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Research Article

Sequence Spaces Defined by Musielak-Orlicz Function over *n*-Normed Spaces

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In the present paper we introduce some sequence spaces over n-normed spaces defined by a Musielak-Orlicz function $\mathcal{M} = (M_k)$. We also study some topological properties and prove some inclusion relations between these spaces.

1. Introduction and Preliminaries

An Orlicz function M is a function, which is continuous, nondecreasing, and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Lindenstrauss and Tzafriri [1] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$; then

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}, \quad (1)$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}. \quad (2)$$

It is shown in [1] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p $(p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$ and for $L > 1$. A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function (see [2, 3]). A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v| u - M_k(u) : u \geq 0\}, \quad k = 1, 2, \ldots, \quad (3)$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows:

$$t_{\mathcal{M}} = \{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\},$$
$$h_{\mathcal{M}} = \{x \in w : I_{\mathcal{M}}(cx) < \infty \ \forall c > 0\}, \quad (4)$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} (M_k)(x_k), \quad x = (x_k) \in t_{\mathcal{M}}. \quad (5)$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$\|x\| = \inf\left\{ k > 0 : I_{\mathcal{M}}\left(\frac{x}{k}\right) \leq 1 \right\} \quad (6)$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf\left\{ \frac{1}{k}\left(1 + I_{\mathcal{M}}(kx)\right) : k > 0 \right\}. \quad (7)$$

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm if

(1) $p(x) \geq 0$ for all $x \in X$,
(2) $p(-x) = p(x)$ for all $x \in X$,

(3) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$,

(4) (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$; then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [4], Theorem 10.4.2, pp. 183). For more details about sequence spaces, see [5–12] and references therein.

A sequence of positive integers $\theta = (k_r)$ is called lacunary if $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r)$ and $q_r = k_r/k_{r-1}$. The space of lacunary strongly convergent sequences N_θ was defined by Freedman et al. [13] as

$$N_\theta = \left\{ x \in w : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left| x_k - l \right| = 0, \text{ for some } l \right\}. \quad (8)$$

Strongly almost convergent sequence was introduced and studied by Maddox [14] and Freedman et al. [13]. Parashar and Choudhary [15] have introduced and examined some properties of four sequence spaces defined by using an Orlicz function M, which generalized the well-known Orlicz sequence spaces $[C, 1, p]$, $[C, 1, p]_0$, and $[C, 1, p]_\infty$. It may be noted here that the space of strongly summable sequences was discussed by Maddox [16] and recently in [17].

Mursaleen and Noman [18] introduced the notion of λ-convergent and λ-bounded sequences as follows.

Let $\lambda = (\lambda_k)_{k=1}^{\infty}$ be a strictly increasing sequence of positive real numbers tending to infinity; that is,

$$0 < \lambda_0 < \lambda_1 < \cdots, \quad \lambda_k \longrightarrow \infty \quad \text{as } k \longrightarrow \infty, \quad (9)$$

and it is said that a sequence $x = (x_k) \in w$ is λ-convergent to the number L, called the λ-limit of x if $\Lambda_m(x) \to L$ as $m \to \infty$, where

$$\lambda_m(x) = \frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) x_k. \quad (10)$$

The sequence $x = (x_k) \in w$ is λ-bounded if $\sup_m |\Lambda_m(x)| < \infty$. It is well known [18] that if $\lim_m x_m = a$ in the ordinary sense of convergence, then

$$\lim_m \left(\frac{1}{\lambda_m} \left(\sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) \left| x_k - a \right| \right) \right) = 0. \quad (11)$$

This implies that

$$\lim_m \left| \Lambda_m(x) - a \right| = \lim_m \left| \frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1})(x_k - a) \right| = 0, \quad (12)$$

which yields that $\lim_m \Lambda_m(x) = a$ and hence $x = (x_k) \in w$ is λ-convergent to a.

The concept of 2-normed spaces was initially developed by Gähler [19] in the mid 1960s, while for that of n-normed spaces one can see Misiak [20]. Since then, many others have studied this concept and obtained various results; see Gunawan ([21, 22]) and Gunawan and Mashadi [23]. Let $n \in \mathbb{N}$ and let X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is the field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $\|\cdot, \ldots, \cdot\|$ on X^n satisfying the following four conditions

(1) $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent in X;

(2) $\|x_1, x_2, \ldots, x_n\|$ is invariant under permutation;

(3) $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \quad \|x_1, x_2, \ldots, x_n\|$ for any $\alpha \in \mathbb{K}$;

(4) $\|x + x', x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|x', x_2, \ldots, x_n\|$

is called an n-norm on X, and the pair $(X, \|\cdot, \ldots, \cdot\|)$ is called an n-normed space over the field $\mathbb{K}$.

For example, if we may take $X = \mathbb{R}^n$ being equipped with the n-norm $\|x_1, x_2, \ldots, x_n\|_E =$ the volume of the n-dimensional parallelepiped spanned by the vectors $x_1, x_2, \ldots, x_n$ which may be given explicitly by the formula

$$\|x_1, x_2, \ldots, x_n\|_E = \left| \det (x_{ij}) \right|, \quad (13)$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \ldots, n$, leting $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \ldots, a_n\}$ be linearly independent set in X, then the following function $\|\cdot, \ldots, \cdot\|_\infty$ on X^{n-1} defined by

$$\begin{aligned} &\|x_1, x_2, \ldots, x_{n-1}\|_\infty \\ &\quad = \max \left\{ \left\| x_1, x_2, \ldots, x_{n-1}, a_i \right\| : i = 1, 2, \ldots, n \right\} \end{aligned} \quad (14)$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \ldots, a_n\}$.

A sequence (x_k) in an n-normed space $(X, \|\cdot, \ldots, \cdot\|)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} \left\| x_k - L, z_1, \ldots, z_{n-1} \right\| = 0 \quad \text{for every } z_1, \ldots, z_{n-1} \in X. \quad (15)$$

A sequence (x_k) in an n-normed space $(X, \|\cdot, \ldots, \cdot\|)$ is said to be Cauchy if

$$\lim_{\substack{k\to\infty \\ p\to\infty}} \left\| x_k - x_p, z_1, \ldots, z_{n-1} \right\| = 0 \quad \text{for every } z_1, \ldots, z_{n-1} \in X. \quad (16)$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

Let $\mathscr{M} = (M_k)$ be a Musielak-Orlicz function, and let $p = (p_k)$ be a bounded sequence of positive real numbers. We define the following sequence spaces in the present paper:

$$\begin{aligned} &w_0^\theta(\mathscr{M},\Lambda,p,s,\|\cdot,\ldots,\cdot\|) \\ &\quad = \Big\{x=(x_k)\in w: \lim_{r\to\infty}\frac{1}{h_r} \\ &\qquad \times\sum_{k\in I_r}k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x)}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}=0, \\ &\qquad \rho>0,\ s\ge 0\Big\}, \\ &w^\theta(\mathscr{M},\Lambda,p,s,\|\cdot,\ldots,\cdot\|) \\ &\quad = \Big\{x=(x_k)\in w: \lim_{r\to\infty}\frac{1}{h_r} \\ &\qquad \times\sum_{k\in I_r}k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x)-L}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k} \\ &\qquad =0,\ \text{for some } L,\ \rho>0,\ s\ge 0\Big\}, \\ &w_\infty^\theta(\mathscr{M},\Lambda,p,s,\|\cdot,\ldots,\cdot\|) \\ &\quad = \Big\{x=(x_k)\in w: \sup_r\frac{1}{h_r} \\ &\qquad \times\sum_{k\in I_r}k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x)}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}<\infty, \\ &\qquad \rho>0,\ s\ge 0\Big\}. \end{aligned} \tag{17}$$

If we take $\mathscr{M}(x) = x$, we get

$$\begin{aligned} &w_0^\theta(\Lambda,p,s,\|\cdot,\ldots,\cdot\|) \\ &\quad = \Big\{x=(x_k)\in w: \lim_{r\to\infty}\frac{1}{h_r} \\ &\qquad \times\sum_{k\in I_r}k^{-s}\left[\left\|\frac{\Lambda_k(x)}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right]^{p_k}=0, \\ &\qquad \rho>0,\ s\ge 0\Big\}, \\ &w^\theta(\Lambda,p,s,\|\cdot,\ldots,\cdot\|) \\ &\quad = \Big\{x=(x_k)\in w: \lim_{r\to\infty}\frac{1}{h_r} \\ &\qquad \times\sum_{k\in I_r}k^{-s}\left[\left\|\frac{\Lambda_k(x)-L}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right]^{p_k}=0, \\ &\qquad \text{for some } L,\ \rho>0,\ s\ge 0\Big\}, \\ &w_\infty^\theta(\Lambda,p,s,\|\cdot,\ldots,\cdot\|) \\ &\quad = \Big\{x=(x_k)\in w: \sup_r\frac{1}{h_r} \\ &\qquad \times\sum_{k\in I_r}k^{-s}\left[\left\|\frac{\Lambda_k(x)}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right]^{p_k}<\infty, \\ &\qquad \rho>0,\ s\ge 0\Big\}. \end{aligned} \tag{18}$$

If we take $p = (p_k) = 1$ for all $k \in \mathbb{N}$, we have

$$\begin{aligned} &w_0^\theta(\mathscr{M},\Lambda,s,\|\cdot,\ldots,\cdot\|) \\ &\quad = \Big\{x=(x_k)\in w: \lim_{r\to\infty}\frac{1}{h_r} \\ &\qquad \times\sum_{k\in I_r}k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x)}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]=0, \\ &\qquad \rho>0,\ s\ge 0\Big\}, \\ &w^\theta(\mathscr{M},\Lambda,s,\|\cdot,\ldots,\cdot\|) \\ &\quad = \Big\{x=(x_k)\in w: \lim_{r\to\infty}\frac{1}{h_r} \\ &\qquad \times\sum_{k\in I_r}k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x)-L}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right] \\ &\qquad =0,\ \text{for some } L,\ \rho>0,\ s\ge 0\Big\}, \\ &w_\infty^\theta(\mathscr{M},\Lambda,s,\|\cdot,\ldots,\cdot\|) \\ &\quad = \Big\{x=(x_k)\in w: \sup_r\frac{1}{h_r} \\ &\qquad \times\sum_{k\in I_r}k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x)}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]<\infty, \\ &\qquad \rho>0,\ s\ge 0\Big\}. \end{aligned} \tag{19}$$

The following inequality will be used throughout the paper. If $0 \le p_k \le \sup p_k = H$, $K = \max(1, 2^{H-1})$, then

$$|a_k+b_k|^{p_k}\le K\left\{|a_k|^{p_k}+|b_k|^{p_k}\right\} \tag{20}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \le \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

In this paper, we introduce sequence spaces defined by a Musielak-Orlicz function over n-normed spaces. We study some topological properties and prove some inclusion relations between these spaces.

2. Main Results

Theorem 1. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, and let $p = (p_k)$ be a bounded sequence of positive real numbers, then the spaces $w_0^\theta(\mathcal{M}, \Lambda, p, s, \|\cdot, \ldots, \cdot\|)$, $w^\theta(\mathcal{M}, \Lambda, p, s, \|\cdot, \ldots, \cdot\|)$, and $w_\infty^\theta(\mathcal{M}, \Lambda, p, s, \|\cdot, \ldots, \cdot\|)$ are linear spaces over the field of complex number $\mathbb{C}$.*

Proof. Let $x = (x_k)$, let $y = (y_k) \in w_0^\theta(\mathcal{M}, \Lambda, p, s, \|\cdot, \ldots, \cdot\|)$, and let $\alpha, \beta \in \mathbb{C}$. In order to prove the result, we need to find some ρ_3 such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} k^{-s} \left[M_k \left(\left\| \frac{\Lambda_k(\alpha x + \beta y)}{\rho_3}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_k} = 0. \tag{21}$$

Since $x = (x_k)$, $y = (y_k) \in w_0^\theta(\mathcal{M}, \Lambda, p, s, \|\cdot, \ldots, \cdot\|)$, there exist positive numbers $\rho_1, \rho_2 > 0$ such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} k^{-s} \left[M_k \left(\left\| \frac{\Lambda_k(x)}{\rho_1}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_k} = 0,$$
$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} k^{-s} \left[M_k \left(\left\| \frac{\Lambda_k(y)}{\rho_2}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_k} = 0. \tag{22}$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since (M_k) is nondecreasing, convex function and by using inequality (20), we have

$$\begin{aligned}
&\frac{1}{h_r} \sum_{k\in I_r} k^{-s} \left[M_k \left(\left\| \frac{\Lambda_k(\alpha x + \beta y)}{\rho_3}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \\
&\le \frac{1}{h_r} \sum_{k\in I_r} k^{-s} \left[M_k \left(\left\| \frac{\alpha\Lambda_k(x)}{\rho_3}, z_1, z_2, \ldots, z_{n-1} \right\| + \left\| \frac{\beta\Lambda_k(y)}{\rho_3}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \\
&\le K \frac{1}{h_r} \sum_{k\in I_r} \frac{1}{2^{p_k}} k^{-s} \left[M_k \left(\left\| \frac{\Lambda_k(x)}{\rho_1}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \\
&\quad + K \frac{1}{h_r} \sum_{k\in I_r} \frac{1}{2^{p_k}} k^{-s} \left[M_k \left(\left\| \frac{\Lambda_k(y)}{\rho_2}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \\
&\le K \frac{1}{h_r} \sum_{k\in I_r} k^{-s} \left[M_k \left(\left\| \frac{\Lambda_k(x)}{\rho_1}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \\
&\quad + K \frac{1}{h_r} \sum_{k\in I_r} k^{-s} \left[M_k \left(\left\| \frac{\Lambda_k(y)}{\rho_2}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \\
&\longrightarrow 0 \text{ as } r \longrightarrow \infty.
\end{aligned} \tag{23}$$

Thus, we have $\alpha x + \beta y \in w_0^\theta(\mathcal{M}, \Lambda, p, s, \|\cdot, \ldots, \cdot\|)$. Hence, $w_0^\theta(\mathcal{M}, \Lambda, p, s, \|\cdot, \ldots, \cdot\|)$ is a linear space. Similarly, we can prove that $w^\theta(\mathcal{M}, \Lambda, p, s, \|\cdot, \ldots, \cdot\|)$ and $w_\infty^\theta(\mathcal{M}, \Lambda, p, s, \|\cdot, \ldots, \cdot\|)$ are linear spaces. □

Theorem 2. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, and let $p = (p_k)$ be a bounded sequence of positive real numbers. Then $w_0^\theta(\mathcal{M}, \Lambda, p, s, \|\cdot, \ldots, \cdot\|)$ is a topological linear space paranormed by*

$$g(x) = \inf \left\{ \rho^{p_r/H} : \left(\frac{1}{h_r} \sum_{k\in I_r} k^{-s} \left[M_k \left(\left\| \frac{\Lambda_k(x)}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \right)^{1/H} \le 1 \right\}, \tag{24}$$

where $H = \max(1, \sup_k p_k) < \infty$.

Proof. Clearly $g(x) \ge 0$ for $x = (x_k) \in w_0^\theta(\mathcal{M}, \Lambda, p, s, \|\cdot, \ldots, \cdot\|)$. Since $M_k(0) = 0$, we get $g(0) = 0$. Again if $g(x) = 0$, then

$$\inf \left\{ \rho^{p_r/H} : \left(\frac{1}{h_r} \sum_{k\in I_r} k^{-s} \left[M_k \left(\left\| \frac{\Lambda_k(x)}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \right)^{1/H} \le 1 \right\} = 0. \tag{25}$$

This implies that for a given $\epsilon > 0$, there exist some $\rho_\epsilon (0 < \rho_\epsilon < \epsilon)$ such that

$$\left(\frac{1}{h_r} \sum_{k\in I_r} k^{-s} \left[M_k \left(\left\| \frac{\Lambda_k(x)}{\rho_\epsilon}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \right)^{1/H} \le 1. \tag{26}$$

Thus,

$$\left(\frac{1}{h_r}\sum_{k\in I_r}k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x)}{\epsilon},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H} \leq \left(\frac{1}{h_r}\sum_{k\in I_r}k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x)}{\rho_\epsilon},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H}. \quad (27)$$

Suppose that $(x_k)\neq 0$ for each $k \in \mathbb{N}$. This implies that $\Lambda_k(x)\neq 0$ for each $k\in\mathbb{N}$. Let $\epsilon \to 0$, then

$$\left\|\frac{\Lambda_k(x)}{\epsilon},z_1,z_2,\ldots,z_{n-1}\right\| \longrightarrow \infty. \quad (28)$$

It follows that

$$\left(\frac{1}{h_r}\sum_{k\in I_r}k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x)}{\epsilon},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H} \longrightarrow \infty, \quad (29)$$

which is a contradiction. Therefore, $\Lambda_k(x) = 0$ for each k, and thus $(x_k) = 0$ for each $k \in \mathbb{N}$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be the case such that

$$\left(\frac{1}{h_r}\sum_{k\in I_r}k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x)}{\rho_1},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H} \leq 1,$$
$$\left(\frac{1}{h_r}\sum_{k\in I_r}k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(y)}{\rho_2},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H} \leq 1. \quad (30)$$

Let $\rho = \rho_1 + \rho_2$; then, by using Minkowski's inequality, we have

$$\begin{aligned}
&\left(\frac{1}{h_r}\sum_{k\in I_r}k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x+y)}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H}\\
&\leq \left(\frac{1}{h_r}\sum_{k\in I_r}k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x)+\Lambda_k(y)}{\rho_1+\rho_2},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H}\\
&\leq \left(\frac{1}{h_r}\sum_{k\in I_r}k^{-s}\left[M_k\left(\frac{\rho_1}{\rho_1+\rho_2}\right)\right.\right.\\
&\qquad\times\left[\left\|\frac{\Lambda_k(x)}{\rho_1},z_1,z_2,\ldots,z_{n-1}\right\|\right]+\left(\frac{\rho_2}{\rho_1+\rho_2}\right)\\
&\qquad\left.\left.\times\left[\left\|\frac{\Lambda_k(y)}{\rho_2},z_1,z_2,\ldots,z_{n-1}\right\|\right]\right]^{p_k}\right)^{1/H}\\
&\leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\\
&\quad\times\left(\frac{1}{h_r}\sum_{k\in I_r}k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x)}{\rho_1},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H}\\
&\quad+\left(\frac{\rho_2}{\rho_1+\rho_2}\right)\\
&\quad\times\left(\frac{1}{h_r}\sum_{k\in I_r}k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(y)}{\rho_2},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H}\\
&\leq 1.
\end{aligned} \quad (31)$$

Since ρ, ρ_1, and ρ_2 are nonnegative, we have

$$\begin{aligned}
&g(x+y)\\
&=\inf\left\{\rho^{p_r/H}:\left(\frac{1}{h_r}\sum_{k\in I_r}k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x+y)}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H}\leq 1\right\}\\
&\leq\inf\left\{(\rho_1)^{p_r/H}:\left(\frac{1}{h_r}\sum_{k\in I_r}k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x)}{\rho_1},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H}\leq 1\right\}
\end{aligned}$$

$$+\inf\left\{(\rho_2)^{p_r/H}:\left(\frac{1}{h_r}\sum_{k\in I_r}k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(y)}{\rho_2},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H}\leq 1\right\}. \tag{32}$$

Therefore, $g(x+y)\leq g(x)+g(y)$. Finally we prove that the scalar multiplication is continuous. Let μ be any complex number. By definition,

$$g(\mu x)=\inf\left\{\rho^{p_r/H}:\left(\frac{1}{h_r}\sum_{k\in I_r}k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(\mu x)}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H}\leq 1\right\}. \tag{33}$$

Thus,

$$g(\mu x)=\inf\left\{(|\mu|t)^{p_r/H}:\left(\frac{1}{h_r}\sum_{k\in I_r}k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x)}{t},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H}\leq 1\right\}, \tag{34}$$

where $1/t=\rho/|\mu|$. Since $|\mu|^{p_r}\leq\max(1,|\mu|^{\sup p_r})$, we have

$$g(\mu x)\leq\max\left(1,|\mu|^{\sup p_r}\right)\times\inf\left\{t^{p_r/H}:\left(\frac{1}{h_r}\sum_{k\in I_r}k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x)}{t},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H}\leq 1\right\}. \tag{35}$$

So the fact that scalar multiplication is continuous follows from the above inequality. This completes the proof of the theorem. □

Theorem 3. *Let $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function. If $\sup_k[M_k(x)]^{p_k}<\infty$ for all fixed $x>0$, then $w_0^\theta(\mathcal{M},\Lambda,p,s,\|\cdot,\ldots,\cdot\|)\subseteq w_\infty^\theta(\mathcal{M},\Lambda,p,s,\|\cdot,\ldots,\cdot\|)$.*

Proof. Let $x=(x_k)\in w_0^\theta(\mathcal{M},\Lambda,p,s,\|\cdot,\ldots,\cdot\|)$; then there exists positive number ρ_1 such that

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x)}{\rho_1},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}=0. \tag{36}$$

Define $\rho=2\rho_1$. Since (M_k) is nondecreasing and convex and by using inequality (20), we have

$$\begin{aligned}
&\sup_r\frac{1}{h_r}\sum_{k\in I_r}k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x)}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\\
&=\sup_r\frac{1}{h_r}\sum_{k\in I_r}k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x)+L-L}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\\
&\leq K\sup_r\frac{1}{h_r}\sum_{k\in I_r}k^{-s}\frac{1}{2^{p_k}}\left[M_k\left(\left\|\frac{\Lambda_k(x)-L}{\rho_1},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\\
&\quad+K\sup_r\frac{1}{h_r}\sum_{k\in I_r}k^{-s}\frac{1}{2^{p_k}}\left[M_k\left(\left\|\frac{L}{\rho_1},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\\
&\leq K\sup_r\frac{1}{h_r}\sum_{k\in I_r}k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x)-L}{\rho_1},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\\
&\quad+K\sup_r\frac{1}{h_r}\sum_{k\in I_r}k^{-s}\left[M_k\left(\left\|\frac{L}{\rho_1},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\\
&<\infty.
\end{aligned} \tag{37}$$

Hence, $x=(x_k)\in w_\infty^\theta(\mathcal{M},\Lambda,p,s,\|\cdot,\ldots,\cdot\|)$. □

Theorem 4. *Let* $0 < \inf p_k = h \le p_k \le \sup p_k = H < \infty$ *and let* $\mathcal{M} = (M_k)$, $\mathcal{M}' = (M_k')$ *be Musielak-Orlicz functions satisfying* Δ_2*-condition, then one has*

(i) $w_0^\theta(\mathcal{M}', \Lambda, p, s, \|\cdot, \dots, \cdot\|) \subset w_0^\theta(\mathcal{M} \circ \mathcal{M}', \Lambda, p, s, \|\cdot, \dots, \cdot\|)$;

(ii) $w^\theta(\mathcal{M}', \Lambda, p, s, \|\cdot, \dots, \cdot\|) \subset w^\theta(\mathcal{M} \circ \mathcal{M}', \Lambda, p, s, \|\cdot, \dots, \cdot\|)$;

(iii) $w_\infty^\theta(\mathcal{M}', \Lambda, p, s, \|\cdot, \dots, \cdot\|) \subset w_\infty^\theta(\mathcal{M} \circ \mathcal{M}', \Lambda, p, s, \|\cdot, \dots, \cdot\|)$.

Proof. Let $x = (x_k) \in w_0^\theta(\mathcal{M}', \Lambda, p, s, \|\cdot, \dots, \cdot\|)$, then we have

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} k^{-s} \left[M_k' \left(\left\| \frac{\Lambda_k(x)}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} = 0. \tag{38}$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_k(t) < \epsilon$ for $0 \le t \le \delta$. Let $(y_k) = M_k'[\|\Lambda_k(x)/\rho, z_1, z_2, \dots, z_{n-1}\|]$ for all $k \in \mathbb{N}$. We can write

$$\frac{1}{h_r} \sum_{k\in I_r} k^{-s} (M_k[y_k])^{p_k} = \frac{1}{h_r} \sum_{\substack{k\in I_r \\ y_k \le \delta}} k^{-s} (M_k[y_k])^{p_k} + \frac{1}{h_r} \sum_{\substack{k\in I_r \\ y_k \ge \delta}} k^{-s} (M_k[y_k])^{p_k}. \tag{39}$$

So, we have

$$\begin{aligned} \frac{1}{h_r} \sum_{\substack{k\in I_r \\ y_k \le \delta}} k^{-s} (M_k[y_k])^{p_k} &\le [M_k(1)]^H \frac{1}{h_r} \sum_{\substack{k\in I_r \\ y_k \le \delta}} k^{-s} (M_k[y_k])^{p_k} \\ &\le [M_k(2)]^H \frac{1}{h_r} \sum_{\substack{k\in I_r \\ y_k \le \delta}} k^{-s} (M_k[y_k])^{p_k}. \end{aligned} \tag{40}$$

For $y_k > \delta$, $y_k < y_k/\delta < 1 + y_k/\delta$. Since $(M_k)'s$ are nondecreasing and convex, it follows that

$$M_k(y_k) < M_k\left(1 + \frac{y_k}{\delta}\right) < \frac{1}{2} M_k(2) + \frac{1}{2} M_k\left(\frac{2y_k}{\delta}\right). \tag{41}$$

Since $\mathcal{M} = (M_k)$ satisfies Δ_2-condition, we can write

$$M_k(y_k) < \frac{1}{2} T \frac{y_k}{\delta} M_k(2) + \frac{1}{2} T \frac{y_k}{\delta} M_k(2) = T \frac{y_k}{\delta} M_k(2). \tag{42}$$

Hence,

$$\begin{aligned} &\frac{1}{h_r} \sum_{\substack{k\in I_r \\ y_k \ge \delta}} k^{-s} M_k[y_k]^{p_k} \\ &\quad \le \max\left(1, \left(T \frac{M_k(2)}{\delta}\right)^H\right) \frac{1}{h_r} \sum_{\substack{k\in I_r \\ y_k \le \delta}} k^{-s} [y_k]^{p_k}. \end{aligned} \tag{43}$$

From (40) and (43), we have $x = (x_k) \in w_0^\theta(\mathcal{M} \circ \mathcal{M}', \Lambda, p, s, \|\cdot, \dots, \cdot\|)$. This completes the proof of (i). Similarly we can prove that

$$\begin{aligned} &w^\theta(\mathcal{M}', \Lambda, p, s, \|\cdot, \dots, \cdot\|) \\ &\quad \subset w^\theta(\mathcal{M} \circ \mathcal{M}', \Lambda, p, s, \|\cdot, \dots, \cdot\|), \\ &w_\infty^\theta(\mathcal{M}', \Lambda, p, s, \|\cdot, \dots, \cdot\|) \\ &\quad \subset w_\infty^\theta(\mathcal{M} \circ \mathcal{M}', \Lambda, p, s, \|\cdot, \dots, \cdot\|). \end{aligned} \tag{44}$$

□

Theorem 5. *Let* $0 < h = \inf p_k = p_k < \sup p_k = H < \infty$. *Then for a Musielak-Orlicz function* $\mathcal{M} = (M_k)$ *which satisfies* Δ_2*-condition, one has*

(i) $w_0^\theta(\Lambda, p, s, \|\cdot, \dots, \cdot\|) \subset w_0^\theta(\mathcal{M}, \Lambda, p, s, \|\cdot, \dots, \cdot\|)$;

(ii) $w^\theta(\Lambda, p, s, \|\cdot, \dots, \cdot\|) \subset w^\theta(\mathcal{M}, \Lambda, p, s, \|\cdot, \dots, \cdot\|)$;

(iii) $w_\infty^\theta(\Lambda, p, s, \|\cdot, \dots, \cdot\|) \subset w_\infty^\theta(\mathcal{M}, \Lambda, p, s, \|\cdot, \dots, \cdot\|)$.

Proof. It is easy to prove, so we omit the details. □

Theorem 6. *Let* $\mathcal{M} = (M_k)$ *be a Musielak-Orlicz function and let* $0 < h = \inf p_k$. *Then* $w_\infty^\theta(\mathcal{M}, \Lambda, p, s, \|\cdot, \dots, \cdot\|) \subset w_0^\theta(\Lambda, p, s, \|\cdot, \dots, \cdot\|)$ *if and only if*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} k^{-s} (M_k(t))^{p_k} = \infty \tag{45}$$

for some $t > 0$.

Proof. Let $w_\infty^\theta(\mathcal{M}, \Lambda, p, s, \|\cdot, \dots, \cdot\|) \subset w_0^\theta(\Lambda, p, s, \|\cdot, \dots, \cdot\|)$. Suppose that (45) does not hold. Therefore, there are subinterval $I_{r(j)}$ of the set of interval I_r and a number $t_0 > 0$, where

$$t_0 = \left\| \frac{\Lambda_k(x)}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \quad \forall k, \tag{46}$$

such that

$$\frac{1}{h_{r(j)}} = \sum_{k\in I_{r(j)}} k^{-s} (M_k(t_0))^{p_k} \le K < \infty, \quad m = 1, 2, 3, \dots. \tag{47}$$

Let us define $x = (x_k)$ as follows:

$$\Lambda_k(x) = \begin{cases} \rho t_0, & k \in I_{r(j)} \\ 0, & k \notin I_{r(j)}. \end{cases} \tag{48}$$

Thus, by (47), $x \in w_\infty^\theta(\mathcal{M}, \Lambda, p, s, \|\cdot, \dots, \cdot\|)$. But $x \notin w_\infty^0(\Lambda, p, s, \|\cdot, \dots, \cdot\|)$. Hence, (45) must hold.

Conversely, suppose that (45) holds and let $x \in w_\infty^\theta(\mathcal{M}, \Lambda, p, s, \|\cdot, \dots, \cdot\|)$. Then for each r,

$$\frac{1}{h_r} \sum_{k\in I_r} k^{-s} \left[M_k \left(\left\| \frac{\Lambda_k(x)}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \le K < \infty. \tag{49}$$

Suppose that $x \notin w_0^\theta(\Lambda, p, s, \|\cdot, \dots, \cdot\|)$. Then for some number $\epsilon > 0$, there is a number k_0 such that for a subinterval $I_{r(j)}$, of the set of interval I_r,

$$\left\| \frac{\Lambda_k(x)}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| > \epsilon \quad \text{for } k \geq k_0. \tag{50}$$

From properties of sequence of Orlicz functions, we obtain

$$\left[M_k \left(\left\| \frac{\Lambda_k(x)}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq M_k(\epsilon)^{p_k}, \tag{51}$$

which contradicts (45), by using (49). Hence, we get

$$w_\infty^\theta (\mathscr{M}, \Lambda, p, s, \|\cdot, \dots, \cdot\|) \subset w_0^\theta (\Lambda, p, s, \|\cdot, \dots, \cdot\|). \tag{52}$$

This completes the proof. □

Theorem 7. *Let $\mathscr{M} = (M_k)$ be a Musielak-Orlicz function. Then the following statements are equivalent:*

(i) $w_\infty^\theta(\Lambda, p, s, \|\cdot, \dots, \cdot\|) \subset w_\infty^\theta(\mathscr{M}, \Lambda, p, s, \|\cdot, \dots, \cdot\|)$;

(ii) $w_0^\theta(\Lambda, p, s, \|\cdot, \dots, \cdot\|) \subset w_\infty^\theta(\mathscr{M}, \Lambda, p, s, \|\cdot, \dots, \cdot\|)$;

(iii) $\sup_r 1/h_r \sum_{k \in I_r} k^{-s}(M_k(t))^{p_k} < \infty$ *for all* $t > 0$.

Proof. (i) ⇒ (ii). Let (i) hold. To verify (ii), it is enough to prove

$$w_0^\theta (\Lambda, p, s, \|\cdot, \dots, \cdot\|) \subset w_\infty^\theta (\mathscr{M}, \Lambda, p, s, \|\cdot, \dots, \cdot\|). \tag{53}$$

Let $x = (x_k) \in w_0^\theta(\Lambda, p, s, \|\cdot, \dots, \cdot\|)$. Then for $\epsilon > 0$, there exists $r \geq 0$, such that

$$\frac{1}{h_r} \sum_{k \in I_r} k^{-s} \left[\left\| \frac{\Lambda_k(x)}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right]^{p_k} < \epsilon. \tag{54}$$

Hence, there exists $K > 0$ such that

$$\sup_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} \left[\left\| \frac{\Lambda_k(x)}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right]^{p_k} < K. \tag{55}$$

So, we get $x = (x_k) \in w_\infty^\theta(\mathscr{M}, \Lambda, p, s, \|\cdot, \dots, \cdot\|)$.

(ii) ⇒ (iii). Let (ii) hold. Suppose (iii) does not hold. Then for some $t > 0$

$$\sup_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} (M_k(t))^{p_k} = \infty, \tag{56}$$

and therefore we can find a subinterval $I_{r(j)}$, of the set of interval I_r, such that

$$\frac{1}{h_{r(j)}} \sum_{k \in I_{r(j)}} k^{-s} \left(M_k \left(\frac{1}{j} \right) \right)^{p_k} > j, \qquad j = 1, 2, 3, \dots \tag{57}$$

Let us define $x = (x_k)$ as follows:

$$\Lambda_k(x) = \begin{cases} \dfrac{\rho}{j}, & k \in I_{r(j)} \\ 0, & k \notin I_{r(j)}. \end{cases} \tag{58}$$

Then $x = (x_k) \in w_0^\theta(\Lambda, p, s, \|\cdot, \dots, \cdot\|)$. But by (57), $x \notin w_\infty^\theta(\mathscr{M}, \Lambda, p, s, \|\cdot, \dots, \cdot\|)$, which contradicts (ii). Hence, (iii) must holds.

(iii) ⇒ (i). Let (iii) hold and suppose that $x = (x_k) \in w_\infty^\theta(\Lambda, p, s, \|\cdot, \dots, \cdot\|)$. Suppose that $x = (x_k) \notin w_\infty^\theta(\mathscr{M}, \Lambda, p, s, \|\cdot, \dots, \cdot\|)$; then

$$\sup_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} \left[M_k \left(\left\| \frac{\Lambda_k(x)}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} = \infty. \tag{59}$$

Let $t = \|\Lambda_k(x)/\rho, z_1, z_2, \dots, z_{n-1}\|$ for each k; then by (59),

$$\sup_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} (M_k(t))^{p_k} = \infty, \tag{60}$$

which contradicts (iii). Hence, (i) must hold. □

Theorem 8. *Let $\mathscr{M} = (M_k)$ be a Musielak-Orlicz function. Then the following statements are equivalent:*

(i) $w_0^\theta(\mathscr{M}, \Lambda, p, s, \|\cdot, \dots, \cdot\|) \subset w_0^\theta(\Lambda, p, s, \|\cdot, \dots, \cdot\|)$;

(ii) $w_0^\theta(\mathscr{M}, \Lambda, p, s, \|\cdot, \dots, \cdot\|) \subset w_\infty^\theta(\Lambda, p, s, \|\cdot, \dots, \cdot\|)$;

(iii) $\inf_r 1/h_r \sum_{k \in I_r} k^{-s}(M_k(t))^{p_k} > 0$ *for all* $t > 0$.

Proof. (i) ⇒ (ii). It is obvious.

(ii) ⇒ (iii). Let (ii) hold. Suppose that (iii) does not hold. Then

$$\inf_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} (M_k(t))^{p_k} = 0 \quad \text{for some } t > 0, \tag{61}$$

and we can find a subinterval $I_{r(j)}$, of the set of interval I_r, such that

$$\frac{1}{h_{r(j)}} \sum_{k \in I_{r(j)}} k^{-s} (M_k(j))^{p_k} < \frac{1}{j}, \qquad j = 1, 2, 3, \dots \tag{62}$$

Let us define $x = (x_k)$ as follows:

$$\Lambda_k(x) = \begin{cases} \rho j, & k \in I_{r(j)} \\ 0, & k \notin I_{r(j)}. \end{cases} \tag{63}$$

Thus, by (62), $x = (x_k) \in w_0^\theta(\mathscr{M}, \Lambda, p, s, \|\cdot, \dots, \cdot\|)$, but $x = (x_k) \notin w_\infty^\theta(\Lambda, p, s, \|\cdot, \dots, \cdot\|)$, which contradicts (ii). Hence, (iii) must hold.

(iii) ⇒ (i). Let (iii) hold. Suppose that $x = (x_k) \in w_0^\theta(\mathscr{M}, \Lambda, p, s, \|\cdot, \dots, \cdot\|)$. Then

$$\frac{1}{h_r} \sum_{k \in I_r} k^{-s} \left[M_k \left(\left\| \frac{\Lambda_k(x)}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_k} \longrightarrow 0 \quad \text{as} \quad r \longrightarrow \infty. \tag{64}$$

Again suppose that $x = (x_k) \notin w_0^\theta(\Lambda, p, s, \|\cdot, \dots, \cdot\|)$; for some number $\epsilon > 0$ and a subinterval $I_{r(j)}$, of the set of interval I_r, we have

$$\left\| \frac{\Lambda_k(x)}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \geq \epsilon \quad \forall k. \tag{65}$$

Then from properties of the Orlicz function, we can write

$$\left[M_k\left(\left\|\frac{\Lambda_k(x)}{\rho}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k} \geq (M_k(\epsilon))^{p_k}. \quad (66)$$

Consequently, by (64), we have

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} k^{-s}(M_k(\epsilon))^{p_k} = 0, \quad (67)$$

which contradicts (iii). Hence, (i) must hold. □

Theorem 9. *Let $0 \leq p_k \leq q_k$ for all k and let (q_k/p_k) be bounded. Then*

$$w^\theta(\mathscr{M}, \Lambda, q, s, \|\cdot, \ldots, \cdot\|) \subseteq w^\theta(\mathscr{M}, \Lambda, p, s, \|\cdot, \ldots, \cdot\|). \quad (68)$$

Proof. Let $x = (x_k) \in w^\theta(\mathscr{M}, \Lambda, q, s, \|\cdot, \ldots, \cdot\|)$; write

$$t_k = \left[M_k\left(\left\|\frac{\Lambda_k(x)}{\rho}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{q_k} \quad (69)$$

and $\mu_k = p_k/q_k$ for all $k \in \mathbb{N}$. Then $0 < \mu_k \leq 1$ for all $k \in \mathbb{N}$. Take $0 < \mu \leq \mu_k$ for $k \in \mathbb{N}$. Define sequences (u_k) and (v_k) as follows.

For $t_k \geq 1$, let $u_k = t_k$ and $v_k = 0$, and for $t_k < 1$, let $u_k = 0$ and $v_k = t_k$. Then clearly for all $k \in \mathbb{N}$, we have

$$t_k = u_k + v_k, \qquad t_k^{\mu_k} = u_k^{\mu_k} + v_k^{\mu_k}. \quad (70)$$

Now it follows that $u_k^{\mu_k} \leq u_k \leq t_k$ and $v_k^{\mu_k} \leq v_k^{\mu}$. Therefore,

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} t_k^{\mu_k} &= \frac{1}{h_r}\sum_{k\in I_r}\left(u_k^{\mu_k} + v_k^{\mu_k}\right)\\ &\leq \frac{1}{h_r}\sum_{k\in I_r} t_k + \frac{1}{h_r}\sum_{k\in I_r} v_k^{\mu}.\end{aligned} \quad (71)$$

Now for each k,

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} v_k^{\mu} &= \sum_{k\in I_r}\left(\frac{1}{h_r}v_k\right)^{\mu}\left(\frac{1}{h_r}\right)^{1-\mu}\\ &\leq \left(\sum_{k\in I_r}\left[\left(\frac{1}{h_r}v_k\right)^{\mu}\right]^{1/\mu}\right)^{\mu}\\ &\quad\times\left(\sum_{k\in I_r}\left[\left(\frac{1}{h_r}\right)^{1-\mu}\right]^{1/(1-\mu)}\right)^{1-\mu}\\ &= \left(\frac{1}{h_r}\sum_{k\in I_r} v_k\right)^{\mu},\end{aligned} \quad (72)$$

and so

$$\frac{1}{h_r}\sum_{k\in I_r} v_k^{\mu} \leq \frac{1}{h_r}\sum_{k\in I_r} t_k + \left(\frac{1}{h_r}\sum_{k\in I_r} v_k\right)^{\mu}. \quad (73)$$

Hence, $x = (x_k) \in w^\theta(\mathscr{M}, \Lambda, p, s, \|\cdot, \ldots, \cdot\|)$. This completes the proof of the theorem. □

Theorem 10. (i) *If $0 < \inf p_k \leq p_k \leq 1$ for all $k \in \mathbb{N}$, then*

$$w^\theta(\mathscr{M}, \Lambda, p, s, \|\cdot, \ldots, \cdot\|) \subseteq w^\theta(\mathscr{M}, \Lambda, s, \|\cdot, \ldots, \cdot\|). \quad (74)$$

(ii) *If $1 \leq p_k \leq \sup p_k = H < \infty$, for all $k \in \mathbb{N}$, then*

$$w^\theta(\mathscr{M}, \Lambda, s, \|\cdot, \ldots, \cdot\|) \subseteq w^\theta(\mathscr{M}, \Lambda, p, s, \|\cdot, \ldots, \cdot\|). \quad (75)$$

Proof. (i) Let $x = (x_k) \in w^\theta(\mathscr{M}, \Lambda, p, s, \|\cdot, \ldots, \cdot\|)$; then

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x) - L}{\rho}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k} = 0. \quad (76)$$

Since $0 < \inf p_k \leq p_k \leq 1$, this implies that

$$\begin{aligned}&\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x) - L}{\rho}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]\\ &\quad\leq \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x) - L}{\rho}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k};\end{aligned} \quad (77)$$

therefore,

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x) - L}{\rho}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right] = 0. \quad (78)$$

Hence,

$$w^\theta(\mathscr{M}, \Lambda, p, s, \|\cdot, \ldots, \cdot\|) \subseteq w^\theta(\mathscr{M}, \Lambda, s, \|\cdot, \ldots, \cdot\|). \quad (79)$$

(ii) Let $p_k \geq 1$ for each k and $\sup p_k < \infty$. Let $x = (x_k) \in w^\theta(\mathscr{M}, \Lambda, s, \|\cdot, \ldots, \cdot\|)$; then for each $\rho > 0$, we have

$$\begin{aligned}&\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x) - L}{\rho},\right.\right.\right.\\ &\qquad\left.\left.\left. z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k} = 0 < 1.\end{aligned} \quad (80)$$

Since $1 \leq p_k \leq \sup p_k < \infty$, we have

$$\begin{aligned}&\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x) - L}{\rho}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k}\\ &\quad\leq \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} k^{-s}\left[M_k\left(\left\|\frac{\Lambda_k(x) - L}{\rho}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]\\ &\quad= 0\\ &\quad< 1.\end{aligned} \quad (81)$$

Therefore, $x = (x_k) \in w^\theta(\mathscr{M}, \Lambda, p, s, \|\cdot, \ldots, \cdot\|)$, for each $\rho > 0$. Hence,

$$w^\theta(\mathscr{M}, \Lambda, s, \|\cdot, \ldots, \cdot\|) \subseteq w^\theta(\mathscr{M}, \Lambda, p, s, \|\cdot, \ldots, \cdot\|). \quad (82)$$

This completes the proof of the theorem. □

Theorem 11. *If* $0 < \inf p_k \leq p_k \leq \sup p_k = H < \infty$*, for all* $k \in \mathbb{N}$*, then*

$$w^{\theta}(\mathscr{M}, \Lambda, p, s, \|\cdot, \ldots, \cdot\|) = w^{\theta}(\mathscr{M}, \Lambda, s, \|\cdot, \ldots, \cdot\|). \quad (83)$$

Proof. It is easy to prove so we omit the details. □

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