**Definition 1** (see [15]). A triple  $(X, d, \leq)$  is called a *partially ordered metric space* if  $(X, d)$  is an MS and  $\leq$  is a partial order on  $X$ .

**Definition 2** (see [10]). An ordered MS  $(X, d, \leq)$  is said to have the *sequential  $g$ -monotone property* if it verifies the following.

- (i) If  $\{x_m\}$  is a nondecreasing sequence and  $\{x_m\} \xrightarrow{d} x$ , then  $gx_m \leq gx$  for all  $m$ .
- (ii) If  $\{y_m\}$  is a nonincreasing sequence and  $\{y_m\} \xrightarrow{d} y$ , then  $gy_m \leq gy$  for all  $m$ .

If  $g$  is the identity mapping, then  $X$  is said to have the *sequential monotone property*.

Henceforth, fix a partition  $\{A, B\}$  of  $\Lambda_n = \{1, 2, \dots, n\}$ ; that is,  $A \cup B = \Lambda_n$  and  $A \cap B = \emptyset$ . We will denote that

$$\begin{aligned} \Omega_{A,B} &= \{\sigma : \Lambda_n \rightarrow \Lambda_n : \sigma(A) \subseteq A, \sigma(B) \subseteq B\}, \\ \Omega'_{A,B} &= \{\sigma : \Lambda_n \rightarrow \Lambda_n : \sigma(A) \subseteq B, \sigma(B) \subseteq A\}. \end{aligned} \tag{2}$$

If  $(X, \leq)$  is a partially ordered space,  $x, y \in X$ , and  $i \in \Lambda_n$ , we will use the following notation:

$$x \leq_i y \iff \begin{cases} x \leq y, & \text{if } i \in A, \\ x \geq y, & \text{if } i \in B. \end{cases} \tag{3}$$

Let  $F : X^n \rightarrow X$  and  $g : X \rightarrow X$  be two mappings.

**Definition 3** (see [35]). We say that  $F$  and  $g$  are *commuting* if  $gF(x_1, \dots, x_n) = F(gx_1, \dots, gx_n)$  for all  $x_1, \dots, x_n \in X$ .

**Definition 4** (see [35]). Let  $(X, \leq)$  be a partially ordered space. We say that  $F$  has the *mixed  $g$ -monotone property* (w.r.t.  $\{A, B\}$ ) if  $F$  is  $g$ -monotone nondecreasing in arguments of  $A$  and  $g$ -monotone nonincreasing in arguments of  $B$ ; that is, for all  $x_1, x_2, \dots, x_n, y, z \in X$  and all  $i$ ,

$$\begin{aligned} gy \leq gz &\implies F(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) \\ &\leq_i F(x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_n). \end{aligned} \tag{4}$$

Henceforth, let  $\sigma_1, \sigma_2, \dots, \sigma_n, \tau : \Lambda_n \rightarrow \Lambda_n$  be  $n + 1$  mappings from  $\Lambda_n$  into itself, and let  $\Phi$  be the  $(n + 1)$ -tuple  $(\sigma_1, \sigma_2, \dots, \sigma_n, \tau)$ .

**Definition 5** (see [35]). A point  $(x_1, x_2, \dots, x_n) \in X^n$  is called a  $\Phi$ -*coincidence point of the mappings  $F$  and  $g$*  if

$$F(x_{\sigma_1(1)}, x_{\sigma_1(2)}, \dots, x_{\sigma_1(n)}) = gx_{\tau(i)} \quad \forall i. \tag{5}$$

If  $g$  is the identity mapping on  $X$ , then  $(x_1, x_2, \dots, x_n) \in X^n$  is called a  $\Phi$ -*fixed point of the mapping  $F$* .

**Remark 6.** If  $F$  and  $g$  are commuting and  $(x_1, x_2, \dots, x_n) \in X^n$  is a  $\Phi$ -coincidence point of  $F$  and  $g$ , then  $(gx_1, gx_2, \dots, gx_n)$  also is a  $\Phi$ -coincidence point of  $F$  and  $g$ .

With regards to coincidence points, it is possible to consider the following simplification. If  $\tau$  is a permutation of  $\Lambda_n$  and we reorder (5), then we deduce that every coincidence point may be seen as a coincidence point associated to the identity mapping on  $\Lambda_n$ .

**Lemma 7.** Let  $\tau$  be a permutation of  $\Lambda_n$ , and let  $\Phi = (\sigma_1, \sigma_2, \dots, \sigma_n, \tau)$  and  $\Phi' = (\sigma_{\tau^{-1}(1)}, \sigma_{\tau^{-1}(2)}, \dots, \sigma_{\tau^{-1}(n)}, I_{\Lambda_n})$ . Then, a point  $(x_1, x_2, \dots, x_n) \in X^n$  is a  $\Phi$ -coincidence point of the mappings  $F$  and  $g$  if and only if  $(x_1, x_2, \dots, x_n)$  is a  $\Phi'$ -coincidence point of the mappings  $F$  and  $g$ .

Therefore, in the sequel, without loss of generality, we will only consider  $Y$ -coincidence points where  $Y = (\sigma_1, \sigma_2, \dots, \sigma_n)$ , that is, that verify  $F(x_{\sigma_1(1)}, x_{\sigma_1(2)}, \dots, x_{\sigma_1(n)}) = gx_i$  for all  $i$ .

If one represents a mapping  $\sigma : \Lambda_n \rightarrow \Lambda_n$  throughout its ordered image, that is,  $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(n))$ , then

- (i) Gnana-Bhaskar and Lakshmikantham’s election in  $n = 2$  is  $\sigma_1 = \tau = (1, 2)$  and  $\sigma_2 = (2, 1)$ ;
- (ii) Berinde and Borcut’s election in  $n = 3$  is  $\sigma_1 = \tau = (1, 2, 3)$ ,  $\sigma_2 = (2, 1, 2)$  and  $\sigma_3 = (3, 2, 1)$ ;
- (iii) Karapnar’s election in  $n = 4$  is  $\sigma_1 = \tau = (1, 2, 3, 4)$ ,  $\sigma_2 = (2, 3, 4, 1)$ ,  $\sigma_3 = (3, 4, 1, 2)$ , and  $\sigma_4 = (4, 1, 2, 3)$ .

For more details, see [35]. We will use the following result about real sequences in the proof of our main theorem.

**Lemma 8.** If  $\{x_m\}_{m \in \mathbb{N}}$  is a sequence in an MS  $(X, d)$  that is not Cauchy, then there exist  $\varepsilon_0 > 0$  and two subsequences  $\{x_{m(k)}\}_{k \in \mathbb{N}}$  and  $\{x_{n(k)}\}_{k \in \mathbb{N}}$  such that, for all  $k \in \mathbb{N}$ ,  $k < m(k) < n(k) < m(k + 1)$ ,  $d(x_{m(k)}, x_{n(k)}) \geq \varepsilon_0$ , and  $d(x_{m(k)}, x_{n(k)-1}) < \varepsilon_0$ .

Meir and Keeler generalized the Banach contraction mapping principle in the following way.

**Definition 9** (Meir and Keeler [34]). A *Meir-Keeler mapping* is a mapping  $T : X \rightarrow X$  on an MS  $(X, d)$  such that for all  $\varepsilon > 0$ , there exists  $\delta > 0$  verifying that if  $x, y \in X$  and  $\varepsilon \leq d(x, y) < \varepsilon + \delta$ , then  $d(Tx, Ty) < \varepsilon$ .

Lim characterized this kind of mappings in terms of a contractivity condition using the following class of functions.

*Definition 10* (Lim [30]). A function  $\phi : [0, \infty[ \rightarrow [0, \infty[$  will be called an *L-function* if (a)  $\phi(0) = 0$ , (b)  $\phi(t) > 0$  for all  $t > 0$ , and (c) for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $\phi(t) \leq \varepsilon$  for all  $t \in [\varepsilon, \varepsilon + \delta]$ .

**Theorem 11** (Lim [30]). *Let  $(X, d)$  be an MS, and let  $T : X \rightarrow X$ . Then  $T$  is a Meir-Keeler mapping if and only if there exists an (nondecreasing, right-continuous) L-map  $\phi$  such that*

$$\begin{aligned} d(T(x), T(y)) &< \phi(d(x, y)) \\ \forall x, y \in X \text{ verifying } d(x, y) &> 0. \end{aligned} \tag{6}$$

Using a result of Chu and Diaz [14], Meir and Keeler [34] proved that every Meir-Keeler mapping on a complete MS has a unique fixed point. Since then, many authors have developed this notion in different ways (e.g., see [2–4, 9, 20, 21, 36, 39]). For instance, in [36], Samet introduces the concept of *generalized Meir-Keeler type function* as follows.

*Definition 12* (see [36]). Let  $(X, d, \preceq)$  be a partially ordered metric space and  $F : X \times X \rightarrow X$  a given mapping. We say that  $F$  is a *generalized Meir-Keeler type function* if for all  $\varepsilon > 0$ , there exists  $\delta(\varepsilon) > 0$  such that

$$\begin{aligned} x \succeq u, \quad y \preceq v, \\ \varepsilon \leq \frac{1}{2} [d(x, u) + d(y, v)] < \varepsilon + \delta(\varepsilon) \\ \implies d(F(x, y), F(u, v)) < \varepsilon. \end{aligned} \tag{7}$$

Then, the author [36] proved some coupled fixed point theorems via generalized Meir-Keeler type mappings. In this paper, we extend the notion of generalized Meir-Keeler type mappings in various ways and get some fixed point results by the help of these notions.

### 3. Multidimensional Meir-Keeler-Type Mappings

Henceforth, let  $(X, d, \preceq)$  be a partially ordered MS and let  $F : X^n \rightarrow X$  and  $g : X \rightarrow X$  be two mappings.

*Definition 13.* We will say that  $F$  is a (multidimensional) *g-Meir-Keeler type mapping*, ((MK) mapping) if it verifies the following two properties.

- (MK1) If  $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in X$  verify  $gx_i = gy_i$  for all  $i$ , then  $F(x_1, x_2, \dots, x_n) = F(y_1, y_2, \dots, y_n)$ .
- (MK2) For all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that if  $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in X$  verify  $gx_i \preceq_i gy_i$  for all  $i$  and

$$\begin{aligned} \varepsilon \leq \max_{1 \leq i \leq n} d(gx_i, gy_i) < \varepsilon + \delta, \\ \text{then } d(F(x_1, x_2, \dots, x_n), F(y_1, y_2, \dots, y_n)) < \varepsilon. \end{aligned} \tag{8}$$

If  $g$  is the identity mapping on  $X$ , we will say that  $F$  is a (*n-dimensional*) *Meir-Keeler type mapping*.

On the one hand, notice that, in a wide sense, property (MK1) may be interpreted as property (MK2) for  $\varepsilon = 0$ . On the other hand, we observe that our definition may not be compared with the original one due to Meir and Keeler since we assume that  $X$  has a partial order. In any case, if  $n = 1$ ,  $(X, d)$  has a partial order and  $g$  is the identity mapping on  $X$ , and we can only establish that if  $F : X \rightarrow X$  is a Meir-Keeler mapping in the sense of Definition 9, then  $F$  is a Meir-Keeler-type mapping in the sense of Definition 13, but the converse does not hold.

*Remark 14.* If  $g$  is an injective mapping on  $X$ , then all mappings  $F$  verify (MK1).

**Lemma 15.** *Let  $F : X^n \rightarrow X$  be a mapping on a partially ordered MS  $(X, d, \preceq)$ , and let  $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in X$  be such that  $gx_i \preceq_i gy_i$  for all  $i$ .*

- (1) *If  $F$  verifies (MK2), then either  $gx_i = gy_i$  for all  $i$  or*

$$d(F(x_1, x_2, \dots, x_n), F(y_1, y_2, \dots, y_n)) < \max_{1 \leq i \leq n} d(gx_i, gy_i). \tag{9}$$

- (2) *If  $F$  is a g-Meir-Keeler type mapping, then*

$$d(F(x_1, x_2, \dots, x_n), F(y_1, y_2, \dots, y_n)) \leq \max_{1 \leq i \leq n} d(gx_i, gy_i), \tag{10}$$

*and the equality is achieved if and only if  $gx_i = gy_i$  for all  $i$ .*

*Proof.* (1) If the condition “ $gx_i = gy_i$  for all  $i$ ” does not hold, then  $\varepsilon = \max_{1 \leq i \leq n} d(gx_i, gy_i) > 0$ . Hence,  $d(F(x_1, x_2, \dots, x_n), F(y_1, y_2, \dots, y_n)) < \varepsilon = \max_{1 \leq i \leq n} d(gx_i, gy_i)$ . (2) If  $F$  is a *g-Meir-Keeler-type mapping*, the case “ $gx_i = gy_i$  for all  $i$ ” means that the equality is achieved.  $\square$

This global contractivity condition (10) is not strong enough to ensure that  $F$  has a fixed point. For instance, if  $n = 1$ , then  $F(x) = x + 1$  for all  $x \in \mathbb{R}$  has no fixed point. In order to characterize this kind of mappings in different ways, we recall some definitions and results.

*Definition 16.* The *g-modulus of uniform continuity* of  $F$  is, for all  $\varepsilon > 0$ ,

$$\begin{aligned} \delta_{g,F}(\varepsilon) = \sup \left( \left\{ \lambda \geq 0 : \left[ \max_{1 \leq i \leq n} d(gx_i, gy_i) < \lambda \right] \right. \right. \\ \left. \left. \implies d(F(x_1, x_2, \dots, x_n), F(y_1, y_2, \dots, y_n)) < \varepsilon \right\} \right). \end{aligned} \tag{11}$$

*Remark 17.* The identity mapping on a set  $X$  will be denoted by  $1_X : X \rightarrow X$ . If  $g : X \rightarrow X$  is a mapping, then

$G : X^n \rightarrow X^n$  will be defined by  $G(x_1, x_2, \dots, x_n) = (gx_1, gx_2, \dots, gx_n)$  for all  $x_1, x_2, \dots, x_n \in X$ . If  $(X, d)$  is a metric space, then  $D : X^n \times X^n \rightarrow [0, \infty[$ , given by  $D(P, Q) = \max_{1 \leq i \leq n} d(p_i, q_i)$  for all  $P = (p_1, p_2, \dots, p_n)$ ,  $Q = (q_1, q_2, \dots, q_n) \in X^n$ , is a metric on  $X^n$ . A partial order  $\leq$  on  $X$  may be induced on  $X^n$  by  $P \leq Q$  if and only if  $p_i \leq q_i$  for all  $i$  (notice that this partial order depends on the partition  $\{A, B\}$  of  $\Lambda_n$ ). Then,  $(X^n, D, \leq)$  also is a partially ordered MS. Furthermore, given any  $\omega = (\omega_2, \omega_3, \dots, \omega_n) \in X^{n-1}$ ,  $F_\omega : X^n \rightarrow X^n$  will denote the mapping defined by  $F_\omega(x_1, x_2, \dots, x_n) = (F(x_1, x_2, \dots, x_n), \omega_2, \omega_3, \dots, \omega_n)$  for all  $x_1, x_2, \dots, x_n \in X$ . It is obvious that

$$\begin{aligned} D(F_\omega(x_1, x_2, \dots, x_n), F_\omega(y_1, y_2, \dots, y_n)) \\ = d(F(x_1, x_2, \dots, x_n), F(y_1, y_2, \dots, y_n)) \end{aligned} \quad (12)$$

for all  $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in X$ .

**Theorem 18.** *Let  $(X, d, \leq)$  be a partially ordered MS, and let  $F : X^n \rightarrow X$  and  $g : X \rightarrow X$  be two mappings. Then, the following statements are equivalent.*

(MK)  $F$  is a  $g$ -Meir-Keeler-type mapping.

(MK3) For all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$\left. \begin{array}{l} x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in X \\ gx_i \leq_i gy_i \quad \forall i \\ \max_{1 \leq i \leq n} d(gx_i, gy_i) < \varepsilon + \delta \end{array} \right\} \quad (13)$$

$$\implies d(F(x_1, x_2, \dots, x_n), F(y_1, y_2, \dots, y_n)) < \varepsilon.$$

(MK4)  $\delta_{g,F}(\varepsilon) > \varepsilon$  for all  $\varepsilon > 0$ .

(MK5)  $F$  and  $g$  verify (MK1), and there exists an (nondecreasing, right-continuous)  $L$ -function  $\phi : [0, \infty[ \rightarrow [0, \infty[$  such that

$$\begin{aligned} d(F(x_1, x_2, \dots, x_n), F(y_1, y_2, \dots, y_n)) \\ < \phi\left(\max_{1 \leq i \leq n} d(gx_i, gy_i)\right) \end{aligned} \quad (14)$$

for all  $x_1, \dots, x_n, y_1, \dots, y_n \in X$  verifying  $gx_i \leq_i gy_i$  for all  $i$  and  $\max_{1 \leq i \leq n} d(gx_i, gy_i) > 0$ .

(MK6) For all  $\omega \in X^{n-1}$ , the mapping  $F_\omega : X^n \rightarrow X^n$  is a  $G$ -Meir-Keeler-type mapping on  $(X^n, D, \leq)$ .

(MK7) There exists  $\omega_0 \in X^{n-1}$  such that the mapping  $F_{\omega_0} : X^n \rightarrow X^n$  is a  $G$ -Meir-Keeler-type mapping on  $(X^n, D, \leq)$ .

*Proof.* [(MK) $\implies$ (MK3)]: Fix  $\varepsilon > 0$ , and let  $\delta > 0$  given by (MK2). Let  $x_1, \dots, x_n, y_1, \dots, y_n \in X$  be such that  $gx_i \leq_i gy_i$  for all  $i$ , and let  $\eta = \max_{1 \leq i \leq n} d(gx_i, gy_i) < \varepsilon + \delta$ . If  $\eta = 0$ , then  $gx_i = gy_i$  for all  $i$ , and so  $d(F(x_1, x_2, \dots, x_n), F(y_1, y_2, \dots, y_n)) = 0 < \varepsilon$  by (MK1). In another case,  $\eta > 0$ . If  $\varepsilon \leq \eta < \varepsilon + \delta$ , then  $d(F(x_1, x_2, \dots, x_n), F(y_1, y_2, \dots, y_n)) < \varepsilon$  by (MK2). Now, suppose that  $0 < \eta < \varepsilon$ . Then,  $\eta \in [\eta, \eta + \delta_\eta[$ , where  $\delta_\eta > 0$

is also given by (MK2), and  $d(F(x_1, \dots, x_n), F(y_1, \dots, y_n)) < \eta < \varepsilon$ . Hence, (MK3) holds.

[(MK3) $\implies$ (MK4)]: Given  $\varepsilon > 0$ , let  $\delta > 0$  verifying (MK3). Then,  $\delta_{g,F}(\varepsilon) \geq \varepsilon + \delta$ , and so  $\delta_{g,F}(\varepsilon) > \varepsilon$ .

[(MK4) $\implies$ (MK)]: On the one hand, if  $gx_i = gy_i$  for all  $i$ , then  $d(F(x_1, x_2, \dots, x_n), F(y_1, y_2, \dots, y_n)) < \varepsilon$  for all  $\varepsilon > 0$ , and so  $F$  verify (MK1). On the other hand, let  $\varepsilon > 0$ , and define  $\delta = (\delta_{g,F}(\varepsilon) - \varepsilon)/2 > 0$ . Therefore,  $\varepsilon + \delta < \delta_{g,F}(\varepsilon)$ . Since  $\delta_{g,F}(\varepsilon)$  is a supremum, there exists  $\lambda_0 \in [\varepsilon + \delta, \delta_{g,F}(\varepsilon)]$  such that if  $gx_i \leq_i gy_i$  for all  $i$  and  $\max_{1 \leq i \leq n} d(gx_i, gy_i) < \lambda_0$ , then  $d(F(x_1, x_2, \dots, x_n), F(y_1, y_2, \dots, y_n)) < \varepsilon$ . In particular, if  $gx_i \leq_i gy_i$  for all  $i$  and  $\varepsilon \leq \max_{1 \leq i \leq n} d(gx_i, gy_i) < \varepsilon + \delta < \lambda_0$ , then  $d(F(x_1, x_2, \dots, x_n), F(y_1, y_2, \dots, y_n)) < \varepsilon$ .

[(MK) $\Leftrightarrow$ (MK5)]: It is possible to follow step by step the proof of Proposition 1 in [39] with slight changes.

[(MK) $\Leftrightarrow$ (MK6) $\Leftrightarrow$ (MK7)]: It is apparent taking into account (12).  $\square$

The following result is a particular case taking  $\phi(t) = kt$  for all  $t \geq 0$ .

**Corollary 19.** *Let  $(X, d, \leq)$  be a partially ordered metric space, and let  $F : X^n \rightarrow X$  and  $g : X \rightarrow X$  be two mappings. Assume that there exists  $k \in (0, 1)$  such that*

$$d(F(x_1, x_2, \dots, x_n), F(y_1, y_2, \dots, y_n)) \leq k \max_{1 \leq i \leq n} d(gx_i, gy_i) \quad (15)$$

for all  $x_1, \dots, x_n, y_1, \dots, y_n \in X$  verifying  $gx_i \leq_i gy_i$  for all  $i$ . Then,  $F$  is a  $g$ -Meir-Keeler-type mapping.

Next, we prove that a generalized Meir Keeler type function in the sense of Samet [36, Definition 12] is a particular case of 2-dimensional Meir-Keeler-type mapping in the sense of Definition 13.

**Lemma 20.** *Every generalized Meir Keeler type function in the sense of Samet is a 2-dimensional Meir-Keeler-type mapping in the sense of Definition 13 taking  $g$  as the identity mapping on the MS.*

*Proof.* Suppose that  $F : X \times X \rightarrow X$  is a generalized Meir Keeler type function in the sense of Samet. Fix  $\varepsilon > 0$  and let  $\delta > 0$  verifying (7). Let  $x, y, u, v \in X$  such that  $x \geq u$ ,  $y \leq v$ , and  $\max(d(x, u), d(y, v)) < \varepsilon + \delta$ . We have to prove that  $d(F(x, y), F(u, v)) < \varepsilon$ . If  $x = u$  and  $y = v$ , there is nothing to prove. Next, suppose that  $\max(d(x, u), d(y, v)) > 0$ . Let

$$M = \frac{1}{2} [d(x, u) + d(y, v)]. \quad (16)$$

If  $M = 0$ , then  $x = u$  and  $y = v$ , which is false. Then,  $M > 0$ . On the other hand,

$$M = \frac{d(x, u) + d(y, v)}{2} \leq \max(d(x, u), d(y, v)) < \varepsilon + \delta. \quad (17)$$

If  $\varepsilon \leq M < \varepsilon + \delta$ , then  $d(F(x, y), F(u, v)) < \varepsilon$  by (7). Finally, if  $0 < M < \varepsilon$ , taking  $\varepsilon' = M$  in (7), we have that  $M \in [\varepsilon', \varepsilon' + \delta_{\varepsilon'}[$  (where  $\delta_{\varepsilon'}$  is taken as in (7)), and

so  $d(F(x, y), F(u, v)) < M < \varepsilon$ . This proves that  $F$  is a 2-dimensional Meir-Keeler type mapping associated to  $g = I_X$ .  $\square$

*Remark 21.* Converse of Lemma 20 does not hold. For instance, let  $X = \mathbb{R}$  be provided with its usual metric  $d(x, y) = |x - y|$  and partial order  $\leq$ . Take  $0 < k < 1$  and consider  $F(x, y) = kx$  for all  $x, y \in \mathbb{R}$ . Then,  $F$  is a 2-dimensional Meir-Keeler-type mapping in the sense of Definition 13 (taking  $g$  as the identity mapping on  $\mathbb{R}$ ), but, if  $k > 1/2$ , it is not a generalized Meir Keeler type function in the sense of Samet.

Indeed, we firstly prove that  $F$  is a 2-dimensional Meir-Keeler-type mapping in the sense of Definition 13 (taking  $g$  as the identity mapping on  $\mathbb{R}$ ). Let  $\varepsilon > 0$ . Consider any  $r \in ]0, 1/k - 1[$  (i.e.,  $k(1 + r) < 1$ ) and define  $\delta = r\varepsilon > 0$ . Consider  $x, y, u, v \in \mathbb{R}$  such that  $x \geq u$  and  $y \leq v$  verifying  $\varepsilon \leq \max(d(x, u), d(y, v)) = \max(|x - u|, |y - v|) < \varepsilon + \delta$ . In particular,  $|x - u| < \varepsilon + \delta$ . Then,

$$\begin{aligned} d(F(x, y), F(u, v)) &= d(kx, ku) = k|x - u| < k(\varepsilon + \delta) \\ &= k(\varepsilon + r\varepsilon) = k(1 + r)\varepsilon < \varepsilon. \end{aligned} \tag{18}$$

It follows that  $F$  is a 2-dimensional Meir-Keeler-type mapping in the sense of Definition 13. Next, we claim that if  $k > 1/2$ , then  $F$  is not a generalized Meir Keeler type function in the sense of Samet. Let  $\varepsilon > 0$ . If  $F$  was a generalized Meir Keeler type function in the sense of Samet, it would be  $\delta > 0$  verifying (7). Take  $x = \varepsilon, u = -\varepsilon$ , and  $y = v = 0$ . Then,  $x \geq u, y \leq v$  and

$$\begin{aligned} \frac{d(x, u) + d(y, v)}{2} &= \frac{|\varepsilon - (-\varepsilon)| + |0 - 0|}{2} \\ &= \frac{2\varepsilon}{2} = \varepsilon \in [\varepsilon, \varepsilon + \delta]. \end{aligned} \tag{19}$$

However,  $d(F(x, y), F(u, v)) = d(k\varepsilon, k(-\varepsilon)) = k|\varepsilon - (-\varepsilon)| = 2k\varepsilon > \varepsilon$  since  $k > 1/2$ .

### 4. Main Results

In the following result, we show sufficient conditions to ensure the existence of  $Y$ -coincidence points, where  $Y = (\sigma_1, \sigma_2, \dots, \sigma_n)$ .

**Theorem 22.** *Let  $(X, d)$  be a complete MS, and let  $\leq$  a partial order on  $X$ . Let  $Y = (\sigma_1, \sigma_2, \dots, \sigma_n)$  be an  $n$ -tuple of mappings from  $\{1, 2, \dots, n\}$  into itself verifying  $\sigma_i \in \Omega_{A,B}$  if  $i \in A$  and  $\sigma_i \in \Omega'_{A,B}$  if  $i \in B$ . Let  $F : X^n \rightarrow X$  and  $g : X \rightarrow X$  be two mappings such that  $F$  is a  $g$ -Meir-Keeler-type mapping and has the mixed  $g$ -monotone property on  $X, F(X^n) \subseteq g(X)$ , and  $g$  is continuous and commuting with  $F$ . Suppose that either  $F$  is continuous or  $(X, d, \leq)$  has the sequential  $g$ -monotone property. If there exist  $x_0^1, x_0^2, \dots, x_0^n \in X$  verifying  $gx_0^i \leq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \dots, x_0^{\sigma_i(n)})$  for all  $i$ , then  $F$  and  $g$  have, at least, one  $Y$ -coincidence point.*

*Proof.* The proof is divided in six steps. We follow the strategy of Theorem 9 in [35].

*Step 1.* There exist  $n$  sequences  $\{x_m^1\}_{m \geq 0}, \{x_m^2\}_{m \geq 0}, \dots, \{x_m^n\}_{m \geq 0}$  such that  $gx_{m+1}^i = F(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \dots, x_m^{\sigma_i(n)})$  for all  $m$  and all  $i$ .

*Step 2.*  $gx_m^i \leq_i gx_{m+1}^i$  for all  $m$  and all  $i$ .

Define  $\delta_m = \max_{1 \leq i \leq n} d(gx_m^i, gx_{m+1}^i) \geq 0$  for all  $m$ . Firstly, suppose that there exists  $m_0 \in \mathbb{N}$  such that  $\delta_{m_0} = 0$ . Then,  $gx_{m_0}^i = gx_{m_0+1}^i = F(x_{m_0}^{\sigma_i(1)}, x_{m_0}^{\sigma_i(2)}, \dots, x_{m_0}^{\sigma_i(n)})$  for all  $i$ , so  $(x_{m_0}^1, x_{m_0}^2, \dots, x_{m_0}^n)$  is a  $Y$ -coincidence point of  $F$  and  $g$  and we have finished. Therefore, we may reduce to the case in which  $\delta_m > 0$  for all  $m$ ; that is,

$$\forall m, \text{ there exists } j \text{ such that } gx_m^j \neq gx_{m+1}^j. \tag{20}$$

*Step 3.* We claim that  $\{d(gx_m^i, gx_{m+1}^i)\}_{m \geq 0} \rightarrow 0$  for all  $i$  (i.e.,  $\{\max_{1 \leq j \leq n} d(gx_m^j, gx_{m+1}^j)\}_{m \geq 0} \rightarrow 0$ ). Indeed, as  $gx_m^i \leq_i gx_{m+1}^i$  for all  $m$  and all  $i$ , then condition (MK2), Lemma 15, and (20) imply that, for all  $m \geq 1$  and all  $i$ ,

$$\begin{aligned} d(gx_m^i, gx_{m+1}^i) &= d\left(F\left(x_{m-1}^{\sigma_i(1)}, x_{m-1}^{\sigma_i(2)}, \dots, x_{m-1}^{\sigma_i(n)}\right), \right. \\ &\quad \left. F\left(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \dots, x_m^{\sigma_i(n)}\right)\right) \\ &< \max_{1 \leq j \leq n} d\left(gx_{m-1}^{\sigma_i(j)}, gx_m^{\sigma_i(j)}\right) \\ &\leq \max_{1 \leq j \leq n} d\left(gx_{m-1}^j, gx_m^j\right) = \delta_{m-1}. \end{aligned} \tag{21}$$

Taking maximum on  $i$ , we deduce that the sequence  $\{\delta_m\}_{m \geq 1}$  is nonincreasing and lower bounded. Therefore, it is convergent; that is, there exists  $\Delta \geq 0$  such that  $\{\delta_m\}_{m \geq 1} \rightarrow \Delta$  (and  $\Delta \leq \delta_m$  for all  $m$ ). We claim that  $\Delta = 0$ . On the contrary, assume that  $\Delta > 0$ . Let  $\delta > 0$  be a positive number associated to  $\varepsilon = \Delta > 0$  by (MK2). Since

$$\left\{ \max_{1 \leq i \leq n} d(gx_m^i, gx_{m+1}^i) \right\}_m = \{\delta_m\}_m \searrow \Delta, \tag{22}$$

there exists  $m_0 \in \mathbb{N}$  such that if  $m \geq m_0$ , then  $\Delta \leq \max_{1 \leq i \leq n} d(gx_m^i, gx_{m+1}^i) < \Delta + \delta$ . By (MK2), it follows that, for all  $i$ ,

$$\begin{aligned} d(gx_{m_0}^i, gx_{m_0+1}^i) &= d\left(F\left(x_{m_0}^{\sigma_i(1)}, x_{m_0}^{\sigma_i(2)}, \dots, x_{m_0}^{\sigma_i(n)}\right), \right. \\ &\quad \left. F\left(x_{m_0+1}^{\sigma_i(1)}, x_{m_0+1}^{\sigma_i(2)}, \dots, x_{m_0+1}^{\sigma_i(n)}\right)\right) < \Delta. \end{aligned} \tag{23}$$

Taking maximum on  $i$ , we deduce that

$$\Delta \leq \delta_{m_0} = \max_{1 \leq i \leq n} d(gx_{m_0}^i, gx_{m_0+1}^i) < \Delta. \tag{24}$$

But this is impossible. Then, we have just proved that  $\Delta = 0$ . Therefore,  $\{\delta_m\}_{m \geq 1} \rightarrow \Delta = 0$ , which means that

$$\lim_{m \rightarrow \infty} \delta_m = \lim_{m \rightarrow \infty} \left( \max_{1 \leq j \leq n} d(gx_m^j, gx_{m+1}^j) \right) = 0. \tag{25}$$

As  $0 \leq d(gx_m^i, gx_{m+1}^i) \leq \delta_m$  for all  $m$  and all  $i$ , then  $\{d(gx_m^i, gx_{m+1}^i)\} \rightarrow 0$  for all  $i$ .

Step 4. Every sequence  $\{gx_m^i\}_{m \geq 0}$  is Cauchy. Suppose that  $\{gx_m^{i_1}\}_{m \geq 0}, \dots, \{gx_m^{i_s}\}_{m \geq 0}$  are not Cauchy and  $\{gx_m^{i_{s+1}}\}_{m \geq 0}, \dots, \{gx_m^{i_n}\}_{m \geq 0}$  are Cauchy, being  $\{i_1, \dots, i_n\} = \{1, \dots, n\}$ . By Lemma 8, for all  $r \in \{1, 2, \dots, s\}$ , there exist  $\varepsilon_r > 0$  and subsequences  $\{gx_{m'_r(k)}^{i_r}\}_{k \in \mathbb{N}}$  and  $\{gx_{n'_r(k)}^{i_r}\}_{k \in \mathbb{N}}$  such that

$$k < m'_r(k) < n'_r(k), \quad d(gx_{m'_r(k)}^{i_r}, gx_{n'_r(k)}^{i_r}) \geq \varepsilon_r, \quad (26)$$

$$d(gx_{m'_r(k)}^{i_r}, gx_{n'_r(k)-1}^{i_r}) < \varepsilon_r, \quad \forall k \in \mathbb{N}.$$

Let  $\varepsilon_0 = \max(\varepsilon_1, \dots, \varepsilon_s)$  and  $\varepsilon'_0 = \min(\varepsilon_1, \dots, \varepsilon_s) > 0$ . Since  $\{gx_m^{i_{s+1}}\}_{m \geq 0}, \dots, \{gx_m^{i_n}\}_{m \geq 0}$  are Cauchy, there exists  $n_0 \in \mathbb{N}$  such that if  $n, m \geq n_0$ , then  $d(gx_m^j, gx_n^j) < \varepsilon'_0/2$  for all  $j \in \{i_{s+1}, \dots, i_n\}$ .

Let  $k_0 \in \mathbb{N}$  such that  $n_0 < \min(m'_1(k_0), m'_2(k_0), \dots, m'_s(k_0))$ , and define  $m(1) = \min(m'_1(k_0), m'_2(k_0), \dots, m'_s(k_0))$ . As  $m(1) = m'_r(k_0)$  for some  $r \in \{1, 2, \dots, s\}$ , there exists  $n'_r(k_0)$  such that  $d(gx_{m'_r(k_0)}^{i_r}, gx_{n'_r(k_0)}^{i_r}) \geq \varepsilon_r \geq \varepsilon_0$ . Thus, we can consider the numbers  $m(1) + 1, m(1) + 2, \dots$  until finding the first positive integer  $n(1) > m(1)$  verifying

$$\max_{1 \leq r \leq s} d(gx_{m(1)}^{i_r}, gx_{n(1)}^{i_r}) \geq \varepsilon_0, \quad (27)$$

$$d(gx_{m(1)}^{i_j}, gx_{n(1)-1}^{i_j}) < \varepsilon_0, \quad \forall j \in \{1, 2, \dots, s\}.$$

Now, let  $k_1 \in \mathbb{N}$  such that  $n(1) < \min(m'_1(k_1), m'_2(k_1), \dots, m'_s(k_1))$  and define  $m(2) = \min(m'_1(k_1), m'_2(k_1), \dots, m'_s(k_1))$ . Since  $m(2) \in \{m'_1(k_1), m'_2(k_1), \dots, m'_s(k_1)\}$ , we can consider the numbers  $m(2) + 1, m(2) + 2, \dots$  until finding the first positive integer  $n(2) > m(2)$  verifying

$$\max_{1 \leq r \leq s} d(gx_{m(2)}^{i_r}, gx_{n(2)}^{i_r}) \geq \varepsilon_0, \quad (28)$$

$$d(gx_{m(2)}^{i_j}, gx_{n(2)-1}^{i_j}) < \varepsilon_0, \quad \forall j \in \{1, 2, \dots, s\}.$$

Repeating this process, we can find sequences such that, for all  $k \geq 1$ ,

$$n_0 < m(k) < n(k) < m(k+1),$$

$$\max_{1 \leq r \leq s} d(gx_{m(k)}^{i_r}, gx_{n(k)}^{i_r}) \geq \varepsilon_0, \quad (29)$$

$$d(gx_{m(k)}^{i_j}, gx_{n(k)-1}^{i_j}) < \varepsilon_0, \quad \forall j \in \{1, 2, \dots, s\}.$$

Since  $n_0 < m(k) < n(k)$ , we know that  $d(gx_{m(k)}^j, gx_{n(k)}^j), d(gx_{m(k)}^j, gx_{n(k)-1}^j), d(gx_{m(k)-1}^j, gx_{n(k)-1}^j) < \varepsilon'_0/2$  for all  $j \in \{i_{s+1}, \dots, i_n\}$ . Therefore, for all  $k$ ,

$$\max_{1 \leq j \leq n} d(gx_{m(k)}^j, gx_{n(k)}^j) = \max_{1 \leq r \leq s} d(gx_{m(k)}^{i_r}, gx_{n(k)}^{i_r}) \geq \varepsilon_0, \quad (30)$$

$$\max_{1 \leq j \leq n} d(gx_{m(k)}^j, gx_{n(k)-1}^j) < \varepsilon'_0.$$

Let  $\delta > 0$  verifying (MK3) using  $\varepsilon_0$ , and let  $k_1 \in \mathbb{N}$  such that if  $k \geq k_1$ , then  $d(gx_{m(k)-1}^j, gx_{m(k)}^j) < \delta$  for all  $j$ . Then, for all  $j$  and all  $k \geq k_1$ ,

$$\begin{aligned} d(gx_{m(k)-1}^j, gx_{n(k)-1}^j) &\leq d(gx_{m(k)-1}^j, gx_{m(k)}^j) \\ &\quad + d(gx_{m(k)}^j, gx_{n(k)-1}^j) < \delta + \varepsilon_0, \end{aligned} \quad (31)$$

Applying (MK3), it follows, for all  $k \geq k_0$  and all  $i$ , that

$$\begin{aligned} d(gx_{m(k)}^i, gx_{n(k)}^i) &= d(F(x_{m(k)-1}^{\sigma_i(1)}, \dots, x_{m(k)-1}^{\sigma_i(n)}), F(x_{n(k)-1}^{\sigma_i(1)}, \dots, x_{n(k)-1}^{\sigma_i(n)})) \\ &< \varepsilon_0, \end{aligned} \quad (32)$$

but this contradicts (30) since  $\max_{1 \leq j \leq n} d(gx_{m(k)}^j, gx_{n(k)}^j) \geq \varepsilon_0$ . This contradiction shows us that every sequence  $\{gx_m^i\}_{m \geq 0}$  is Cauchy.

Existence of a fixed point is derived by standard techniques. Indeed, since  $(X, d)$  is complete, there exist  $x_1, x_2, \dots, x_n \in X$  such that  $x_i = \lim_{m \rightarrow \infty} gx_m^i$  for all  $i$ . As  $g$  is continuous and  $F$  commutes with  $g$ ,

$$\begin{aligned} \lim_{m \rightarrow \infty} F(gx_m^{\sigma_i(1)}, gx_m^{\sigma_i(2)}, \dots, gx_m^{\sigma_i(n)}) &= \lim_{m \rightarrow \infty} gF(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \dots, x_m^{\sigma_i(n)}) \\ &= \lim_{m \rightarrow \infty} ggx_{m+1}^i = gx_i \quad \forall i. \end{aligned} \quad (33)$$

Step 5. Suppose that  $F$  is continuous. In this case, since  $\{gx_m^{\sigma_i(j)}\}_m \rightarrow x_{\sigma_i(j)}$  for all  $i, j$  and  $F$  is continuous,

$$\begin{aligned} \lim_{m \rightarrow \infty} ggx_{m+1}^i &= \lim_{m \rightarrow \infty} F(gx_m^{\sigma_i(1)}, gx_m^{\sigma_i(2)}, \dots, gx_m^{\sigma_i(n)}) \\ &= F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \dots, x_{\sigma_i(n)}) \end{aligned} \quad (34)$$

for all  $i$ . Then,  $F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \dots, x_{\sigma_i(n)}) = gx_i$  for all  $i$ ; that is,  $(x_1, x_2, \dots, x_n)$  is a  $Y$ -coincidence point of  $F$  and  $g$ .

Step 6. Suppose that  $(X, d, \preceq)$  has the sequential  $g$ -monotone property. In this case, by step 2, we know that  $gx_m^i \preceq_i gx_{m+1}^i$  for all  $m$  and all  $i$ . This means that the sequence  $\{gx_m^i\}_{m \geq 0}$  is monotone. As  $x_i = \lim_{m \rightarrow \infty} gx_m^i$ , we deduce that  $ggx_m^i \preceq_i gx_i$  for all  $m$  and all  $i$ . This condition implies that, for all  $m$  and all  $j$ ,

$$\begin{aligned} &\text{either } [ggx_m^{\sigma_j(i)} \preceq_i gx_{\sigma_j(i)} \quad \forall i] \\ &\text{or } [gx_{\sigma_j(i)} \preceq_i ggx_m^{\sigma_j(i)} \quad \forall i] \end{aligned} \quad (35)$$

(the first case occurs when  $j \in A$  and the second one when  $j \in B$ ). Fix  $j \in \{1, 2, \dots, n\}$ , and we claim that  $\lim_{m \rightarrow \infty} F(gx_m^{\sigma_j(1)}, gx_m^{\sigma_j(2)}, \dots, gx_m^{\sigma_j(n)}) = F(x_{\sigma_j(1)}, x_{\sigma_j(2)}, \dots, x_{\sigma_j(n)})$ . Indeed, let  $\varepsilon > 0$  arbitrary, and let  $\delta > 0$  verifying

(MK3). Since  $\{ggx_m^{\sigma_j(i)}\}_m \rightarrow gx_{\sigma_j(i)}$  for all  $i$ , there exists  $m_2 \in \mathbb{N}$  such that if  $m \geq m_2$ , then  $d(ggx_m^{\sigma_j(i)}, gx_{\sigma_j(i)}) < \varepsilon + \delta$  for all  $i$ . Applying (MK3) and (35),

$$d\left(F\left(gx_m^{\sigma_j(1)}, gx_m^{\sigma_j(2)}, \dots, gx_m^{\sigma_j(n)}\right), F\left(x_{\sigma_j(1)}, x_{\sigma_j(2)}, \dots, x_{\sigma_j(n)}\right)\right) < \varepsilon. \tag{36}$$

Therefore,  $F(x_{\sigma_j(1)}, x_{\sigma_j(2)}, \dots, x_{\sigma_j(n)}) = \lim_{m \rightarrow \infty} F(gx_m^{\sigma_j(1)}, gx_m^{\sigma_j(2)}, \dots, gx_m^{\sigma_j(n)}) = \lim_{m \rightarrow \infty} ggx_{m+1}^j = gx_j$  for all  $j$ . In conclusion,  $(x_1, x_2, \dots, x_n)$  is a  $Y$ -coincidence point of  $F$  and  $g$ .  $\square$

### 5. Uniqueness of $Y$ -Coincidence Points

For the uniqueness of a fixed point, we need the following notion. Consider on the product space  $X^n$  the following partial order: for  $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in X^n$ ,

$$(x_1, x_2, \dots, x_n) \leq (y_1, y_2, \dots, y_n) \iff x_i \preceq_i y_i, \quad \forall i. \tag{37}$$

We say that two points  $(x_1, x_2, \dots, x_n)$  and  $(y_1, y_2, \dots, y_n)$  are *comparable* if  $(x_1, x_2, \dots, x_n) \leq (y_1, y_2, \dots, y_n)$  or  $(x_1, x_2, \dots, x_n) \geq (y_1, y_2, \dots, y_n)$ .

By following the lines of Theorem 11 in [35], we prove the uniqueness of the coincidence point.

**Theorem 23.** *Under the hypothesis of Theorem 22, assume that for all  $Y$ -coincidence points  $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in X^n$  of  $F$  and  $g$ , there exists  $(u_1, u_2, \dots, u_n) \in X^n$  such that  $(gu_1, gu_2, \dots, gu_n)$  is comparable, at the same time, to  $(gx_1, gx_2, \dots, gx_n)$  and to  $(gy_1, gy_2, \dots, gy_n)$ .*

*Then,  $F$  and  $g$  have a unique  $Y$ -coincidence point  $(z_1, z_2, \dots, z_n) \in X^n$  such that  $gz_i = z_i$  for all  $i$ .*

It is natural to say that  $g$  is *injective* on the set of all  $Y$ -coincidence points of  $F$  and  $g$  when  $gx_i = gy_i$  for all  $i$  implies  $x_i = y_i$  for all  $i$  when  $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in X^n$  are two  $Y$ -coincidence points of  $F$  and  $g$ . For example, this is true that  $g$  is injective on  $X$ .

**Corollary 24.** *In addition to the hypotheses of Theorem 23, suppose that  $g$  is injective on the set of all  $Y$ -coincidence points of  $F$  and  $g$ . Then,  $F$  and  $g$  have a unique  $Y$ -coincidence point.*

*Proof.* If  $(x_1, x_2, \dots, x_n)$  and  $(y_1, y_2, \dots, y_n)$  are two  $Y$ -coincidence points of  $F$  and  $g$ , we have proved in (A.1) that  $gx_i = gy_i$  for all  $i$ . As  $g$  is injective on these points, then  $x_i = y_i$  for all  $i$ .  $\square$

**Corollary 25.** *In addition to the hypotheses of Theorem 23, suppose that  $(z_{\sigma_j(1)}, z_{\sigma_j(2)}, \dots, z_{\sigma_j(n)})$  is comparable to  $(z_{\sigma_j(1)}, z_{\sigma_j(2)}, \dots, z_{\sigma_j(n)})$  for all  $i, j$ . Then,  $z_1 = z_2 = \dots = z_n$ .*

*In particular, there exists a unique  $z \in X$  such that  $F(z, z, \dots, z) = z$ , which verifies  $gz = z$ .*

*Proof.* Let  $M = \max_{1 \leq i, j \leq n} d(z_i, z_j)$  and we are going to show that  $M = 0$  by contradiction. Assume that  $M > 0$  and let  $j_0, s_0 \in \{1, 2, \dots, n\}$  such that  $d(z_{j_0}, z_{s_0}) = M$ . As  $(z_{\sigma_{j_0}(1)}, z_{\sigma_{j_0}(2)}, \dots, z_{\sigma_{j_0}(n)})$  is comparable to  $(z_{\sigma_{s_0}(1)}, z_{\sigma_{s_0}(2)}, \dots, z_{\sigma_{s_0}(n)})$ , then either  $z_{\sigma_{j_0}(i)} \preceq_i z_{\sigma_{s_0}(i)}$  for all  $i$  or  $z_{\sigma_{s_0}(i)} \preceq_i z_{\sigma_{j_0}(i)}$  for all  $i$ . Since  $gz_i = z_i$  for all  $i$ , we know that either  $gz_{\sigma_{j_0}(i)} \preceq_i gz_{\sigma_{s_0}(i)}$  for all  $i$  or  $gz_{\sigma_{s_0}(i)} \preceq_i gz_{\sigma_{j_0}(i)}$  for all  $i$ . Now, we have to distinguish between two cases.

If  $gz_{\sigma_{j_0}(i)} = gz_{\sigma_{s_0}(i)}$  for all  $i$  (i.e.,  $z_{\sigma_{j_0}(i)} = z_{\sigma_{s_0}(i)}$  for all  $i$ ), then

$$z_{j_0} = gz_{j_0} = F\left(z_{\sigma_{j_0}(1)}, z_{\sigma_{j_0}(2)}, \dots, z_{\sigma_{j_0}(n)}\right) = F\left(z_{\sigma_{s_0}(1)}, z_{\sigma_{s_0}(2)}, \dots, z_{\sigma_{s_0}(n)}\right) = gz_{s_0} = z_{s_0}, \tag{38}$$

which is impossible since  $d(z_{j_0}, z_{s_0}) = M > 0$ . Now, suppose that  $\max_{1 \leq i \leq n} d(gz_{\sigma_{j_0}(i)}, gz_{\sigma_{s_0}(i)}) > 0$ . In this case, item 1 of Lemma 15 guarantees that

$$M = d(z_{j_0}, z_{s_0}) = d(gz_{j_0}, gz_{s_0}) = d\left(F\left(z_{\sigma_{j_0}(1)}, z_{\sigma_{j_0}(2)}, \dots, z_{\sigma_{j_0}(n)}\right), F\left(z_{\sigma_{s_0}(1)}, z_{\sigma_{s_0}(2)}, \dots, z_{\sigma_{s_0}(n)}\right)\right) < \max_{1 \leq i \leq n} d(gz_{\sigma_{j_0}(i)}, gz_{\sigma_{s_0}(i)}) \leq \max_{1 \leq i, j \leq n} d(gz_i, gz_j) = \max_{1 \leq i, j \leq n} d(z_i, z_j) = M,$$

which also is impossible. This contradiction proves that  $M = 0$ ; that is,  $z_i = z_j$  for all  $i, j$ .  $\square$

**Remark 26.** Notice that a *mixed strict monotone mapping*  $F : X \times X \rightarrow X$  in the sense of [36, Definition 2.1] is always a mixed monotone mapping in our sense (where  $n = 2$  and  $g$  is the identity mapping on  $X$ ). Then, Theorems 2.1, 2.2, 2.3, and 2.4 in [36] (and, by extension, theorems by Gnana Bhaskar and Lakshmikantham [10]) are consequence of our main results.

**Example 27.** Let  $X = \mathbb{R}$  and  $d(x, y) = |x - y|$  be usual metric on  $\mathbb{R}$ . Consider the mapping  $F(x_1, x_2, x_3, x_4, x_5, x_6) = (2x_1 - 3x_2 + x_3 - x_4 + x_5 - x_6)/12$  and  $g(x) = x$ . It is clear that  $F$  is monotone nonincreasing in odd arguments and  $F$  is monotone nondecreasing in even arguments. All conditions of Theorems 22 and 23 are satisfied. It is clear that  $(0, 0, 0, 0, 0, 0)$  is the unique fixed point.

**Example 28.** Let  $X = \mathbb{R}$  be provided with its usual partial order  $\leq$  and its usual metric  $d(x, y) = |x - y|$ . Let  $n \in \mathbb{N}$ , and let  $a_1, a_2, \dots, a_n \in \mathbb{R} \setminus \{0\}$  real numbers such that there exist  $i_0, j_0 \in \{1, 2, \dots, n\}$  verifying  $a_{i_0} < 0 < a_{j_0}$ . Let  $N > |a_1| + |a_2| + \dots + |a_n|$ , and consider  $F(x_1, x_2, \dots, x_n) = (a_1x_1 + a_2x_2 + \dots + a_nx_n)/N$  and  $gx = x$ , for all  $x, x_1, x_2, \dots, x_n \in X$ . Then,  $F$  is monotone nondecreasing in those arguments for which  $a_i > 0$  and monotone nonincreasing in those arguments for

which  $a_i < 0$ . Furthermore, taking  $k = (|a_1| + |a_2| + \dots + |a_n|)/N \in (0, 1)$ , Corollary 19 shows that  $F$  is a  $g$ -Meir-Keeler-type mapping. Actually, all conditions of Theorems 22 and 23 are satisfied. Indeed, it is clear that  $(0, 0, \dots, 0)$  is the unique fixed point of  $F$ .

## Appendix

### Proof of Theorem 23

*Proof.* From Theorem 22, the set of  $Y$ -coincidence points of  $F$  and  $g$  is nonempty. The proof is divided in two steps.

*Step 1.* We claim that if  $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in X^n$  are two  $Y$ -coincidence points of  $F$  and  $g$ , then

$$gx_i = gy_i \quad \forall i. \quad (\text{A.1})$$

Let  $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in X^n$  be two  $Y$ -coincidence points of  $F$  and  $g$ , and let  $(u_1, u_2, \dots, u_n) \in X^n$  be a point such that  $(gu_1, gu_2, \dots, gu_n)$  is comparable, at the same time, to  $(gx_1, gx_2, \dots, gx_n)$  and to  $(gy_1, gy_2, \dots, gy_n)$ . Using  $(u_1, u_2, \dots, u_n)$  define the following sequences. Let  $u_0^i = u_i$  for all  $i$ . Reasoning as in Theorem 22, we can determine sequences  $\{u_m^1\}_{m \geq 0}, \{u_m^2\}_{m \geq 0}, \dots, \{u_m^n\}_{m \geq 0}$  such that  $gu_{m+1}^i = F(u_m^{\sigma_i(1)}, u_m^{\sigma_i(2)}, \dots, u_m^{\sigma_i(n)})$  for all  $m$  and all  $i$ . We are going to prove that  $gx_i = \lim_{m \rightarrow \infty} gu_m^i = gy_i$  for all  $i$ , and so (A.1) will be true.

Firstly, we reason with  $(gu_1, gu_2, \dots, gu_n)$  and  $(gx_1, gx_2, \dots, gx_n)$ , and the same argument will be true for  $(gu_1, gu_2, \dots, gu_n)$  and  $(gy_1, gy_2, \dots, gy_n)$ . As  $(gu_1, gu_2, \dots, gu_n)$  and  $(gx_1, gx_2, \dots, gx_n)$  are comparable, we can suppose that  $(gu_1, gu_2, \dots, gu_n) \leq (gx_1, gx_2, \dots, gx_n)$  (the other case is similar); that is,  $gu_0^i = gu_i \leq_i gx_i$  for all  $i$ . Using that  $F$  has the mixed  $g$ -monotone property and reasoning as in Theorem 22, it is possible to prove that  $gu_m^i \leq_i gx_i$  for all  $m \geq 1$  and all  $i$ . This condition implies that, for all  $j$  and all  $m \geq 1$ ,

$$\begin{aligned} &\text{either } \left[ gu_m^{\sigma_j(i)} \leq_i gx_{\sigma_j(i)} \quad \forall i \right] \\ &\text{or } \left[ gx_{\sigma_j(i)} \leq_i gu_m^{\sigma_j(i)} \quad \forall i \right]. \end{aligned} \quad (\text{A.2})$$

Define  $\beta_m = \max_{1 \leq i \leq n} d(gu_m^i, gx_i)$  for all  $m$ . Reasoning as in Theorem 22, one can observe that  $\{\beta_m\}_{m \geq 1} \rightarrow 0$ , which means that  $\lim_{m \rightarrow \infty} \beta_m = \lim_{m \rightarrow \infty} (\max_{1 \leq i \leq n} d(gu_m^i, gx_i)) = 0$ . As  $0 \leq d(gu_m^i, gx_i) \leq \beta_m$  for all  $m$  and all  $i$ , we deduce that  $\{d(gu_m^i, gx_i)\}_{m \geq 1} \rightarrow 0$  for all  $i$ ; that is,

$$\lim_{m \rightarrow \infty} gu_m^i = gx_i \quad \forall i. \quad (\text{A.3})$$

If we had supposed that  $(gx_1, gx_2, \dots, gx_n) \leq (gu_1, gu_2, \dots, gu_n)$ , we would have obtained the same property (A.3). And as  $(gu_1, gu_2, \dots, gu_n)$  also is comparable to  $(gy_1, gy_2, \dots, gy_n)$ , we can reason in the same way to prove that  $gy_i = \lim_{m \rightarrow \infty} gu_m^i = gx_i$  for all  $i$ .

Let  $(x_1, x_2, \dots, x_n) \in X^n$  be a  $Y$ -coincidence point of  $F$  and  $g$ , and define  $z_i = gx_i$  for all  $i$ . As  $(z_1, z_2, \dots, z_n) =$

$(gx_1, gx_2, \dots, gx_n)$ , Remark 6 assures us that  $(z_1, z_2, \dots, z_n)$  also is a  $Y$ -coincidence point of  $F$  and  $g$ .

*Step 2.* We claim that  $(z_1, z_2, \dots, z_n)$  is the unique  $\Phi$ -coincidence point of  $F$  and  $g$  such that  $gz_i = z_i$  for all  $i$ . It is similar to Step 2 in Theorem 11 in [35].  $\square$

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