Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 398164, 11 pages
http://dx.doi.org/10.1155/2013/398164

Research Article

Existence of Prescribed L^2-Norm Solutions for a Class of Schrödinger-Poisson Equation

Yisheng Huang,[1] Zeng Liu,[1] and Yuanze Wu[2]

[1] Department of Mathematics, Soochow University, Suzhou, Jiangsu 215006, China
[2] Cumt College of Sciences, China University of Mining and Technology, Xuzhou, Jiangsu 221116, China

Correspondence should be addressed to Zeng Liu; luckliuz@163.com

Received 26 May 2013; Accepted 15 July 2013

Academic Editor: Kanishka Perera

Copyright © 2013 Yisheng Huang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

By using the standard scaling arguments, we show that the infimum of the following minimization problem: $I_{\rho^2} = \inf\{(1/2)\int_{\mathbb{R}^3}|\nabla u|^2dx + (1/4)\iint_{\mathbb{R}^3}(|u(x)|^2|u(y)|^2/|x-y|)dx\,dy - (1/p)\int_{\mathbb{R}^3}|u|^p dx : u \in B_\rho\}$ can be achieved for $p \in (2,3)$ and $\rho > 0$ small, where $B_\rho := \{u \in H^1(\mathbb{R}^3) : \|u\|_2 = \rho\}$. Moreover, the properties of I_{ρ^2}/ρ^2 and the associated Lagrange multiplier λ_ρ are also given if $p \in (2, 8/3]$.

1. Introduction

In this paper, we consider the nonlinear Schrödinger-Poisson type equation:

$$-\Delta u + \left(|x|^{-1} * |u|^2\right)u - |u|^{p-2}u = \lambda u, \quad \text{in } \mathbb{R}^3, \tag{1}$$

where $\lambda \in \mathbb{R}$ is a parameter, $p \in (2,6)$, and $*$ denotes the convolution. Problems like (1) have attracted considerable attentions recently since a pair (u,λ), solution of (1), corresponds to a solitary wave of the form $\psi(x,t) = e^{-i\lambda t}u(x)$ of the evolution equation:

$$i\psi_t + \Delta\psi - \left(|x|^{-1} * |\psi|^2\right)\psi + |\psi|^{p-2}\psi = 0, \quad \text{in } \mathbb{R}^3 \times \mathbb{R}^+, \tag{2}$$

which was obtained by approximation of a special case of Hatree-Fock equation with the frequency λ describing a quantum mechanical system of many particles. For more mathematical and physical background of (2), we refer to [1–4] and the references therein.

In the case that the frequency λ is a fixed and assigned parameter, the critical points of the following functional defined in $H^1(\mathbb{R}^3;\mathbb{R})$:

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx - \frac{\lambda}{2}\int_{\mathbb{R}^3}|u|^2dx + \frac{1}{4}\iint_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}dx\,dy - \frac{1}{p}\int_{\mathbb{R}^3}|u|^pdx, \tag{3}$$

are the solutions of (1), where $E(u)$ is obviously well defined and is a C^1 functional for each $p \in (2,6)$ (cf. [5]). Such case has been extensively studied by using variational methods in the past decades including the existence, nonexistence, and multiplicity of solutions; see, for example, [5–12] and the references therein.

On the other hand, the physicists are often interested in the solutions with prescribed L^2-norm and unknown frequency λ, such a solution is called a "normalized solution," which is associated with the existence of stable standing waves. Precisely, by a "normalized solution" (u_ρ, λ_ρ) of (1), we mean that

$$\left(u_\rho, \lambda_\rho\right) \in H^1\left(\mathbb{R}^3;\mathbb{C}\right) \times \mathbb{R} \text{ solves (1) with } \left\|u_\rho\right\|_2 = \rho. \tag{4}$$

Clearly, this kind of solutions can be obtained as the constrained critical points of the C^1 functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{4}\iint_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dx\, dy - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx, \tag{5}$$

on the constraint

$$B_\rho = \left\{u \in H^1\left(\mathbb{R}^3; \mathbb{C}\right) : \|u\|_2 = \rho\right\}. \tag{6}$$

Thus, the frequency $\lambda_\rho \in \mathbb{R}$ cannot be fixed any longer and it will appear as a Lagrange multiplier associated with the critical point u_ρ on B_ρ. Among all the critical points of I constrained on B_ρ, we are interested in the ones with minimal energy since the corresponded standing waves are orbitally stable under the flow of (2) and can provide us some information on the dynamics of (2). Therefore, we are reduced to study the minimization problem

$$I_{\rho^2} = \min_{u \in B_\rho} I(u), \tag{7}$$

for $p \in (2, 10/3)$. Here we note that, for each $\rho > 0$, $I_{\rho^2} \in (-\infty, 0]$ if $p \in (2, 10/3)$, and $I_{\rho^2} = -\infty$ if $p \in (10/3, 6)$ (cf. [13, Remark 1.1] or (15) below). When $p \in (10/3, 6)$ (now $I_{\rho^2} = -\infty$), by using a mountain pass argument, it was proved in [14] that I has a critical point constrained on B_ρ at a strictly positive energy level for $\rho > 0$ small, and this critical point is orbitally unstable.

The main difficulty of considering (7) is the lack of compactness for the (bounded) minimizing sequence $\{u_n\} \subset B_\rho$. We recall that the necessary and sufficient condition due to Lions [15, 16] in order that any minimizing sequence for (7) is relatively compact is the strong subadditivity inequality:

$$I_{\rho^2} < I_{\mu^2} + I_{\rho^2 - \mu^2}, \quad \forall 0 < \mu < \rho. \tag{8}$$

In the range $p \in \{8/3\} \cup (3, 10/3)$, by using the standard scaling arguments, Bellazzini and Siciliano in [17] proved that (8) holds for $\rho > 0$ large. In the range $p \in (2, 3)$, Bellazzini and Siciliano also showed in [18] that (8) holds for $\rho > 0$ small, where they developed a new abstract theorem which guarantees the following condition (MD) for $s > 0$ small:

(MD) The function $s \mapsto I_{s^2}/s^2$ is monotone decreasing.

We remark that their abstract theorem heavily relies on the behavior of I_{ρ^2} near zero; that is, to use the abstract theorem, one has to verify some extra conditions, such as

$$\rho \longmapsto I_{\rho^2} \text{ is continuous}; \quad \lim_{\rho \to 0} \frac{I_{\rho^2}}{\rho^2} = 0; \tag{9}$$

these are unnecessary if one can show (8) by using the standard scaling arguments like [17]. However, as mentioned in [18], the authors were not sure whether (8) can be proved or not by using the standard scaling arguments if $p \in (2, 3)$. Therefore, the first aim of this paper is to show that (8) holds for $\rho > 0$ small when $p \in (2, 3)$ by using the standard scaling arguments. To achieve this aim, we introduce a new subset $B_\rho \cap \mathcal{P}$ of B_ρ (see details in Section 3), then we consider the minimization problem (7) constrained on $B_\rho \cap \mathcal{P}$ instead of B_ρ, and we use the standard scaling arguments to prove that (8) holds for $\rho > 0$ small. Moreover, we can get a specific estimate on ρ that allows us to obtain the sign and the behavior of the Lagrange multiplier λ_ρ if $p \in (2, 8/3]$; these are not considered in [18].

The other aim of this paper is to study the properties of the Lagrange multiplier λ_ρ and the ratio I_{ρ^2}/ρ^2 corresponding to the solution (u_ρ, λ_ρ) of (1) with $\|u_\rho\|_2 = \rho$. It is known that λ_ρ and I_{ρ^2}/ρ^2 are interpreted in physics as the frequency and the ratio between the infimum of the energy of the standing waves with fixed charge and the charge itself, respectively, and the relevance of the energy/charge ratio for the existence of standing waves in field theories has been discussed under a general framework in [19].

Our main results read as follows.

Theorem 1. *All the minimizing sequences for* (7) *are precompact in* $H^1(\mathbb{R}^3; \mathbb{C})$ *up to translations provided that one of the following conditions holds*

(1) $p \in (2, 8/3]$ *and* $0 < \rho < \overline{\rho}_1 := 3^{(3p-8)/8(3-p)} \pi^{(3p-10)/8(3-p)} S^{3/2}((p-2)/2p)^{1/4(3-p)}$, *where* S *is defined by* (12)*;*

(2) $p \in (8/3, 3)$ *and* $0 < \rho < \overline{\rho}_2$ *for some* $\overline{\rho}_2 > 0$.

In particular, (1) *has a solution* $(u_\rho, \lambda_\rho) \in H^1(\mathbb{R}^3; \mathbb{C}) \times \mathbb{R}$ *such that* $I(u_\rho) = I_{\rho^2}$ *and* $\|u_\rho\|_2 = \rho$*. Moreover, if the above assumption (1) holds and* (u_ρ, λ_ρ) *is a solution of* (1) *with* $\|u_\rho\|_2 = \rho > 0$ *and* $I(u_\rho) = I_{\rho^2}$*, then* $\lambda_\rho < 0$, $\lambda_\rho \to 0$ *and* $I_{\rho^2}/\rho^2 \to 0$ *as* $\rho \to 0$*, respectively.*

Theorem 2. *Let* $p \in (2, 12/5]$ *and let* $\rho > 0$*. If* (u_ρ, λ_ρ) *is a solution of* (1) *with* $\| u_\rho\|_2 = \rho$*, then we have*

(i) $\lambda_\rho < 0$, $I(u_\rho) < 0$, $\lambda_\rho \to 0$ *as* $\rho \to 0$ *and there exists a positive constant* C_1*, independent of* ρ*, such that* $\lambda_\rho \in (-C_1, 0)$*;*

(ii) *there exists a positive constant* C_2*, independent of* ρ*, such that* $I(u_\rho)/\rho^2 \in (-C_2, 0)$*. In particular, if* $I(u_\rho) = I_{\rho^2}$*, then* $I_{\rho^2}/\rho^2 \in (-C_2, 0)$*.*

Remarks. (a) We point out that parts of Theorem 1 are already contained in [18, Theorem 4.1]. In the proof of Theorem 1, with $\overline{\rho}_1$ in hand, we can obtain some additional information of the Lagrange multiplier λ_ρ and the ratio I_{ρ^2}/ρ^2 when $p \in (2, 8/3]$, and these are not contained in [18, Theorem 4.1]. However, we do not know whether $\overline{\rho}_1$ is optimal or not.

(b) Theorem 2(i) shows that (1) has only the zero solution if $p \in (2, 12/5]$ and $\lambda \geq 0$. In the case of $p \in (2, 3)$, it was shown in [5, 20] (see also [13, Remark 1.4]) that there exists $\lambda_0 < 0$ such that (1) has only the zero solution for $\lambda \in (-\infty, \lambda_0)$. The nonexistence results of nonzero solutions of (1) were also discussed in [13] for $p \in [3, 10/3]$.

(c) As we have anticipated, the existence of minimizers for I_{ρ^2} is related to the existence and stability of the standing wave solutions to (2). For the existence of stable standing wave solutions to (2), we refer to [4, 14, 17, 18, 20, 21] and the references therein.

This paper is organized as follows. In Section 2, we give some preliminaries. Section 3 is devoted to the proof of the main theorems, especially in the proof of Theorem 1, we first define a new subset of B_ρ and then analyze the properties of minimizing sequences for I_{ρ^2} constrained on the new subset, and finally, we prove that (8) holds when $p \in (2, 8/3]$ and $p \in (8/3, 3)$, respectively.

2. Preliminaries

Throughout this paper, all the functions, unless otherwise stated, are complex valued, but for simplicity we will write $L^q(\mathbb{R}^3)$, $H^1(\mathbb{R}^3)$ and $\mathcal{D}^{1,2}(\mathbb{R}^3)$ defined in the following:

(i) $L^q(\mathbb{R}^3)$ is the usual Lebesgue space endowed with the norm $\|u\|_q := (\int_{\mathbb{R}^3} |u|^q dx)^{1/q}$, where $q \in [1, \infty)$;

(ii) $H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with the norm

$$\|u\| := \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + |u|^2\right) dx\right)^{1/2}; \tag{10}$$

(iii) $\mathcal{D}^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|\nabla u\|_2 = \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{1/2}; \tag{11}$$

(iv) S is the best Sobolev imbedding constant of $\mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ defined as

$$S := \inf\left\{\|\nabla u\|_2 : u \in \mathcal{D}^{1,2}\left(\mathbb{R}^3\right), \|u\|_6 = 1\right\}. \tag{12}$$

Moreover, the letter C will denote a suitable positive constant, whose value may change in the same line, and the symbol $o(1)$ denotes a quantity which goes to zero. We also use $O(1)$ to denote a bounded quantity.

Let $\phi_u(x) = |x-y|^{-1} * |u|^2$, and then, for each $u \in H^1(\mathbb{R}^3)$, ϕ_u is the unique solution of the Poisson equation $-\Delta\phi = 4\pi|u|^2$ and is usually interpreted as the Coulombian potential of the electrostatic field generated by the charge density $|u|^2$. Evidently, see, for example [5],

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla\phi_u|^2 dx &= 4\pi \int_{\mathbb{R}^3} \phi_u |u|^2 dx \\ &= 4\pi \iint_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dx\, dy,\end{aligned} \tag{13}$$

$$\int_{\mathbb{R}^3} \phi_u |u|^2 dx \le \|\phi_u\|_6 \|u\|_{12/5}^2 \le \frac{4\pi}{S^2} \|u\|_{12/5}^4. \tag{14}$$

For each $\rho > 0$, let $u \in B_\rho$ and $u^t(x) = t^{3/2}u(tx)$ $(t > 0)$, and we have $\|u^t\|_2 = \|u\|_2 = \rho$, that is, $u^t \in B_\rho$. Let

$$\begin{aligned} f_u(t) = I\left(u^t\right) = \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{t}{4}\int_{\mathbb{R}^3} \phi_u |u|^2 dx \\ - \frac{t^{(3/2)(p-2)}}{p}\int_{\mathbb{R}^3} |u|^p dx;\end{aligned} \tag{15}$$

it is clear that $I_{\rho^2} \le 0$ for all $\rho > 0$ since $f_u(t) \to 0$ as $t \to 0$.

We now recall an abstract result on the constrained minimization problem

$$J_{\rho^2} = \inf_{u \in B_\rho} J(u), \quad \left(\text{we agree } J_0 = 0\right), \tag{16}$$

where $\rho > 0$, $B_\rho = \{u \in H^1(\mathbb{R}^3) : \|u\|_2 = \rho\}$, $J_{\rho^2} > -\infty$ is assumed, and

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 + T(u), \tag{17}$$

for some functional $T \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$.

Lemma 3 (see [17, 18, Lemma 2.1]). *Let $T \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$. Let $\rho > 0$ and $\{u_n\} \subset B_\rho$ be a minimizing sequence for J_{ρ^2} weakly convergent, up to translations, to a nonzero function $\bar{u}$. Assume that* (8) *holds and that*

$$T\left(u_n - \bar{u}\right) + T\left(\bar{u}\right) = T\left(u_n\right) + o(1); \tag{18}$$

$$\begin{aligned} T\left(\alpha_n\left(u_n - \bar{u}\right)\right) - T\left(u_n - \bar{u}\right) = o(1) \\ \text{where } \alpha_n = \frac{\rho^2 - \|\bar{u}\|_2^2}{\|u_n - \bar{u}\|_2};\end{aligned} \tag{19}$$

$$\langle T'\left(u_n\right), u_n\rangle = O(1); \tag{20}$$

$$\left\langle T'\left(u_n\right) - T'\left(u_m\right), u_n - u_m\right\rangle = o(1) \quad \text{as } n, m \longrightarrow \infty. \tag{21}$$

Then $\|u_n - \bar{u}\| \to 0$. In particular it follows that $\bar{u} \in B_\rho$ and $J(\bar{u}) = J_{\rho^2}$.

As pointed out in [18], Lemma 3 is a variant of the concentration-compactness principle of Lions [15, 16]. Assumption (18) shows that T possesses the Brizis-Lieb splitting property and (19) is the homogeneity of T. If, in addition, the condition (8) holds, then one can show that dichotomy does not occur; that is, $\bar{u} \in B_\rho$. Furthermore, if (20) and (21) are also fulfilled, then $\{u_n\}$ strongly converges to $\bar{u}$ in $H^1(\mathbb{R}^3)$. Finally we recall the following results obtained in [17, 18].

Lemma 4 (see [18]). *If $p \in (2, 3)$, then $I_{\rho^2} < 0$ for all $\rho > 0$.*

Lemma 5 (see [17, Lemma 3.1]). *If $p \in (2, 10/3)$, then, for every $\rho > 0$, the functional I is bounded below and coercive on B_ρ.*

Remark 6. For $p \in (2,3)$, it follows from Lemmas 4 and 5 that each minimizing sequence for I_{ρ^2} is bounded from below and above by two positive constants in $\mathscr{D}^{1,2}(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)$, up to a subsequence, respectively.

3. Proof of the Main Theorems

Before proving our main theorems, we need some preliminary lemmas. First, we set

$$\mathscr{P} := \left\{u \in H^1\left(\mathbb{R}^3\right) : Q(u) = 0, I(u) = \min_{t>0} I\left(u^t\right)\right\}, \quad (22)$$

where $u^t(x) = t^{3/2}u(tx)$ with $t > 0$ and $Q(u)$ is a functional on $H^1(\mathbb{R}^3)$ defined as

$$Q(u) = \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u |u|^2 dx - \frac{3(p-2)}{2p}\int_{\mathbb{R}^3} |u|^p dx. \quad (23)$$

It was shown in [13, Lemma 2.1] that if u_ρ is a constrained critical point of I on B_ρ associated with the Lagrange multiplier λ_ρ,then $Q(u_\rho) = 0$, which is nothing but a linear combination of $\langle E'(u_\rho), u_\rho\rangle = 0$ (recall that $E(u)$ is given by (3)) and the following Pohozaev identity for (1) (cf. [5, 9])

$$\frac{1}{2}\left\|\nabla u_\rho\right\|_2^2 - \frac{3\lambda_\rho}{2}\left\|u_\rho\right\|_2^2 + \frac{5}{4}\int_{\mathbb{R}^3} \phi_{u_\rho}\left|u_\rho\right|^2 dx - \frac{3}{p}\left\|u_\rho\right\|_p^p = 0. \quad (24)$$

The following lemma shows that $B_\rho \cap \mathscr{P}$ is well defined.

Lemma 7. *Let $p \in (2,3)$ and let $\rho > 0$. For each $u \in B_\rho$ with $I(u) < 0$, there exists a unique $t_u > 0$ such that $I(u^{t_u}) = \min\{I(u^t) : t > 0\}$; moreover, $u^{t_u} \in B_\rho \cap \mathscr{P}$.*

Proof. We divide the proof into two cases.

Case 1 ($p \in (2, 8/3)$). Let $u \in B_\rho$, for simplicity, and we will write $f_u'(t)$, $f_u''(t)$ and $f_u'''(t)$, the derivatives of $f_u(t)$ on t, instead of $df_u(t)/dt$, $d^2 f_u(t)/dt^2$ and $d^3 f_u(t)/dt^3$. From (15), we have

$$f_u'(t) = t\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u |u|^2 dx - \frac{3(p-2)}{2p} t^{(3p-8)/2}\int_{\mathbb{R}^3} |u|^p dx. \quad (25)$$

Noting that $(3p-8)/2 \in (-1, 0)$ since $p \in (2, 8/3)$, then, by (25), $\lim_{t\to 0} f_u'(t) = -\infty$ and $\lim_{t\to\infty} f_u'(t) = +\infty$; thus there exists $t_u > 0$ such that $f_u'(t_u) = 0$. If there exists another $s_u > 0$ such that $f_u'(s_u) = 0$, without loss of generality, we may assume that $s_u > t_u$, and then we get

$$\begin{aligned} 0 &= f_u'(s_u) - f_u'(t_u) \\ &= (s_u - t_u)\int_{\mathbb{R}^3} |\nabla u|^2 dx \\ &\quad + \frac{3(p-2)}{2p}\left(t_u^{(3p-8)/2} - s_u^{(3p-8)/2}\right)\int_{\mathbb{R}^3} |u|^p dx > 0, \end{aligned} \quad (26)$$

a contradiction. Therefore, t_u is unique and it is clear that $I(u^{t_u}) = \min\{I(u^t) : t > 0\}$. Moreover, $u^{t_u} \in B_\rho \cap \mathscr{P}$ because of $f_u'(t_u)t_u = 0$.

Case 2 ($p \in [8/3, 3)$). By Lemma 4, we know that the set $A_\rho := \{u \in B_\rho : I(u) < 0\} \neq \emptyset$. Let $u \in A_\rho$, if $f_u'(t) > 0$ for all $t > 0$; that is, $f_u(t)$ is strictly increasing, then we obtain that $f_u(t) < f_u(1) = I(u) < 0$ for all $t \in (0, 1)$. However, it is easy to see that $\lim_{t\to 0} f_u(t) = 0$; this is a contradiction. On the other hand, we know that $f_u(t) \to \infty$ as $t \to \infty$; hence there is a $t_u > 0$ such that $f_u'(t_u) = 0$, $u^{t_u} \in B_\rho \cap \mathscr{P}$ and

$$f_u(t_u) = \min\{f_u(t) : t > 0\} \le f_u(1) = I(u) < 0. \quad (27)$$

Next, we will show that t_u is unique. Arguing by contradiction, suppose that there is another $s_u > 0$ such that $f_u(t_u) = f_u(s_u) = \min\{f_u(t) : t > 0\}$, without loss of generality, we may assume that $s_u > t_u$, and then we have

$$f_u'(t_u) = f_u'(s_u) = 0, \quad f_u''(t_u) \ge 0, \quad f_u''(s_u) \ge 0. \quad (28)$$

According to (28), there exists $\omega_u \in (t_u, s_u)$ such that $f_u''(\omega_u) = 0$. After a simple calculation, we get

$$f_u''(t) = \|\nabla u\|_2^2 - \frac{3(p-2)(3p-8)}{4p} t^{(3p-10)/2}\int_{\mathbb{R}^3} |u|^p dx, \quad (29)$$

$$f_u'''(t) = -\frac{3(p-2)(3p-8)(3p-10)}{8p} t^{(3p-12)/2} \times \int_{\mathbb{R}^3} |u|^p dx. \quad (30)$$

If $p = 8/3$,then, by (29), $f_u''(t) > 0$ for all $t > 0$, which contradicts $f_u''(\omega_u) = 0$. If $p \in (8/3, 3)$, then, by (30), $f_u'''(t) > 0$ for all $t > 0$. Noting that $\omega_u \in (t_u, s_u)$, we have

$$0 \le f_u''(t_u) < f_u''(\omega_u) = 0, \quad (31)$$

again a contradiction. Therefore, $t_u > 0$ is unique. □

Lemma 8. *Let $p \in (2,3)$ and $\rho > 0$. For each $\{u_n\} \subset B_\rho$ such that $I(u_n) \to I_{\rho^2} < 0$ as $n \to \infty$ and $I(u_n) < 0$ for all $n \in \mathbb{N}$, there exists a bounded sequence $\{t_n\} \subset \mathbb{R}^+$ such that $\{u_n^{t_n}\} \subset B_\rho \cap \mathscr{P}$ and $I(u_n^{t_n}) \to I_{\rho^2}$ as $n \to \infty$ with $I(u_n^{t_n}) < 0$ for all $n \in \mathbb{N}$; that is, $\{u_n^{t_n}\}$ is also a minimizing sequence for I_{ρ^2} constrained on B_ρ.*

Proof. It follows from Lemma 7 that, for each u_n, there exists $t_n > 0$ such that $u_n^{t_n} \in B_\rho \cap \mathcal{P}$ and $I(u_n^{t_n}) = \min\{I(u_n^t) : t > 0\} \leq I(u_n) < 0$; therefore, we have

$$I_{\rho^2} \leq I\left(u_n^{t_n}\right) \leq I\left(u_n\right) \longrightarrow I_{\rho^2}, \tag{32}$$

as $n \to \infty$, that is, $\{u_n^{t_n}\}$ is a minimizing sequence. Next, we will show that $\{t_n\}$ is bounded. Indeed, from Remark 6, $\{u_n\}$ and $\{u_n^{t_n}\}$ are bounded from below and above by two positive constants in $\mathscr{D}^{1,2}(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)$, respectively. Noting that $\int_{\mathbb{R}^3} |\nabla u_n^{t_n}|^2 dx = t_n^2 \int_{\mathbb{R}^3} |\nabla u_n|^2 dx$; therefore, $\{t_n\}$ is bounded from below and above by two positive constants. □

Remark 9. Thanks to the Lemma 8, we know that $I_{\rho^2} = \inf\{I(u) : u \in B_\rho\} = \inf\{I(u) : u \in B_\rho \cap \mathcal{P}\}$, and, in the following, we will consider the minimization problem (7) restricted to $B_\rho \cap \mathcal{P}$ instead of B_ρ. By Lemmas 4 and 8, for each $\rho > 0$, if $\{u_n\} \subset B_\rho \cap \mathcal{P}$ satisfying $I(u_n) \to I_{\rho^2}$ as $n \to \infty$, then, up to a subsequence, we may assume that $I(u_n) < 0$. It follows from Lemma 5 that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$; by the results of [17, 18], we may assume that $u_n \rightharpoonup u \neq 0$ as $n \to \infty$ in $H^1(\mathbb{R}^3)$.

The following estimates of the elements of $B_\rho \cap \mathcal{P}$ are crucial to proving the strong subadditivity inequality (8).

Lemma 10. *Let $p \in (2,3)$ and $\rho > 0$. For each $u \in B_\rho \cap \mathcal{P}$, it holds*

$$\|\nabla u\|_2^2 \leq \left(\frac{3(p-2)}{2p}\right)^{4/(10-3p)} S^{-6(p-2)/(10-3p)} \rho^{2(6-p)/(10-3p)},$$
$$\|u\|_p^p \leq \left(\frac{3(p-2)}{2p}\right)^{3(p-2)/(10-3p)} S^{-6(p-2)/(10-3p)} \rho^{2(6-p)/(10-3p)}. \tag{33}$$

Proof. Since $u \in B_\rho \cap \mathcal{P}$,

$$\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u |u|^2 dx - \frac{3(p-2)}{2p}\|u\|_p^p = 0. \tag{34}$$

Noting that $\int_{\mathbb{R}^3} \phi_u |u|^2 dx \geq 0$ (see (13)), by using the Hölder inequality, we get

$$\begin{aligned}\|\nabla u\|_2^2 &\leq \frac{3(p-2)}{2p}\|u\|_p^p \leq \frac{3(p-2)}{2p}\|u\|_2^{(6-p)/2}\|u\|_6^{3(p-2)/2} \\ &\leq \frac{3(p-2)}{2p} S^{-3(p-2)/2}\|u\|_2^{(6-p)/2}\|\nabla u\|_2^{3(p-2)/2},\end{aligned} \tag{35}$$

which implies that

$$\|\nabla u\|_2^2 \leq \left(\frac{3(p-2)}{2p}\right)^{4/(10-3p)} S^{-6(p-2)/(10-3p)} \rho^{2(6-p)/(10-3p)}. \tag{36}$$

On the other hand, we have

$$\begin{aligned}\|u\|_p^p &\leq \|u\|_2^{(6-p)/2}\|u\|_6^{3(p-2)/2} \\ &\leq S^{-3(p-2)/2}\|u\|_2^{(6-p)/2}\|\nabla u\|_2^{3(p-2)/2} \\ &\leq \left(\frac{3(p-2)}{2p}\right)^{3(p-2)/(10-3p)} \\ &\quad \times S^{-6(p-2)/(10-3p)} \rho^{2(6-p)/(10-3p)};\end{aligned} \tag{37}$$

this concludes the proof of this lemma. □

Remark 11. Let $p \in (3, 10/3)$. It was shown in [13, Theorem 1.1] that $I_{\rho^2} < 0$ if and only if $\rho \in (\overline{\rho}, \infty)$, where the positive number $\overline{\rho}$ is defined as

$$\overline{\rho} = \inf\left\{\rho > 0 : I_{\rho^2} < 0\right\}. \tag{38}$$

Therefore, after a simple calculation, we can show that both of Lemmas 7 and 10 hold if $p \in (3, 10/3)$ and $\rho \in (\overline{\rho}, \infty)$.

Motivated by [17], we will use the standard scaling arguments to prove that the strong subadditivity inequality (8) holds for $p \in (2,3)$. First, we consider the case of $p \in (2, 8/3]$.

Lemma 12. *For $p \in (2, 8/3]$, let*

$$\overline{\rho}_1 = 3^{(3p-8)/8(3-p)} \pi^{(3p-10)/8(3-p)} S^{3/2}\left(\frac{p-2}{2p}\right)^{1/4(3-p)} > 0. \tag{39}$$

Then

$$I_{\rho^2} < I_{\mu^2} + I_{\rho^2-\mu^2} \quad \forall 0 < \mu < \rho < \overline{\rho}_1. \tag{40}$$

Proof. By Lemma 8 and Remark 9, for each $\{u_n\} \subset B_\rho \cap \mathcal{P}$ satisfying $I(u_n) \to I_{\rho^2} < 0$ as $n \to \infty$, we may assume that, for all n, $I(u_n) \leq I_{\rho^2}/2$, which implies that

$$\|u_n\|_p^p \geq -\frac{p}{2} I_{\rho^2}. \tag{41}$$

Noting that $tu_n \in B_{t\rho}$ $(t > 0)$, we have

$$\begin{aligned}I(tu_n) &= \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{t^4}{4}\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^2 dx \\ &\quad - \frac{t^p}{p}\int_{\mathbb{R}^3} |u_n|^p dx \\ &= t^2\left(I(u_n) + \frac{t^2-1}{4}\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^2 dx\right. \\ &\quad \left. - \frac{t^{p-2}-1}{p}\int_{\mathbb{R}^3} |u_n|^p dx\right) \\ &= t^2\left(I(u_n) + g(t,u_n)\right),\end{aligned} \tag{42}$$

where

$$g(t,u) = \frac{t^2-1}{4}\int_{\mathbb{R}^3} \phi_u |u|^2 dx - \frac{t^{p-2}-1}{p}\int_{\mathbb{R}^3} |u|^p dx. \tag{43}$$

We calculate the derivative of $g(t,u)$ on t:

$$\frac{dg(t,u)}{dt} = \frac{t}{2}\int_{\mathbb{R}^3}\phi_u|u|^2dx - \frac{p-2}{p}t^{p-3}\|u\|_p^p. \quad (44)$$

Letting $dg(t,u)/dt = 0$, we see from (14) that

$$t^{p-4} = \frac{p\int_{\mathbb{R}^3}\phi_u|u|^2dx}{2(p-2)\|u\|_p^p} \le \frac{2\pi p\|u\|_{12/5}^4}{S^2(p-2)\|u\|_p^p}. \quad (45)$$

Furthermore,

$$\frac{d^2g(t,u)}{dt^2} = \frac{1}{2}\int_{\mathbb{R}^3}\phi_u|u|^2dx - \frac{(p-2)(p-3)}{p}t^{p-4}\|u\|_p^p > 0. \quad (46)$$

Now we divide the value of p into two cases to discuss $dg(t,u_n)/dt$.

Case 1 ($p \in (2,12/5)$). It follows from Lemma 10, (14), and the Hölder inequality that

$$\begin{aligned}\left.\frac{dg(t,u_n)}{dt}\right|_{t=1} &= \frac{1}{2}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^2dx - \frac{p-2}{p}\int_{\mathbb{R}^3}|u_n|^pdx \\ &\le \frac{2\pi}{S^2}\|u_n\|_{12/5}^4 - \frac{p-2}{p}\|u_n\|_p^p \\ &\le \frac{2\pi}{S^2}\|u_n\|_p^{6p/(6-p)}\|u_n\|_6^{2(12-5p)/(6-p)} - \frac{p-2}{p}\|u_n\|_p^p \\ &= \|u_n\|_p^p\left(2\pi S^{12(p-3)/(6-p)}\|\nabla u_n\|_2^{2(12-5p)/(6-p)} \times \|u_n\|_p^{p^2/(6-p)} - \frac{p-2}{p}\right) \\ &\le \|u_n\|_p^p\left(2\pi S^{12(p-3)/(10-3p)}\left(\frac{3(p-2)}{2p}\right)^{(8-3p)/(10-3p)} \times \rho^{8(3-p)/(10-3p)} - \frac{p-2}{p}\right).\end{aligned} \quad (47)$$

Case 2 ($p \in [12/5,8/3]$). Again by Lemma 10, (14), and the Hölder inequality, we have

$$\begin{aligned}\left.\frac{dg(t,u_n)}{dt}\right|_{t=1} &= \frac{1}{2}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^2dx - \frac{p-2}{p}\int_{\mathbb{R}^3}|u_n|^pdx \\ &\le \frac{2\pi}{S^2}\|u_n\|_{12/5}^4 - \frac{p-2}{p}\|u_n\|_p^p \\ &\le \frac{2\pi}{S^2}\|u_n\|_2^{2(5p-12)/3(p-2)}\|u_n\|_p^{2p/3(p-2)} - \frac{p-2}{p}\|u_n\|_p^p \\ &\le \|u_n\|_p^p \times \left(\frac{2\pi}{S^2}\|u_n\|_2^{2(5p-12)/3(p-2)}\|u_n\|_p^{p(8-3p)/3(p-2)} - \frac{p-2}{p}\right) \\ &\le \|u_n\|_p^p\left(2\pi S^{12(p-3)/(10-3p)}\left(\frac{3(p-2)}{2p}\right)^{(8-3p)/(10-3p)} \times \rho^{8(3-p)/(10-3p)} - \frac{p-2}{p}\right).\end{aligned} \quad (48)$$

Let

$$\overline{\rho}_1 = 3^{(3p-8)/8(3-p)}\pi^{(3p-10)/8(3-p)}S^{3/2}\left(\frac{p-2}{2p}\right)^{1/4(3-p)}. \quad (49)$$

Then by (47), (48), and (49), we know that, for each $0 < \rho < \overline{\rho}_1$, there holds

$$\begin{aligned}\widetilde{\rho}_1(\rho) &:= \frac{2\pi p}{p-2}S^{12(p-3)/(10-3p)} \times \left(\frac{3(p-2)}{2p}\right)^{(8-3p)/(10-3p)}\rho^{8(3-p)/(10-3p)} < 1, \\ &\left.\frac{dg(t,u_n)}{dt}\right|_{t=1} < 0,\end{aligned} \quad (50)$$

for all $n \in \mathbb{N}$. On the other hand, for each $\varepsilon \in (0, 1-\widetilde{\rho}_1(\rho))$, it follows from (41), (44), (45), (46), and Lemma 10 that, for all $t \in (1, (\widetilde{\rho}_1(\rho)/(1-\varepsilon))^{1/(p-4)})$ and all $n \in \mathbb{N}$,

$$\begin{aligned}\frac{dg(t,u_n)}{dt} &= \frac{p-2}{p}\|u_n\|_p^p t^{p-3}\left(\frac{p\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^2dx}{2(p-2)\|u_n\|_p^p}t^{4-p} - 1\right) \\ &\le \frac{\varepsilon(p-2)}{2}I_{\rho^2} < 0.\end{aligned} \quad (51)$$

This, together with the mean value theorem and (41), yields that for all $t \in (1, ((1-\varepsilon)^{-1}\widetilde{\rho}_1(\rho))^{1/(p-4)})$ and all $n \in \mathbb{N}$,

$$g(t,u_n) = g(1,u_n) + \left.\frac{dg(t,u_n)}{dt}\right|_{t=\theta_{t_n}}(t-1) < -C(t-1), \quad (52)$$

where $\theta_{t_n} \in (1,t)$ and $C > 0$ depend only on ε, p,and ρ. By (42), we have

$$\begin{aligned}I_{(t\rho)^2} &\le I(tu_n) = t^2(I(u_n) + g(t,u_n)) \\ &\le t^2I(u_n) - Ct^2(t-1) \\ &= t^2I_{\rho^2} - Ct^2(t-1) + o(1),\end{aligned} \quad (53)$$

then

$$I_{(t\rho)^2} < t^2 I_{\rho^2} \quad \forall t \in \left(1, \left(\frac{\widetilde{\rho}_1(\rho)}{1-\varepsilon}\right)^{1/(p-4)}\right), \ \rho \in (0, \overline{\rho}_1). \tag{54}$$

Clearly, $\widetilde{\rho}_1$ (cf. (50)) is strictly increasing on ρ, and then $(\widetilde{\rho}_1/(1-\varepsilon))^{1/(p-4)}$ is strictly decreasing on ρ since $p \in (2,3)$.

Let

$$h(\rho) := \left(\frac{\widetilde{\rho}_1(\rho)}{1-\varepsilon}\right)^{1/(p-4)}. \tag{55}$$

For each $\rho \in (0, \overline{\rho}_1)$, let $\mu \in (0,\rho)$ without loss of generality, we may assume that $\mu > \sqrt{\rho^2-\mu^2}$. Choosing $\varepsilon \in (0, \min\{\overline{\rho}_1 - \rho, 1-\widetilde{\rho}_1(\rho), \mu-\sqrt{\rho^2-\mu^2}\})$, then by (50) we know that $h(\sqrt{\rho^2-\mu^2}) > h(\mu) > h(\rho) > 1$.

(a) If$\rho/\mu \in (1, h(\mu))$, then by (54)

$$\begin{aligned} I_{\rho^2} &= I_{(\rho^2/\mu^2)\mu^2} < \frac{\rho^2}{\mu^2} I_{\mu^2} = I_{\mu^2} + \frac{\rho^2-\mu^2}{\mu^2} I_{\mu^2} \\ &= I_{\mu^2} + \frac{\rho^2-\mu^2}{\mu^2} I_{(\mu^2/(\rho^2-\mu^2))(\rho^2-\mu^2)}. \end{aligned} \tag{56}$$

(b) If$\rho/\mu \notin (1, h(\mu))$, then there exists $k \in \mathbb{N}$ such that $(\rho/\mu)^{1/k} \in (1, h(\rho))$. Therefore

$$\left(\frac{\rho}{\mu}\right)^{1/k} \in \left(1, h\left(\left(\frac{\rho}{\mu}\right)^{(k-i)/k}\mu\right)\right), \quad \forall i = 1,2,\ldots,k. \tag{57}$$

It follows from (54) that

$$\begin{aligned} I_{\rho^2} &= I_{(\rho/\mu)^{2/k}((\rho/\mu)^{2(k-1)/k}\mu^2)} \\ &< \left(\frac{\rho}{\mu}\right)^{2/k} I_{(\rho/\mu)^{2(k-1)/k}\mu^2} < \cdots < \frac{\rho^2}{\mu^2} I_{\mu^2}. \end{aligned} \tag{58}$$

Combining the above cases (a) and (b), we can show that

$$I_{\rho^2} < I_{\mu^2} + I_{\rho^2-\mu^2} \quad \forall 0 < \mu < \rho < \overline{\rho}_1. \tag{59}$$

Thus the conclusion of this lemma holds. □

Remark 13. For the case of $p = 8/3$, it has been proved in [4, 17] that the strong subadditivity inequality (8) holds for $\rho > 0$ small. By using the result of [17], we can give a specific estimate of lower bound of ρ such that (8) holds; that is, (8) holds for all $\rho \in (0, (8\pi)^{-3/4}S^{3/2})$. However, if we plug $p = 8/3$ into (49), then we have $\overline{\rho}_1 = (8\pi)^{-3/4}S^{3/2}$, which coincides with the one given in [17].

Next, we will show (8) for $p \in (8/3, 3)$. We point out that the case of $p \in (8/3,3)$ is quite different from the case of $p \in (2, 8/3]$ since the inequality (48) does not hold anymore. Inspired by [18], we will give some estimates for I_{ρ^2} in Lemmas 14 and 15, and these are crucial for the proof of (8) if $p \in (8/3, 3)$.

Lemma 14. *Let $p \in (8/3,3)$ and $\rho > 0$ be fixed. If there exists $u \in B_\rho \cap \mathscr{P}$ such that $I(u) \le I_{\rho^2}/2$ and*

$$\|u\|_p^p > 3\|\nabla u\|_2^2, \tag{60}$$

then there exist positive constants C_3 and C_4 dependent on p and ρ, such that

$$I_{\mu^2} \le -C_3 \mu^{2(6-p)/(10-3p)} + C_4 \mu^{2(18-5p)/(10-3p)} \quad \forall \mu > 0. \tag{61}$$

Proof. From the assumptions of the lemma, we see that

$$\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u |u|^2 dx - \frac{3(p-2)}{2p}\|u\|_p^p = 0, \tag{62}$$

$$\frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u |u|^2 dx - \frac{1}{p}\|u\|_p^p = I(u) \le \frac{I_{\rho^2}}{2}. \tag{63}$$

By (60), (62), and (63), we deduce that

$$\frac{I_{\rho^2}}{2} \ge I(u) = -\frac{1}{2}\|\nabla u\|_2^2 + \frac{3p-8}{2p}\|u\|_p^p > \frac{4(p-3)}{p}\|\nabla u\|_2^2. \tag{64}$$

Combining (62) and (64), and using Lemma 10, we also obtain

$$\begin{aligned} \frac{1}{4}\int_{\mathbb{R}^3} \phi_u |u|^2 dx &= \frac{3(p-2)}{2p}\|u\|_p^p - \|\nabla u\|_2^2 \\ &\le \left(\frac{3(p-2)}{2p}\right)^{4/(10-3p)} S^{-6(p-2)/(10-3p)} \\ &\quad \times \rho^{2(6-p)/(10-3p)} + \frac{p}{8(3-p)} I_{\rho^2}. \end{aligned} \tag{65}$$

For each $t > 0$, let $u_t(x) = t^{4/(10-3p)}u(t^{2(p-2)/(10-3p)}x)$, we have $\|u_t\|_2 = t\|u\|_2 = t\rho$. It follows from (60), (64), and (65) that

$$\begin{aligned} I_{(t\rho)^2} \le I(u_t) &= \frac{1}{2}t^{2(6-p)/(10-3p)}\|\nabla u\|_2^2 \\ &\quad + \frac{1}{4}t^{2(18-5p)/(10-3p)}\int_{\mathbb{R}^3} \phi_u |u|^2 dx \\ &\quad - \frac{1}{p}t^{2(6-p)/(10-3p)}\|u\|_p^p \\ &\le \left(\frac{1}{2} - \frac{3}{p}\right) t^{2(6-p)/(10-3p)}\|\nabla u\|_2^2 \end{aligned}$$

$$+\frac{1}{4}t^{2(18-5p)/(10-3p)}\int_{\mathbb{R}^3}\phi_u|u|^2dx$$
$$\le \frac{6-p}{16(3-p)}\frac{I_{\rho^2}}{\rho^{2(6-p)/(10-3p)}}(t\rho)^{2(6-p)/(10-3p)}$$
$$+\left(\left(\frac{3(p-2)}{2p}\right)^{4/(10-3p)}S^{-6(p-2)/(10-3p)}\right.$$
$$\times\rho^{8(p-3)/(10-3p)}$$
$$\left.+\frac{p}{8(3-p)}\frac{I_{\rho^2}}{\rho^{2(18-5p)/(10-3p)}}\right)$$
$$\times(t\rho)^{2(18-5p)/(10-3p)}. \tag{66}$$

Set $t\rho=\mu$, then $\mu\in(0,\infty)$ since $t\in(0,\infty)$ and ρ is a fixed positive constant. From the above inequality, we see that

$$I_{\mu^2}\le -C_3\mu^{2(6-p)/(10-3p)}+C_4\mu^{2(18-5p)/(10-3p)}, \tag{67}$$

for some positive constants C_3 and C_4 depending on p and ρ. □

Lemma 15. *Suppose that $p\in(8/3,3)$ and $\{u_k\}\subset B_{\rho_k}\cap\mathscr{P}$ satisfying $\|u_k\|_p^p>3\|\nabla u_k\|_2^2$ and $I(u_k)\le I_{\rho_k^2}/2$ for all $k\in\mathbb{N}$. Then there exists a positive constant C dependent on p, such that*

$$I_{\rho_k^2}\ge -C\rho_k^{2(5p-12)/(3p-8)}\quad\forall k\in\mathbb{N}. \tag{68}$$

Proof. Following the line of the proof of Lemma 14, we arrive that

$$\frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_k}|u_k|^2dx=\frac{3(p-2)}{2p}\|u_k\|_p^p-\|\nabla u_k\|_2^2$$
$$>\frac{7p-18}{6p}\|u_k\|_p^p, \tag{69}$$

which, together with (14) and the Höder inequality, implies that

$$\|u_k\|_p^p\le\frac{3p}{2(7p-18)}\int_{\mathbb{R}^3}\phi_{u_k}|u_k|^2dx$$
$$\le\frac{6\pi p}{(7p-18)S^2}\|u_k\|_{12/5}^4 \tag{70}$$
$$\le\frac{6\pi p}{(7p-18)S^2}\|u_k\|_2^{2(5p-12)/3(p-2)}\|u_k\|_p^{2p/3(p-2)},$$

and then

$$\|u_k\|_p^p\le\left(\frac{6\pi p}{(7p-18)S^2}\right)^{3(p-2)/(3p-8)}\|u_k\|_2^{2(5p-12)/(3p-8)}$$
$$=\left(\frac{6\pi p}{(7p-18)S^2}\right)^{3(p-2)/(3p-8)}\rho_k^{2(5p-12)/(3p-8)}. \tag{71}$$

Combining (62), (69), and (71), we have

$$\frac{I_{\rho_k^2}}{2}>I(u_k)=\frac{1}{2}\|\nabla u_k\|_2^2+\frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_k}|u_k|^2dx-\frac{1}{p}\|u_k\|_p^p$$
$$=\frac{3p-10}{4p}\|u_k\|_p^p+\frac{1}{8}\int_{\mathbb{R}^3}\phi_{u_k}|u_k|^2dx$$
$$\ge\frac{4(p-3)}{3p}\|u_k\|_p^p \tag{72}$$
$$\ge\frac{4(p-3)}{3p}\left(\frac{6\pi p}{(7p-18)S^2}\right)^{3(p-2)/(3p-8)}$$
$$\times\rho_k^{2(5p-12)/(3p-8)},$$

and this completes the proof. □

Lemma 16. *If $p\in(8/3,3)$, then there exists a positive constant $\overline{\rho}_2$ such that*

$$I_{\rho^2}<I_{\mu^2}+I_{\rho^2-\mu^2}\quad\forall 0<\mu<\rho<\overline{\rho}_2. \tag{73}$$

Proof. Suppose that $\rho>0$ and $\{u_n\}\subset B_\rho\cap\mathscr{P}$ satisfying $I(u_n)\to I_{\rho^2}$ as $n\to\infty$. It follows from Remark 9 that, up to a subsequence, $I(u_n)\le I_{\rho^2}/2<0$ for all $n\in\mathbb{N}$. By Lemma 5, it is easy to see that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Noting that $tu_n\in B_{t\rho}$, then, by (42), we have

$$I(tu_n)=t^2(I(u_n)+g(t,u_n)), \tag{74}$$

where $g(t,u)$ is given by (43). Obviously,

$$\left.\frac{dg(t,u_n)}{dt}\right|_{t=1}=\frac{1}{2}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^2dx-\frac{(p-2)}{p}\|u_n\|_p^p$$
$$=I(u_n)-\frac{1}{2}\|\nabla u_n\|_2^2$$
$$+\frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^2dx+\frac{(3-p)}{p}\|u_n\|_p^p \tag{75}$$
$$=I(u_n)-\frac{3}{2}\|\nabla u_n\|_2^2+\frac{1}{2}\|u_n\|_p^p,$$

since $u_n\in B_\rho\cap\mathscr{P}$ and (34) holds. Moreover,

$$\frac{d^2g(t,u_n)}{dt^2}=\frac{1}{2}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^2dx$$
$$-\frac{(p-2)(p-3)}{p}t^{p-4}\|u_n\|_p^p>0, \tag{76}$$

for all $t>0$ and $n\in\mathbb{N}$.

We claim that there exists $\overline{\rho}_2>0$ such that for each $\rho\in(0,\overline{\rho}_2)$ and for each $\{u_n\}\subset B_\rho\cap\mathscr{P}$ satisfying $I(u_n)\le I_{\rho^2}/2<0$ and $I(u_n)\to I_{\rho^2}$ as $n\to\infty$, we have

$$\|u_n\|_p^p\le 3\|\nabla u_n\|_2^2. \tag{77}$$

Indeed, if not, we can find $\{\rho_k\}$ and $\{u_n^k\} \subset B_{\rho_k} \cap \mathscr{P}$ such that $\rho_k \to 0$ as $k \to \infty$ and for each $k \in \mathbb{N}$, $I(u_n^k) \to I_{\rho_k^2} < 0$ as $n \to \infty$, but $\|u_n^k\|_p^p > 3\|\nabla u_n^k\|_2^2$. For $k = 1$, there exists $n_1 > 0$ such that $I(u_{n_1}^1) < I_{\rho_1^2}/2 < 0$, and it can be deduced from Lemma 14 that

$$I_{\mu^2} \leq -C_3\mu^{2(6-p)/(10-3p)} + C_4\mu^{2(18-5p)/(10-3p)} \quad \forall \mu > 0, \tag{78}$$

where C_3 and C_4 are positive constants dependent on p and ρ_1. On the other hand, we know that for each $k \in \mathbb{N}$ there exists $n_k > 0$ such that $I(u_{n_k}^k) < I_{\rho_k^2}/2 < 0$. Then by Lemma 15, we obtain

$$I_{\rho_k^2} \geq -C\rho_k^{2(5p-12)/(3p-8)} \quad \forall k \in \mathbb{N}, \tag{79}$$

where C is a positive constant depending only on p. Noting that (78) holds for all $\mu > 0$, by (78) and (79), we deduce that

$$\begin{aligned} -C\rho_k^{2(5p-12)/(3p-8)} \leq I_{\rho_k^2} \leq & -C_3\rho_k^{2(6-p)/(10-3p)} \\ & + C_4\rho_k^{2(18-5p)/(10-3p)}, \end{aligned} \tag{80}$$

which is a contradiction for k large since $p \in (8/3, 3)$ implies

$$\frac{2(5p-12)}{3p-8} > \frac{2(6-p)}{10-3p}. \tag{81}$$

Thus we have shown the claim. Now for each $\rho \in (0, \bar{\rho}_2)$ and for all $\{u_n\} \subset B_\rho \cap \mathscr{P}$ with $I(u_n) \leq I_{\rho^2}/2$ and $I(u_n) \to I_{\rho^2}$ as $n \to \infty$, using (77), we have

$$\left.\frac{dg(t,u_n)}{dt}\right|_{t=1} \leq I(u_n) \leq \frac{I_{\rho^2}}{2} < 0. \tag{82}$$

By (76), similarly as in the proofs of (45) and (51), we get that

$$\begin{aligned} t^{4-p} &= \frac{2(p-2)}{p}\frac{\|u_n\|_p^p}{\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^2dx} \\ &= \frac{2(p-2)}{p}\frac{\|u_n\|_p^p}{(6(p-2)/p)\|u_n\|_p^p - 4\|\nabla u_n\|_2^2} \\ &\geq \frac{3(p-2)}{7p-18} > 1. \end{aligned} \tag{83}$$

Now, we can choose $\varepsilon > 0$ so small that there exists a positive constant C dependent on p, ρ,and ε, such that

$$\frac{dg(t,u_n)}{dt} \leq -C < 0 \quad \forall t \in \left(1, \left(\frac{3(1-\varepsilon)(p-2)}{7p-18}\right)^{1/(4-p)}\right). \tag{84}$$

Since, for each n, $g(1,u_n) = 0$, it follows that, for each $t \in (1, (3(1-\varepsilon)(p-2)/(7p-18))^{1/(4-p)})$,

$$\begin{aligned} I_{(t\rho)^2} &\leq I(tu_n) = t^2(I(u_n) + g(t,u_n)) \\ &\leq t^2\left(I(u_n) + \left.\frac{dg(t,u_n)}{dt}\right|_{t=\theta_{t_n}}(t-1)\right) \\ &\leq t^2(I(u_n) - C(t-1)) \\ &= t^2 I_{\rho^2} - Ct^2(t-1) + o(1), \end{aligned} \tag{85}$$

where $\theta_{t_n} \in (1,t)$, namely, $I_{(t\rho)^2} < t^2 I_{\rho^2}$ for all $t \in (1, (3(1-\varepsilon)(p-2)/(7p-18))^{1/(4-p)})$. Thus we complete the proof of this lemma by using the arguments in the proof of Lemma 12. □

Lemma 17. *Let $\rho > 0$. Assume that (u_ρ, λ_ρ) is a solution of (1) with $\|u_\rho\|_2 = \rho$.*

(a) *If $p \in (2, 12/5]$, then $\lambda_\rho < 0$.*
(b) *If $p \in (12/5, 8/3]$ and $\lambda_\rho \geq 0$, then*

$$\begin{aligned} \rho \geq \bar{\rho}_3 := & \left(\frac{6-p}{6\pi p}\right)^{(10-3p)/8(3-p)} \\ & \times S^{3/2}\left(\frac{3(p-2)}{2p}\right)^{(8-3p)/8(3-p)}. \end{aligned} \tag{86}$$

Proof. Since (u_ρ, λ_ρ) is a solution of (1), it follows that

$$\|\nabla u_\rho\|_2^2 + \int_{\mathbb{R}^3}\phi_{u_\rho}|u_\rho|^2dx - \|u_\rho\|_p^p = \lambda_\rho\|u_\rho\|_2^2, \tag{87}$$

$$\|\nabla u_\rho\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_\rho}|u_\rho|^2dx - \frac{3(p-2)}{2p}\|u_\rho\|_p^p = 0. \tag{88}$$

Thus, from (87) and (88), after a simple calculation, we have

$$\lambda_\rho\|u_\rho\|_2^2 = \frac{p-6}{3(p-2)}\|\nabla u_\rho\|_2^2 + \frac{5p-12}{6(p-2)}\int_{\mathbb{R}^3}\phi_{u_\rho}|u_\rho|^2dx, \tag{89}$$

which yields that (a) holds. Moreover, if $p \in (12/5, 8/3]$ and $\lambda_\rho \geq 0$, then (89) implies that

$$\frac{5p-12}{6(p-2)}\int_{\mathbb{R}^3}\phi_{u_\rho}|u_\rho|^2dx \geq \frac{6-p}{3(p-2)}\|\nabla u_\rho\|_2^2. \tag{90}$$

Thus we get from (88) that

$$\begin{aligned} \frac{3(p-2)}{2p}\|u_\rho\|_p^p &= \|\nabla u_\rho\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_\rho}|u_\rho|^2dx \\ &\leq \frac{9(p-2)}{4(6-p)}\int_{\mathbb{R}^3}\phi_{u_\rho}|u_\rho|^2dx \\ &\leq \frac{9(p-2)\pi}{(6-p)S^2}\|u_\rho\|_{12/5}^4. \end{aligned} \tag{91}$$

By using the Hölder inequality, it can be deduced from (91) and Lemma 10 that

$$\begin{aligned}&\frac{6-p}{6\pi p}S^2\|u_\rho\|_p^p\\&\leq \|u_\rho\|_{12/5}^4 \leq \|u_\rho\|_2^{2(5p-12)/3(p-2)}\|u_\rho\|_p^{2p/3(p-2)}\\&\leq \left(S^{-2(8-3p)/(10-3p)}\left(\frac{3(p-2)}{2p}\right)^{(8-3p)/(10-3p)}\right.\\&\qquad\left.\times\rho^{8(3-p)/(10-3p)}\right)\|u_\rho\|_p^p,\end{aligned} \tag{92}$$

and this means that

$$\rho \geq \left(\frac{6-p}{6\pi p}\right)^{(10-3p)/8(3-p)} S^{3/2}\left(\frac{3(p-2)}{2p}\right)^{(3p-8)/8(3-p)}. \tag{93}$$

Thus (b) holds. At this point, the lemma is proved. □

Proof of Theorem 1. It follows from Lemmas 12 and 16 that (8) holds. Let $T(u) = \int_{\mathbb{R}^3}\phi_u|u|^2dx/4 - \|u\|_p^p/p$. From the results of [17, 18], we know that (18), (19), (20), and (21) hold. Therefore, by Lemma 3, all the minimizing sequences for (7) are precompact and then (1) has a solution (u_ρ, λ_ρ). Lemma 17 shows that, for $p \in (2, 8/3]$, $\lambda_\rho < 0$ since $\overline{\rho}_1 < \overline{\rho}_3$, where $\overline{\rho}_1$ and $\overline{\rho}_3$ are given by (49) and (86), respectively.

To complete the proof of Theorem 1, we need to show that $\lambda_\rho \to 0$ and $I_{\rho^2}/\rho^2 \to 0$ as $\rho \to 0$ provided that the assumption (1) of Theorem 1 holds. Indeed, since (u_ρ, λ_ρ) is the solution of (1), it follows from (87), (88), and Lemma 10 that

$$\begin{aligned}0 &\leq \frac{3}{4}\int_{\mathbb{R}^3}\phi_{u_\rho}|u_\rho|^2dx - \lambda_\rho\rho^2\\&= \frac{6-p}{2p}\|u_\rho\|_p^p \leq C\rho^{2(6-p)/(10-3p)},\end{aligned} \tag{94}$$

which implies that

$$0 \leq -\lambda_\rho \leq C\rho^{4(p-2)/(10-3p)}, \tag{95}$$

and that is, $\lambda_\rho \to 0$ as $\rho \to 0$. On the other hand, we have

$$I(u_\rho) = \frac{1}{2}\|\nabla u_\rho\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_\rho|u_\rho|^2dx - \frac{1}{p}\|u_\rho\|_p^p, \tag{96}$$

this, together with (87) and (88), gives

$$\|u_\rho\|_p^p = \frac{3p}{2(p-3)}I(u_\rho) + \frac{p}{4(3-p)}\lambda_\rho\rho^2. \tag{97}$$

Therefore, $I(u_\rho) < 0$ since $\lambda_\rho < 0$ and $p \in (2, 8/3)$. Noting that $I(u_\rho) = I_{\rho^2}$, by Lemma 10 and (97), we obtain

$$\begin{aligned}0 &< \frac{3p}{2(p-3)}\frac{I_{\rho^2}}{\rho^2}\\&\leq \left(\frac{3(p-2)}{2p}\right)^{3(p-2)/(10-3p)} S^{-6(p-2)/(10-3p)}\\&\quad\times\rho^{4(p-2)/(10-3p)} - \frac{p}{4(3-p)}\lambda_\rho \longrightarrow 0 \quad \text{as } \rho \longrightarrow 0.\end{aligned} \tag{98}$$

□

Proof of Theorem 2. Suppose that $p \in (2, 12/5]$ and (u_ρ, λ_ρ) is a solution of (1) with $\|u_\rho\|_2 = \rho$. Then Lemma 17 and the above proof of Theorem 1 show that $\lambda_\rho < 0$, $I(u_\rho) < 0$ and $\lambda_\rho \to 0$ as $\rho \to 0$. It was proved in [5, 20] (see also [13, Remark 1.4]) that there exists $\lambda_0 < 0$ such that (1) has only the zero solution when $p \in (2, 3)$ and $\lambda \in (-\infty, \lambda_0)$. Therefore, λ_ρ must be bounded; that is, (i) holds. For (ii), it is clear that (87), (88), and (96) hold; after a simple calculation, we have

$$\begin{aligned}\|\nabla u_\rho\|_2^2 &= \frac{5p-12}{2(p-3)}I(u_\rho) + \frac{3p-8}{4(3-p)}\lambda_\rho\rho^2,\\\int_{\mathbb{R}^3}\phi_{u_\rho}|u_\rho|^2dx &= \frac{6-p}{p-3}I(u_\rho) + \frac{10-3p}{2(3-p)}\lambda_\rho\rho^2,\\\|u_\rho\|_p^p &= \frac{3p}{2(p-3)}I(u_\rho) + \frac{p}{4(3-p)}\lambda_\rho\rho^2.\end{aligned} \tag{99}$$

On the other hand, since ϕ_{u_ρ} is the solution of the Poisson equation $-\Delta\phi = 4\pi|u_\rho|^2$, multiplying this equation by $|u_\rho|$ and integrating, we obtain

$$\begin{aligned}4\pi\|u_\rho\|_3^3 &= \int_{\mathbb{R}^3}\nabla\phi_\rho\nabla|u_\rho|\,dx\\&\leq \frac{1}{2}\|\nabla\phi_\rho\|_2^2 + \frac{1}{2}\|\nabla u_\rho\|_2^2\\&= 2\pi\int_{\mathbb{R}^3}\phi_{u_\rho}|u_\rho|^2dx + \frac{1}{2}\|\nabla u_\rho\|_2^2.\end{aligned} \tag{100}$$

It follows from (99) and (100) that

$$\begin{aligned}0 &< \frac{3p}{2(p-3)}I(u_\rho) + \frac{p}{4(3-p)}\lambda_\rho\rho^2\\&= \|u_\rho\|_p^p \leq \|u_\rho\|_2^{2(3-p)}\|u_\rho\|_3^{3(p-2)}\\&\leq \|u_\rho\|_2^{2(3-p)}\left(\frac{1}{2}\int_{\mathbb{R}^3}\phi_{u_\rho}|u_\rho|^2dx + \|\nabla u_\rho\|_2^2\right)^{p-2}\\&= \left(\frac{2}{p}\right)^{p-2}\|u_\rho\|_2^{2(3-p)}\\&\quad\times\left(\frac{p(2p-3)}{2(p-3)}I(u_\rho) + \frac{p}{4(3-p)}\lambda_\rho\rho^2\right)^{p-2}\end{aligned}$$

$$\leq \left(\frac{2}{p}\right)^{p-2} \|u_\rho\|_2^{2(3-p)} \times \left(\frac{3p}{2(p-3)} I(u_\rho) + \frac{p}{4(3-p)} \lambda_\rho \rho^2\right)^{p-2}, \quad (101)$$

which implies that

$$0 < \frac{3p}{2(p-3)} I(u_\rho) + \frac{p}{4(3-p)} \lambda_\rho \rho^2 \leq \left(\frac{2}{p}\right)^{(p-2)/(3-p)} \|u_\rho\|_2^2 = \left(\frac{2}{p}\right)^{(p-2)/(3-p)} \rho^2. \quad (102)$$

Therefore we get

$$0 < \frac{3p}{2(p-3)} \frac{I(u_\rho)}{\rho^2} + \frac{p}{4(3-p)} \lambda_\rho \leq \left(\frac{2}{p}\right)^{(p-2)/(3-p)}, \quad (103)$$

so that there exists $C > 0$ such that $I(u_\rho)/\rho^2 \in (-C, 0)$ since, by (i), $\lambda_\rho < 0$ is bounded. □

Acknowledgments

The authors would like to thank the referees for carefully reading the paper and helpful comments, which greatly improve the presentation of the paper, in particular, the proof of Lemma 17 that was simplified. This paper is supported by Natural Science Foundation of China (11071180, 11171247) and GIP of Jiangsu Province (CXZZ12_0802).

References

[1] O. Bokanowski and N. J. Mauser, "Local approximation for the Hartree-Fock exchange potential: a deformation approach," *Mathematical Models & Methods in Applied Sciences*, vol. 9, no. 6, pp. 941–961, 1999.

[2] O. Bokanowski, J. L. López, and J. Soler, "On an exchange interaction model for quantum transport: the Schrödinger-Poisson-Slater system," *Mathematical Models & Methods in Applied Sciences*, vol. 13, no. 10, pp. 1397–1412, 2003.

[3] T. Cazenave, *Semilinear Schrödinger Equations*, vol. 10 of *Courant Lecture Notes in Mathematics*, New York University, Courant Institute of Mathematical Sciences, New York, NY, USA; American Mathematical Society, Providence, RI, USA, 2003.

[4] O. Sánchez and J. Soler, "Long-time dynamics of the Schrödinger-Poisson-Slater system," *Journal of Statistical Physics*, vol. 114, no. 1-2, pp. 179–204, 2004.

[5] D. Ruiz, "The Schrödinger-Poisson equation under the effect of a nonlinear local term," *Journal of Functional Analysis*, vol. 237, no. 2, pp. 655–674, 2006.

[6] A. Ambrosetti, "On Schrödinger-Poisson systems," *Milan Journal of Mathematics*, vol. 76, pp. 257–274, 2008.

[7] A. Azzollini, P. d'Avenia, and A. Pomponio, "On the Schrödinger-Maxwell equations under the effect of a general nonlinear term," *Annales de l'Institut Henri Poincaré*, vol. 27, no. 2, pp. 779–791, 2010.

[8] G. Cerami and G. Vaira, "Positive solutions for some non-autonomous Schrödinger-Poisson systems," *Journal of Differential Equations*, vol. 248, no. 3, pp. 521–543, 2010.

[9] T. d'Aprile and D. Mugnai, "Non-existence results for the coupled Klein-Gordon-Maxwell equations," *Advanced Nonlinear Studies*, vol. 4, no. 3, pp. 307–322, 2004.

[10] T. d'Aprile and D. Mugnai, "Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations," *Proceedings of the Royal Society of Edinburgh A*, vol. 134, no. 5, pp. 893–906, 2004.

[11] Y. Jiang and H.-S. Zhou, "Schrödinger-Poisson system with steep potential well," *Journal of Differential Equations*, vol. 251, no. 3, pp. 582–608, 2011.

[12] L. Zhao and F. Zhao, "On the existence of solutions for the Schrödinger-Poisson equations," *Journal of Mathematical Analysis and Applications*, vol. 346, no. 1, pp. 155–169, 2008.

[13] L. Jeanjean and T. Luo, "Sharp nonexistence results of prescribed L^2-norm solutions for some class of Schrödinger-Poisson and quasi-linear equations," *Zeitschrift für angewandte Mathematik und Physik*, vol. 64, no. 4, pp. 937–954, 2013.

[14] J. Bellazzini, L. Jeanjean, and T. Luo, "Existence and instability of standing waves with prescribed norm for a class of Schrödinger-Poisson equations," *Proceedings of London Mathematical Society*, 2013.

[15] P.-L. Lions, "The concentration-compactness principle in the-calculus of variation. The locally compact case, part I," *Annales de l'Institut Henri Poincaré*, vol. 1, no. 2, pp. 109–145, 1984.

[16] P.-L. Lions, "The concentration-compactness principle in the calculus of variation. The locally compact case, part II," *Annales de l'Institut Henri Poincaré*, vol. 1, no. 4, pp. 223–283, 1984.

[17] J. Bellazzini and G. Siciliano, "Stable standing waves for a class of nonlinear Schrödinger-Poisson equations," *Zeitschrift für Angewandte Mathematik und Physik*, vol. 62, no. 2, pp. 267–280, 2011.

[18] J. Bellazzini and G. Siciliano, "Scaling properties of functionals and existence of constrained minimizers," *Journal of Functional Analysis*, vol. 261, no. 9, pp. 2486–2507, 2011.

[19] V. Benci and D. Fortunato, "Hylomorphic solitons on lattices," *Discrete and Continuous Dynamical Systems A*, vol. 28, no. 3, pp. 875–897, 2010.

[20] H. Kikuchi, "Existence and stability of standing waves for Schrödinger-Poisson-Slater equation," *Advanced Nonlinear Studies*, vol. 7, no. 3, pp. 403–437, 2007.

[21] V. Georgiev, F. Prinari, and N. Visciglia, "On the radiality of constrained minimizers to the Schrödinger-Poisson-Slater energy," *Annales de l'Institut Henri Poincaré*, vol. 29, no. 3, pp. 369–376, 2012.