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Research Article

Some Curvature Properties of $(\mathrm{LCS})_n$-Manifolds

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The object of the present paper is to study $(\mathrm{LCS})_n$-manifolds with vanishing quasi-conformal curvature tensor. $(\mathrm{LCS})_n$-manifolds satisfying Ricci-symmetric condition are also characterized.

1. Introduction

Recently, in [1], Shaikh introduced and studied Lorentzian concircular structure manifolds (briefly (LCS)-manifold) which generalizes the notion of LP-Sasakian manifolds, introduced by Matsumoto [2].

Generalizing the notion of LP-Sasakian manifold in 2003 [1], Shaikh introduced the notion of $(\mathrm{LCS})_n$-manifolds along with their existence and applications to the general theory of relativity and cosmology. Also, Shaikh and his coauthors studied various types of $(\mathrm{LCS})_n$-manifolds by imposing the curvature restrictions (see [3–6]). In [7, 8], the authors also studied $(\mathrm{LCS})_{2n+1}$-manifolds.

The submanifold of an $(\mathrm{LCS})_n$-manifold is studied by Atceken and Hui [9, 10] and Shukla et al. [11]. In [12], Yano and Sawaki introduced the quasi-conformal curvature tensor, and later it was studied by many authors with curvature restrictions on various structures [13].

After then, the same author studied weakly symmetric $(\mathrm{LCS})_n$-manifolds by several examples and obtain various results in such manifolds. In [7], authors shown that a pseudo projectively flat and pseudo projectively recurrent $(\mathrm{LCS})_n$ manifolds are η-Einstein manifold.

On the other hand, in [5], authors proved the existence of ϕ-recurrent $(\mathrm{LCS})_3$ manifold which is neither locally symmetric nor locally ϕ-symmetric by nontrivial examples. Furthermore, they also give the necessary and sufficient conditions for a $(\mathrm{LCS})_n$-manifold to be locally ϕ-recurrent.

In this study, we have investigated the quasi-conformal flat $(\mathrm{LCS})_n$-manifolds satisfying properties such as Ricci-symmetric, locally symmetric, and η-Einstein. Finally, we give an example for η-Einstein manifolds.

2. Preliminaries

An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric tensor g, that is, M admits a smooth symmetric tensor field g of the type $(2,0)$ such that, for each $p \in M$,

$$g_p : T_M(p) \times T_M(p) \longrightarrow \mathbb{R} \tag{1}$$

is a nondegenerate inner product of signature $(-,+,+,\ldots,+)$. In such a manifold, a nonzero vector $X_p \in T_M(p)$ is said to be timelike (resp., nonspacelike, null, and spacelike) if it satisfies the condition $g_p(X_p, X_p) < 0$ (resp., ≤ 0, $=0$, >0). These cases are called casual character of the vectors.

Definition 1. In a Lorentzian manifold (M, g), a vector field P defined by

$$g(X, P) = A(X) \tag{2}$$

for any $X \in \Gamma(TM)$ is said to be a concircular vector field if

$$(\nabla_X A) Y = \alpha \{ g(X, Y) + w(X) A(Y) \} \tag{3}$$

for $Y \in \Gamma(TM)$, where α is a nonzero scalar function, A is a 1-form, w is also closed 1-form, and ∇ denotes the Levi-Civita connection on M [7].

Let M be a Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi,\xi) = -1. \tag{4}$$

Since ξ is a unit concircular unit vector field, there exists a nonzero 1-form η such that

$$g(X,\xi) = \eta(X). \tag{5}$$

The equation of the following form holds:

$$(\nabla_X \eta) Y = \alpha\{g(X,Y) + \eta(X)\eta(Y)\}, \quad \alpha \neq 0 \tag{6}$$

for all $X, Y \in \Gamma(TM)$, where α is a nonzero scalar function satisfying

$$\nabla_X \alpha = X(\alpha) = d\alpha(X) = \rho\eta(X), \tag{7}$$

ρ being a certain scalar function given by $\rho = -\xi(\alpha)$.

Let us put

$$\nabla_X \xi = \alpha\phi X, \tag{8}$$

then from (6) and (8), we can derive

$$\phi X = X + \eta(X)\xi, \tag{9}$$

which tell us that ϕ is a symmetric $(1,1)$-tensor. Thus the Lorentzian manifold M together with the unit timelike concircular vector field ξ, its associated 1-form η, and $(1,1)$-type tensor field ϕ is said to be a Lorentzian concircular structure manifold.

A differentiable manifold M of dimension n is called (LCS)-manifold if it admits a $(1,1)$-type tensor field ϕ, a covariant vector field η, and a Lorentzian metric g which satisfy

$$\eta(\xi) = g(\xi,\xi) = -1, \tag{10}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{11}$$

$$g(X,\xi) = \eta(X), \tag{12}$$

$$\phi\xi = 0, \qquad \eta \circ \phi = 0 \tag{13}$$

for all $X \in \Gamma(TM)$. Particularly, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [2].

Also, in an $(\mathrm{LCS})_n$-manifold M, the following relations are satisfied (see [3–6]):

$$\eta(R(X,Y)Z) = (\alpha^2 - \rho)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{14}$$

$$R(\xi,X)Y = (\alpha^2 - \rho)[g(X,Y)\xi - \eta(Y)X], \tag{15}$$

$$R(X,Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{16}$$

$$(\nabla_X \phi)Y = \alpha[g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X], \tag{17}$$

$$S(X,\xi) = (n-1)(\alpha^2 - \rho)\eta(X), \tag{18}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)(\alpha^2 - \rho)\eta(X)\eta(Y) \tag{19}$$

for all vector fields X, Y, Z on M, where R and S denote the Riemannian curvature tensor and Ricci curvature, respectively, Q is also the Ricci operator given by $S(X,Y) = g(QX,Y)$.

Now let (M,g) be an n-dimensional Riemannian manifold; then the concircular curvature tensor $\widetilde{C}$, the Weyl conformal curvature tensor C, and the pseudo projective curvature tensor $\widetilde{P}$ are, respectively, defined by

$$\widetilde{C}(X,Y)Z = R(X,Y)Z - \frac{\tau}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{20}$$

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z &- \frac{1}{n-2} \times [S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY] \\ &+ \frac{\tau}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{21}$$

$$\begin{aligned} \widetilde{P}(X,Y)Z = aR(X,Y)Z &+ b[S(Y,Z)X - S(X,Z)Y] \\ &- \frac{\tau}{n}\left[\frac{a}{n-1} + b\right][g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{22}$$

where a and b are constants such that $a, b \neq 0$, and τ is also the scalar curvature of M [7].

For an n-dimensional $(\mathrm{LCS})_n$-manifold the quasi-conformal curvature tensor $\widetilde{\mathscr{C}}$ is given by

$$\begin{aligned} \widetilde{\mathscr{C}}(X,Y)Z = aR(X,Y)Z &+ b[S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY] \\ &- \frac{\tau}{n}\left[\frac{a}{n-1} + 2b\right][g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{23}$$

for all $X, Y, Z \in \Gamma(TM)$ [14].

The notion of quasi-conformal curvature tensor was defined by Yano and Swaki [12]. If $a = 1$ and $b = -1/(n-1)$, then quasi-conformal curvature tensor reduces to conformal curvature tensor.

3. Quasi-Conformally Flat $(\mathrm{LCS})_n$-Manifolds and Some of Their Properties

For an n-dimensional quasi-conformally flat $(\mathrm{LCS})_n$-manifold, we know for $Z = \xi$ from (23),

$$\begin{aligned} &aR(X,Y)\xi + b[S(Y,\xi)X - S(X,\xi)Y \\ &\qquad + g(Y,\xi)QX - g(X,\xi)QY] \\ &\quad - \frac{\tau}{n}\left[\frac{a}{n-1} + 2b\right][g(Y,\xi)X - g(X,\xi)Y] = 0. \end{aligned} \tag{24}$$

Here, taking into account of (16), we have

$$\begin{aligned} &[\eta(Y)X - \eta(X)Y]\Big[a(\alpha^2 - \rho) + b(n-1)(\alpha^2 - \rho) \\ &\qquad -\frac{\tau}{n}\left(\frac{a}{n-1} + 2b\right)\Big] \\ &\quad + b[\eta(Y)QX - \eta(X)QY] = 0. \end{aligned} \tag{25}$$

Let $Y = \xi$ be in (25); then also by using (18) we obtain

$$\begin{aligned} &[-X - \eta(X)\xi]\Big[a(\alpha^2 - \rho) - \frac{\tau}{n}\left(\frac{a}{n-1} + 2b\right) \\ &\qquad + b(n-1)(\alpha^2 - \rho)\Big] \\ &\quad + b[-QX - \eta(X)(n-1)(\alpha^2 - \rho)\xi] = 0. \end{aligned} \tag{26}$$

Taking the inner product on both sides of the last equation by Y, we obtain

$$\begin{aligned} &[g(X,Y) + \eta(X)\eta(Y)]\Big[a(\alpha^2 - \rho) + b(n-1) \\ &\qquad \times (\alpha^2 - \rho) - \frac{\tau}{n}\left(\frac{a}{n-1} + 2b\right)\Big] \\ &\quad + b[S(X,Y) + \eta(X)\eta(Y)(\alpha^2 - \rho)(n-1)] = 0, \end{aligned} \tag{27}$$

that is,

$$\begin{aligned} S(X,Y) &= g(X,Y) \\ &\quad \times \left[\frac{\tau}{nb}\left(\frac{a}{n-1} + 2b\right) - (\alpha^2 - \rho)\left(\frac{a}{b} + (n-1)\right)\right] \\ &\quad + \eta(X)\eta(Y)\Big[\frac{\tau}{nb}\left(\frac{a}{n-1} + 2b\right) \\ &\qquad - (\alpha^2 - \rho)\left(\frac{a}{b} + 2(n-1)\right)\Big]. \end{aligned} \tag{28}$$

Now we are in a proposition to state the following.

Theorem 2. *If an n-dimensional $(LCS)_n$-manifold M is quasi-conformally flat, then M is an η-Einstein manifold.*

Now, let $\{e_1, e_2, \ldots, e_{n-1}, \xi\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = Y = e_i, \xi$ in (28), and taking summation for $1 \le i \le n-1$, we have

$$\tau = n(n-1)(\alpha^2 - \rho) \quad \text{if } a + (n-2)b \neq 0. \tag{29}$$

In view of (28) and (29), we obtain

$$S(X,Y) = (n-1)(\alpha^2 - \rho)g(X,Y), \tag{30}$$

which is equivalent to

$$QX = (n-1)(\alpha^2 - \rho)X \tag{31}$$

for any $X \in \Gamma(TM)$.

By using (29) and (31) in (23) for a quasi-conformally flat $(\mathrm{LCS})_n$-manifold M, we get

$$R(X,Y)Z = (\alpha^2 - \rho)\{g(Y,Z)X - g(X,Z)Y\}, \tag{32}$$

for all $X, Y, Z \in \Gamma(TM)$. If we consider Schur's Theorem, we can give the following the theorem.

Theorem 3. *A quasi-conformally flat $(LCS)_n$-manifold M $(n > 1)$ is a manifold of constant curvature $(\alpha^2 - \rho)$ provided that $a + b(n-2) \neq 0$.*

Now let us consider an $(\mathrm{LCS})_n$-manifold M which is conformally flat. Thus we have from (21) that

$$\begin{aligned} R(X,Y)Z &= \frac{1}{n-2}\{S(Y,Z)X - S(X,Z)Y \\ &\qquad + g(Y,Z)QX - g(X,Z)QY\} \\ &\quad - \frac{\tau}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned} \tag{33}$$

for all vector fields X, Y, Z tangent to M. Setting $Z = \xi$ in (33) and using (16), (18) we have

$$\begin{aligned} &\left[\frac{\tau}{n-1} - (\alpha^2 - \rho)\right][\eta(Y)X - \eta(X)Y] \\ &\quad = [\eta(Y)QX - \eta(X)QY]. \end{aligned} \tag{34}$$

If we put $Y = \xi$ in (34) and also using (18), we obtain

$$QX = \left[\frac{\tau}{n-1} - (\alpha^2 - \rho)\right]X + \left[\frac{\tau}{n-1} - n(\alpha^2 - \rho)\right]\eta(X)\xi. \tag{35}$$

Corollary 4. *A conformally flat $(LCS)_n$-manifold is an η-Einstein manifold.*

Generalizing the notion of a manifold of constant curvature, Chen and Yano [15] introduced the notion of a manifold of quasi-constant curvature which can be defined as follows:

Definition 5. A Riemannian manifold is said to be a manifold of quasi-constant curvature if it is conformally flat and its curvature tensor $\widetilde{R}$ of type $(0,4)$ is of the form

$$\begin{aligned} &\widetilde{R}(X,Y,Z,W) \\ &\quad = a\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ &\qquad + b\{g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W) \\ &\qquad\quad + g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z)\}, \end{aligned} \tag{36}$$

for all $X, Y, Z, W \in \Gamma(TM)$, where a, b are scalars of which $b \neq 0$ and A is a nonzero 1-form (for more details, we refer to [13, 16]).

Thus we have the following theorem for $(LCS)_n$-conformally flat manifolds.

Theorem 6. *A conformally flat $(LCS)_n$-manifold is a manifold of quasi-constant curvature.*

Proof. From (33) and (35), we obtain

$$\begin{aligned}
\widetilde{R}(X,Y,Z,W) &= \left(\frac{\tau - 2(n-1)(\alpha^2-\rho)}{(n-1)(n-2)}\right) \\
&\quad \times \{g(X,W)g(Y,Z) - g(Y,W)g(X,Z)\} \\
&\quad + \left(\frac{\tau - n(n-1)(\alpha^2-\rho)}{(n-1)(n-2)}\right) \\
&\quad \times \{g(X,W)\eta(Y)\eta(Z) - g(Y,W)\eta(X)\eta(Z) \\
&\qquad + g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W)\}.
\end{aligned} \tag{37}$$

This implies (36) for

$$a = \frac{\tau - 2(n-1)(\alpha^2-\rho)}{(n-1)(n-2)}, \qquad b = \frac{\tau - n(n-1)(\alpha^2-\rho)}{(n-1)(n-2)}, \qquad A = \eta. \tag{38}$$

This proves our assertion. □

Next, differentiating the (19) covariantly with respect to W, we get

$$\begin{aligned}
\nabla_W S(\phi X, \phi Y) &= \nabla_W S(X,Y) + (n-1)W(\alpha^2-\rho) \\
&\quad + (n-1)(\alpha^2-\rho)W[\eta(X)\eta(Y)],
\end{aligned} \tag{39}$$

for any $X, Y \in \Gamma(TM)$. Making use of the definition of ∇S and (8), we have

$$\begin{aligned}
(\nabla_W S)(\phi X, \phi Y) &+ S(\nabla_W \phi X, \phi Y) + S(\phi X, \nabla_W \phi Y) \\
&= (\nabla_W S)(X,Y) + S(\nabla_W X, Y) + S(X, \nabla_W Y) \\
&\quad + (n-1)W(\alpha^2-\rho)\eta(X)\eta(Y) \\
&\quad + (n-1)(\alpha^2-\rho)\eta(Y)\{\eta(\nabla_W X) + \alpha g(X, \phi W)\} \\
&\quad + (n-1)(\alpha^2-\rho)\eta(X)\{\eta(\nabla_W Y) + \alpha g(Y, \phi W)\}.
\end{aligned} \tag{40}$$

Thus we have

$$\begin{aligned}
(\nabla_W S)&(\phi X, \phi Y) - (\nabla_W S)(X,Y) \\
&= -S((\nabla_W \phi)X + \phi \nabla_W X, \phi Y) \\
&\quad - S(\phi X, (\nabla_W \phi)Y + \phi \nabla_W Y) + S(\nabla_W X, Y) \\
&\quad + S(X, \nabla_W Y) + (n-1)W(\alpha^2-\rho)\eta(X)\eta(Y) \\
&\quad + (n-1)(\alpha^2-\rho)\eta(Y)\{\eta(\nabla_W X) + \alpha g(X, \phi W)\} \\
&\quad + (n-1)(\alpha^2-\rho)\eta(X)\{\eta(\nabla_W Y) + \alpha g(Y, \phi W)\}.
\end{aligned} \tag{41}$$

Here taking account of (17), we arrive at

$$\begin{aligned}
(\nabla_W S)&(\phi X, \phi Y) - (\nabla_W S)(X,Y) \\
&= -S(\alpha\{g(X,W)\xi + 2\eta(X)\eta(W)\xi + \eta(X)W\}, \phi Y) \\
&\quad - S(\phi X, \alpha\{g(Y,W)\xi + 2\eta(Y)\eta(W)\xi + \eta(Y)W\}) \\
&\quad - S(\phi X, \phi \nabla_W Y) + S(\nabla_W X, Y) \\
&\quad + S(X, \nabla_W Y) + (n-1)W(\alpha^2-\rho)\eta(X)\eta(Y) \\
&\quad - S(\phi \nabla_W X, \phi Y) + (n-1)(\alpha^2-\rho)\eta(Y) \\
&\quad \times \{\eta(\nabla_W X) + \alpha g(X, \phi W)\} + (n-1)(\alpha^2-\rho)\eta(X) \\
&\quad \times \{\eta(\nabla_W Y) + \alpha g(Y, \phi W)\} \\
&= -\alpha\{g(X,W)S(\xi, \phi Y) + 2\eta(X)\eta(W)S(\xi, \phi Y) \\
&\qquad + \eta(X)S(W, \phi Y)\} \\
&\quad - \alpha\{g(Y,W)S(\phi X, \xi) + 2\eta(Y)\eta(W)S(\phi X, \xi) \\
&\qquad + \eta(Y)S(\phi X, W)\} \\
&\quad - S(\phi X, \phi \nabla_W Y) + S(\nabla_W X, Y) + S(X, \nabla_W Y) \\
&\quad - S(\phi \nabla_W X, \phi Y) + (n-1)W(\alpha^2-\rho)\eta(X)\eta(Y) \\
&\quad + (n-1)(\alpha^2-\rho)\eta(Y)\{\eta(\nabla_W X) + \alpha g(X, \phi W)\} \\
&\quad + (n-1)(\alpha^2-\rho)\eta(X)\{\eta(\nabla_W Y) + \alpha g(Y, \phi W)\}.
\end{aligned} \tag{42}$$

Again, by using (13), (18), and (19), we reach

$$\begin{aligned}
(\nabla_W S)&(\phi X, \phi Y) - (\nabla_W S)(X,Y) \\
&= -\alpha\eta(X)S(W, \phi Y) - \alpha\eta(Y)S(\phi X, W) \\
&\quad - (n-1)(\alpha^2-\rho)\eta(X)\eta(\nabla_W X) \\
&\quad - (n-1)(\alpha^2-\rho)\eta(Y)\eta(\nabla_W X)
\end{aligned}$$

$$
\begin{aligned}
&+ (n-1) W\left(\alpha^2-\rho\right)\eta(X)\eta(Y) \\
&+ (n-1)\left(\alpha^2-\rho\right) \\
&\times \{\eta(\nabla_W X)\eta(Y) + \alpha\eta(Y) g(X,\phi W) \\
&\quad +\eta(\nabla_W Y)\eta(X) + \alpha\eta(X) g(Y,\phi W)\} \\
= &-\alpha\eta(X) S(W,\phi Y) - \alpha\eta(Y) S(\phi X, W) \\
&+ \alpha(n-1)\left(\alpha^2-\rho\right) \\
&\times \{\eta(Y) g(X,\phi W) + \eta(X) g(Y,\phi W)\} \\
&+ (n-1) W\left(\alpha^2-\rho\right)\eta(X)\eta(Y).
\end{aligned} \tag{43}
$$

Thus we have the following theorem.

Theorem 7. *If an* $(\mathrm{LCS})_n$*-manifold* M *is Ricci-symmetric; then* $\alpha^2-\rho$ *is constant.*

Proof. If $n > 1$-dimensional $(\mathrm{LCS})_n$-manifold M is Ricci-symmetric, then from (43) we conclude that

$$
\begin{aligned}
&\alpha(n-1)\left(\alpha^2-\rho\right)\{\eta(Y) g(X,\phi W) + \eta(X) g(Y,\phi W)\} \\
&\quad + (n-1) W\left(\alpha^2-\rho\right)\eta(X)\eta(Y) \\
&\quad - \alpha\eta(X) S(W,\phi Y) - \alpha\eta(Y) S(\phi X, W) = 0.
\end{aligned} \tag{44}
$$

It follows that

$$
\begin{aligned}
&\alpha(n-1)\left(\alpha^2-\rho\right)\{g(X,\phi W)\xi - \eta(X)\phi W\} \\
&\quad + (n-1) W\left(\alpha^2-\rho\right)\eta(X)\xi \\
&\quad - \alpha\eta(X)\phi Q W - \alpha S(\phi X, W)\xi = 0,
\end{aligned} \tag{45}
$$

from which

$$
\begin{aligned}
&-\alpha(n-1)\left(\alpha^2-\rho\right) g(X,\phi W) \\
&\quad - (n-1) W\left(\alpha^2-\rho\right)\eta(X) + S(\phi X, W) = 0,
\end{aligned} \tag{46}
$$

which is equivalent to

$$
\begin{aligned}
&-\alpha(n-1)\left(\alpha^2-\rho\right)\phi W - (n-1) W\left(\alpha^2-\rho\right)\xi \\
&\quad + \alpha\phi Q W = 0,
\end{aligned} \tag{47}
$$

that is,

$$
W\left(\alpha^2-\rho\right) = 0, \tag{48}
$$

which proves our assertion. □

Since $\nabla R = 0$ implies that $\nabla S = 0$, we can give the following corollary.

Corollary 8. *If an n-dimensional* $(LCS)_n$*-manifold* M *is locally symmetric, then* $\alpha^2-\rho$ *is constant.*

Now, taking the covariant derivation of the both sides of (18) with respect to Y, we have

$$
YS(X,\xi) = (n-1) W\left[\left(\alpha^2-\rho\right)\eta(X)\right]. \tag{49}
$$

From the definition of the covariant derivation of Ricci-tensor, we have

$$
\begin{aligned}
(\nabla_Y S)(X,\xi) &= \nabla_Y S(X,\xi) - S(\nabla_Y X,\xi) - S(X,\nabla_Y\xi) \\
&= (n-1)\left\{Y\left(\alpha^2-\rho\right)\eta(X) + \left(\alpha^2-\rho\right)\right. \\
&\qquad\left.\times\left[\eta(\nabla_Y X) + \alpha g(X,\phi Y)\right]\right\} \\
&\quad - (n-1)\left(\alpha^2-\rho\right)\eta(\nabla_Y X) - \alpha S(X,\phi Y) \\
&= (n-1) Y\left(\alpha^2-\rho\right)\eta(X) \\
&\quad + \alpha(n-1)\left(\alpha^2-\rho\right) g(X,\phi Y) - \alpha S(X,\phi Y).
\end{aligned} \tag{50}
$$

If an $(LCS)_n$-manifold M Ricci symmetric, then Theorem 7 and (43) imply that

$$
S(X,\phi Y) = (n-1)\left(\alpha^2-\rho\right) g(\phi Y, X). \tag{51}
$$

This leads us to state the following.

Theorem 9. *If an* $(LCS)_n$*-manifold* M *is Ricci symmetric, then it is an Einstein manifold.*

Corollary 10. *If an* $(LCS)_n$*-manifold* M *is locally symmetric, then it is an Einstein manifold.*

In this section, an example is used to demonstrate that the method presented in this paper is effective. But this example is a special case of Example 6.1 of [6].

Example 11. Now, we consider the 3-dimensional manifold

$$
M = \left\{(x,y,z)\in\mathbb{R}^3, z\neq 0\right\}, \tag{52}
$$

where (x,y,z) denote the standard coordinates in $\mathbb{R}^3$. The vector fields

$$
\begin{gathered}
e_1 = e^z\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \qquad e_2 = e^z\frac{\partial}{\partial y}, \\
e_3 = \frac{\partial}{\partial z}
\end{gathered} \tag{53}
$$

are linearly independent of each point of M. Let g be the Lorentzian metric tensor defined by

$$
\begin{gathered}
g(e_1,e_1) = g(e_2,e_2) = -g(e_3,e_3) = 1, \\
g\left(e_i,e_j\right) = 0, \quad i\neq j,
\end{gathered} \tag{54}
$$

for $i, j = 1, 2, 3$. Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \Gamma(TM)$. Let ϕ be the (1,1)-tensor field defined by

$$\phi e_1 = e_1, \qquad \phi e_2 = e_2, \qquad \phi e_3 = 0. \tag{55}$$

Then using the linearity of ϕ and g, we have $\eta(e_3) = -1$,

$$\begin{gathered} \phi^2 Z = Z + \eta(Z) e_3, \\ g(\phi Z, \phi W) = g(Z, W) + \eta(Z) \eta(W), \end{gathered} \tag{56}$$

for all $Z, W \in \Gamma(TM)$. Thus for $\xi = e_3$, (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M.

Now, let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g, and let R be the Riemannian curvature tensor of g. Then we have

$$[e_1, e_2] = -e^z e_2, \qquad [e_1, e_3] = -e_1, \qquad [e_2, e_3] = -e_2. \tag{57}$$

Making use of the Koszul formulae for the Lorentzian metric tensor g, we can easily calculate the covariant derivations as follows:

$$\begin{gathered} \nabla_{e_1} e_1 = -e_3, \qquad \nabla_{e_2} e_1 = e^z e_2, \qquad \nabla_{e_1} e_3 = -e_1, \\ \nabla_{e_2} e_3 = -e_2, \qquad \nabla_{e_2} e_2 = -e^z e_1 - e_3, \\ \nabla_{e_1} e_2 = \nabla_{e_3} e_1 = \nabla_{e_3} e_2 = \nabla_{e_3} e_3 = 0. \end{gathered} \tag{58}$$

From the previously mentioned, it can be easily seen that (ϕ, ξ, η, g) is an $(\mathrm{LCS})_3$-structure on M, that is, M is an $(\mathrm{LCS})_3$-manifold with $\alpha = -1$ and $\rho = 0$. Using the previous relations, we can easily calculate the components of the Riemannian curvature tensor as follows:

$$\begin{gathered} R(e_1, e_2) e_1 = \left(e^{2z} - 1\right) e_2, \qquad R(e_1, e_2) e_2 = \left(1 - e^{2z}\right) e_1, \\ R(e_1, e_3) e_1 = -e_3, \qquad R(e_1, e_3) e_3 = -e_1, \\ R(e_2, e_3) e_2 = -e_3, \qquad R(e_2, e_3) e_3 = -e_2, \\ R(e_1, e_2) e_3 = R(e_1, e_3) e_2 = R(e_2, e_3) e_1 = 0. \end{gathered} \tag{59}$$

By using the properties of R and definition of the Ricci tensor, we obtain

$$\begin{gathered} S(e_1, e_1) = S(e_2, e_2) = -e^{2z}, \qquad S(e_3, e_3) = -2, \\ S(e_1, e_2) = S(e_1, e_3) = S(e_2, e_3) = 0. \end{gathered} \tag{60}$$

Thus the scalar curvature τ of M is given by

$$\tau = \sum_{i=1}^{3} g(e_i, e_i) S(e_i, e_i) = 2\left(1 - e^{2z}\right). \tag{61}$$

On the other hand, for any $Z, W \in \Gamma(TM)$, Z and W can be written as $Z = \sum_{i=1}^{3} f_i e_i$ and $W = \sum_{j=1}^{3} g_j e_j$, where f_i and g_i are smooth functions on M. By direct calculations, we have

$$\begin{aligned} S(Z, W) &= -e^{2z}(f_1 g_1 + f_2 g_2) - 2 f_3 g_3 \\ &= -e^{2z}(f_1 g_1 + f_2 g_2 - f_3 g_3) - f_3 g_3 \left(e^{2z} + 2\right). \end{aligned} \tag{62}$$

Since $\eta(Z) = -f_3$ and $\eta(W) = -g_3$ and $g(Z, W) = f_1 g_1 + f_2 g_2 - f_3 g_3$, we have

$$S(Z, W) = -e^{2z} g(Z, W) - \left(e^{2z} + 2\right) \eta(Z) \eta(W). \tag{63}$$

This tell us that M is an η-Einstein manifold.

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