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Research Article

On the q-Bernstein Polynomials of Unbounded Functions with $q > 1$

Sofiya Ostrovska and Ahmet Yaşar Özban

Department of Mathematics, Atilim University, Incek, 06836 Ankara, Turkey

Correspondence should be addressed to Sofiya Ostrovska; ostrovsk@atilim.edu.tr

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The aim of this paper is to present new results related to the q-Bernstein polynomials $B_{n,q}(f;x)$ of unbounded functions in the case $q > 1$ and to illustrate those results using numerical examples. As a model, the behavior of polynomials $B_{n,q}(f;x)$ is examined both theoretically and numerically in detail for functions on $[0,1]$ satisfying $f(x) \sim Kx^{-\alpha}$ as $x \to 0^+$, where $\alpha > 0$ and $K \neq 0$ are real numbers.

1. Introduction

In 1912, precisely a century ago, Bernstein [1] published his famous proof of the Weierstrass approximation theorem by introducing polynomials, known today as the *Bernstein polynomials*. Later, it was found that these polynomials possess many remarkable properties, which made them an area of intensive research with wide range of applications. See, for example [2, 3]. The importance of the Bernstein polynomials led to the discovery of their numerous generalizations aimed to provide appropriate tools for various areas of mathematics, such as approximation theory, computer-aided geometric design, and the statistical inference.

Due to the speedy development of the q-calculus, recent generalizations based on the q-integers have emerged. A. Lupaş was the person who pioneered the work on the q-versions of the Bernstein polynomials. In 1987, he introduced (cf. [4]) a q-analogue of the Bernstein operator and investigated its approximation and shape-preserving properties. See also [5]. Subsequently, another generalization, called the q-Bernstein polynomials, was brought into the spotlight by Phillips [6] and was studied afterwards by a number of authors from different perspectives.

To define these polynomials, let us recall some notions related to the q-calculus. See, for example, [7], Ch. 10. Let $q > 0$. For any nonnegative integer k, the *q-integer* $[k]_q$ is defined by

$$[k]_q := 1 + q + \cdots + q^{k-1} \quad (k = 1, 2, \ldots), \ [0]_q := 0, \tag{1}$$

and the *q-factorial* $[k]_q!$ is defined by

$$[k]_q! := [1]_q[2]_q \cdots [k]_q \quad (k = 1, 2, \ldots), \ [0]_q! := 1. \tag{2}$$

For integers k, n with $0 \le k \le n$, the *q-binomial* coefficient $\left[{n \atop k} \right]_q$ is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_g!}{[k]_q![n-k]_q!}. \tag{3}$$

We also use the following standard notations:

$$(a;q)_0 := 1, \qquad (a;q)_k := \prod_{s=0}^{k-1} (1 - aq^s),$$
$$(a;q)_\infty := \prod_{s=0}^{\infty} (1 - aq^s). \tag{4}$$

Definition 1. Let $f : [0,1] \to \mathbb{R}$. The *q-Bernstein polynomial* of f is

$$B_{n,q}(f;x) := \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) p_{nk}(q;x), \quad n = 1, 2, \ldots, \tag{5}$$

where the *q-Bernstein basic polynomials* $p_{nk}(q;x)$ are given by

$$p_{nk}(q;x) := \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (x;q)_{n-k}, \quad k = 0, 1, \ldots n. \tag{6}$$

Polynomials $p_{n0}(q;x), p_{n1}(q;x), \ldots$, and $p_{nn}(q;x)$ form the *q-Bernstein basis* in the linear space of the polynomials of degree at most n.

Note that for $q = 1$, $B_{n,q}(f;x)$ is the classical Bernstein polynomial $B_n(f;x)$. Conventionally, the name *q-Bernstein polynomials* is reserved for the case $q \neq 1$.

Over the past years, the q-Bernstein polynomials have remained under intensive study, and new researches concerning not only the properties of the q-Bernstein polynomials, but also their various generalizations are constantly coming out (see, e.g., papers [8–14]). A detailed review of the results on the q-Bernstein polynomials along with the extensive bibliography has been provided in [10].

The popularity of the q-Bernstein polynomials is attributed to the fact that they are closely related to the q-binomial and the q-deformed Poisson probability distributions; see [15]. The q-binomial distribution plays an important role in the q-boson theory, providing a q-deformation for the quantum harmonic formalism. More specifically, it has been used to construct the binomial state for the q-boson. Meanwhile, its limit form called the q-deformed Poisson distribution defines the distribution of energy in a q-analogue of the coherent state; see [16].

It has been known that the q-Bernstein polynomials inherit some of the properties of the classical Bernstein polynomials. For example, they possess the following endpoint interpolation property:

$$B_{n,q}(f;0) = f(0), \quad B_{n,q}(f;1) = f(1), \\ n = 1, 2, \ldots, \quad q > 0, \tag{7}$$

and have linear functions as their fixed points:

$$B_{n,q}(ax+b;x) = ax+b, \quad n = 1, 2, \ldots, \quad q > 0. \tag{8}$$

The latter follows from the identity:

$$\sum_{k=0}^{n} p_{nk}(q;x) = 1 \quad \forall n = 1, 2, \ldots, \text{and all } q > 0. \tag{9}$$

Nevertheless, the convergence properties of the q-Bernstein polynomials for $q \neq 1$ are essentially *different* from those of the classical ones. What is more, the cases $0 < q < 1$ and $q > 1$ in terms of convergence are not similar to each other. See, for example [10]. This lack of similarity stems from the fact that while for $0 < q < 1$, the q-Bernstein operators given by

$$B_{n,q} : f \longmapsto B_{n,q}(f;\cdot) \tag{10}$$

are *positive* linear operators on $C[0,1]$; this is no longer valid for $q > 1$. In addition, the case $q > 1$ is aggravated by the rather irregular behavior of basic polynomials (6), which, in this case, combine the fast increase in magnitude with the sign oscillations. For details see [17], where it has been shown that the norm $\|B_{n,q}\|$ increases rather rapidly in both n and q, namely,

$$\|B_{n,q}\| \sim \frac{2}{e} \cdot \frac{q^{(n(n-1)/2)}}{n} \quad \text{as } n \longrightarrow \infty,\ q \longrightarrow +\infty. \tag{11}$$

In the present paper, the q-Bernstein polynomials in the case $q > 1$ are studied for unbounded functions—a problem which has not been considered before. The approximation of unbounded functions by the classical Bernstein polynomials was investigated by Lorentz in [2] and, recently, by Weba in [18].

Throughout the paper, $q > 1$ is assumed to be fixed. In presenting the results, the notation $\mathbb{J}_q$ is used for the time scale:

$$\mathbb{J}_q := \{0\} \cup \left\{q^{-j}\right\}_{j=0}^{\infty}. \tag{12}$$

First, we prove that for a certain class of unbounded functions on $[0,1]$, their sequence of the q-Bernstein polynomials is approximating on $\mathbb{J}_q$. Further, the behavior of $B_{n,q}(f;x)$ has been considered for functions $f : [0,1] \to \mathbb{R}$ which are continuous on $(0,1]$ and satisfy

$$f(x) \sim Kx^{-\alpha} \quad \text{as } x \longrightarrow 0^+, \tag{13}$$

where $\alpha > 0$ and $K \in \mathbb{R} \setminus \{0\}$. Previously, the q-Bernstein polynomials with $q > 1$ of the power functions x^α, where $\alpha > 0$, were studied in [19]. There, it was proved that in the case $\alpha > 0$, $B_{n,q}(x^\alpha;x) \to x^\alpha$ uniformly on $[0,1]$ if and only if $\alpha \in \mathbb{N}$.

Numerical examples are used both to illustrate Theorems 3 and 6 and also to discuss the significance of the assumptions therein. All the numerical results have been calculated in a Maple 8 environment.

2. The q-Bernstein Polynomials of Unbounded Functions

It has been known that for $q > 1$, all functions continuous on $[0,1]$ are approximated by their q-Bernstein polynomials on the time scale $\mathbb{J}_q$. Here, it will be proved that this fact remains true for functions which are continuous from the left at all points of $\mathbb{J}_q$.

For $f : [0,1] \to \mathbb{R}$, $j \in \mathbb{Z}_+$ and $t \in [0, q^{-j}]$, we set

$$\Omega_{f,q^{-j}}(t) = \sup_{x \in [q^{-j}-t,\, q^{-j}]} \left| f(x) - f\left(q^{-j}\right)\right|. \tag{14}$$

Lemma 2. *Let $f : [0,1] \to \mathbb{R}$ be bounded on $[q^{-j}, 1]$, where $j \in \mathbb{N}$, and let $M_j = \sup_{x\in[q^{-j},1]}|f(x)|$. Then, for n large enough, the following estimate holds:*

$$\left|B_{n,q}\left(f;q^{-j}\right) - f\left(q^{-j}\right)\right| \leq \Omega_{f,q^{-j}}\left(\frac{1}{q^n - 1}\right) + \frac{2M_j[j]_q}{q^n}. \tag{15}$$

Proof. For $j = 0$, the statement is obvious due to the endpoint interpolation property (7). Therefore, it is assumed that $j \geq 1$.

First, it should be noticed that $p_{n,n-k}(q;q^{-j}) \geq 0$ with $p_{n,n-k}(q;q^{-j}) = 0$ for $k > j$, and that, by virtue of (9), $\sum_{k=0}^{j} p_{n,n-k}(q;q^{-j}) = 1$. Then, for $n > j$, the following equality is true:

$$B_{n,q}\left(f;q^{-j}\right) = \sum_{k=0}^{j} f\left(\frac{[n-k]_q}{[n]_q}\right) p_{n,n-k}\left(q;q^{-j}\right). \tag{16}$$

Then

$$\begin{aligned} &\left|B_{n,q}\left(f;q^{-j}\right) - f\left(q^{-j}\right)\right| \\ &\quad\leq \sum_{k=0}^{j}\left|f\left(\frac{[n-k]_q}{[n]_q}\right) - f\left(q^{-j}\right)\right| p_{n,n-k}\left(q;q^{-j}\right) \\ &\quad\leq \left|f\left(\frac{[n-j]_q}{[n]_q}\right) - f\left(q^{-j}\right)\right| \cdot p_{n,n-j}\left(q;q^{-j}\right) \\ &\qquad + 2M_j \cdot \sum_{k=0}^{j-1} p_{n,n-k}\left(q;q^{-j}\right) \end{aligned} \tag{17}$$

for n large enough to satisfy $[n-j+1]_q/[n]_q > q^{-j}$. Since

$$\begin{gathered} \left|\frac{[n-j]_q}{[n]_q} - q^{-j}\right| = \frac{q^j - 1}{q^j\left(q^n - 1\right)} \leq \frac{1}{q^n - 1}, \\ p_{n,n-j}\left(q;q^{-j}\right) = \left(1 - \frac{1}{q^n}\right)\cdots\left(1 - \frac{1}{q^{n-j+1}}\right) \leq 1, \end{gathered} \tag{18}$$

the first term can be estimated as follows:

$$\begin{aligned} &\left|f\left(\frac{[n-k]_q}{[n]_q}\right) - f\left(q^{-j}\right)\right| \cdot p_{n,n-j}\left(q;q^{-j}\right) \\ &\quad\leq \Omega_{f,q^{-j}}\left(\frac{1}{q^n - 1}\right). \end{aligned} \tag{19}$$

To estimate the second term, one can write

$$\begin{aligned} \sum_{k=0}^{j-1} p_{n,n-k}\left(q;q^{-j}\right) &= 1 - p_{n,n-j}\left(q;q^{-j}\right) \\ &= 1 - \left(1 - \frac{1}{q^n}\right)\cdots\left(1 - \frac{1}{q^{n-j+1}}\right). \end{aligned} \tag{20}$$

By virtue of the inequality $1 - (1-\alpha_1)\cdots(1-\alpha_m) \leq \sum_{k=1}^{m} \alpha_k$, which is satisfied for all $\alpha_1, \ldots, \alpha_m \in (0,1)$, it follows that

$$\sum_{k=0}^{j-1} p_{n,n-k}\left(q;q^{-j}\right) \leq \frac{1}{q^n}\sum_{k=0}^{j-1} q^k = \frac{[j]_q}{q^n}. \tag{21}$$

□

With the help of Lemma 2, the following result can be derived easily.

Theorem 3. *If $f : [0,1] \to \mathbb{R}$ is continuous from the left at $x = q^{-j}$ and bounded on $[q^{-j}, 1]$, then $B_{n,q}(f;q^{-j}) \to f(q^{-j})$ as $n \to \infty$.*

Proof. The statement follows immediately from estimate (15). Indeed, $\lim_{t\to 0^+}\Omega_{f,q^{-j}}(t) = 0$ since f is continuous from the left at q^{-j}, and $\lim_{n\to\infty}((2M_j[j]_q)/(q^n)) = 0$ since $M_j < \infty$. □

Corollary 4. *If $f : [0,1] \to \mathbb{R}$ is continuous from the left at every q^{-j}, $j \in \mathbb{N}$, then $B_{n,q}(f;q^{-j}) \to f(q^{-j})$ as $n \to \infty$ for all $q^{-j} \in \mathbb{J}_q$.*

Remark 5. The conditions of the theorem are essential and cannot be left out entirely, as it will be shown by Example 8. Furthermore, it is not difficult to see that if $f(x)$ is bounded on $[q^{-j}, 1]$, then the condition

$$\lim_{n\to\infty} f\left(\frac{[n-j]_q}{[n]_q}\right) = f\left(q^{-j}\right) \tag{22}$$

is necessary for the approximation at $x = q^{-j}$.

The power functions $x^{-\alpha}$, $\alpha > 0$ supply a natural family of functions discontinuous at 0 satisfying the conditions of Theorem 3 for all $q^{-j} \in \mathbb{J}_q \setminus \{0\}$. Therefore, it is interesting to consider the q-Bernstein polynomials of functions whose behavior at 0 is similar to that of the power functions. The next theorem investigates the behavior of the q-Bernstein polynomials of such functions on the set $\mathbb{R} \setminus \mathbb{J}_q$.

Theorem 6. *Let $f : [0,1] \to \mathbb{R}$ so that $f(x) \in C(0,1]$ and $\lim_{x\to 0^+} x^\alpha f(x) = K \neq 0$. Then, for $q \geq 2$,*

$$B_{n,q}(f;x) \longrightarrow \infty \quad \text{as } n \longrightarrow \infty\ \forall x \in \mathbb{R} \setminus \mathbb{J}_q. \tag{23}$$

Proof. We set $u(x) = x^\alpha f(x)$, $x \in (0,1]$ with $u(0) = K$. It can be readily seen that

$$\begin{aligned} &B_{n,q}(f;x) \\ &\quad= f(0)\,p_{n0}(q;x) + \sum_{k=1}^{n} u\left(\frac{[k]_q}{[n]_q}\right)\frac{[n]_q^\alpha}{[k]_q^\alpha} p_{nk}(q;x) \\ &\quad= (-1)^n [n]_q^\alpha q^{n(n-1)/2} x^n \\ &\qquad\cdot \left\{\frac{f(0)}{[n]_q^\alpha} + \sum_{k=1}^{n}\frac{u\left([k]_q/[n]_q\right)(-1)^k q^k (q^{-n};q)_k}{[k]_q^\alpha\left(q^k - 1\right)\cdots(q-1)}\right. \\ &\qquad\quad\left.\times\left(\frac{1}{x};\frac{1}{q}\right)_{n-k}\right\} \\ &\quad= (-1)^n [n]_q^\alpha q^{n(n-1)/2} x^n \cdot \left\{\frac{f(0)}{[n]_q^\alpha} + \sum_{k=1}^{\infty} c_{kn}(q;x)\right\}, \end{aligned} \tag{24}$$

where

$$c_{kn}(q;x) = \begin{cases} \dfrac{u\left([k]_q/[n]_q\right)(-1)^k q^k (q^{-n};q)_k}{[k]_q^\alpha (q^k-1)\cdots(q-1)} \cdot \left(\dfrac{1}{x};\dfrac{1}{q}\right)_{n-k} & \text{if } k \le n, \\ 0 & \text{if } k > n. \end{cases} \quad (25)$$

Since, if $x \neq 0$, $[n]_q^\alpha q^{n(n-1)/2} x^n \to \infty$ and $(f(0)/[n]_q^\alpha) \to 0$ as $n \to \infty$, it suffices to prove that $\lim_{n\to\infty} \sum_{k=1}^{\infty} c_{kn}(q;x) \neq 0$ whenever $x \notin \mathbb{J}_q$. Let $\max_{x\in[0,1]}|u(x)| = M$. Then, it is not difficult to see that

$$|c_{kn}(q;x)| \le \frac{Mq^k}{[k]_q^\alpha (q^k-1)\cdots(q-1)} \cdot \left(-\frac{1}{|x|};\frac{1}{q}\right)_\infty =: d_k(q;x) \quad (26)$$

and that by the ratio test, $\sum_{k=1}^{\infty} d_k(q;x) < \infty$. Hence, by the Lebesgue dominated convergence theorem,

$$\begin{aligned} \lim_{n\to\infty} \sum_{k=1}^{\infty} c_{kn}(q;x) &= \sum_{k=1}^{\infty} \left(\lim_{n\to\infty} c_{kn}(q;x)\right) \\ &= \left(\frac{1}{x};\frac{1}{q}\right)_\infty \cdot \sum_{k=1}^{\infty} \frac{K(-1)^k q^k}{[k]_q^\alpha (q^k-1)\cdots(q-1)} \\ &=: K\left(\frac{1}{x};\frac{1}{q}\right)_\infty \cdot \sum_{k=1}^{\infty} (-1)^k a_k. \end{aligned} \quad (27)$$

Obviously, $K(1/x;1/q)_\infty \neq 0$ if $x \notin \mathbb{J}_q$, and it is left to show that

$$\sum_{k=1}^{\infty} (-1)^k a_k \neq 0 \quad \text{for } q \ge 2. \quad (28)$$

Consider

$$\frac{a_{k+1}}{a_k} = \frac{q}{q^{k+1}-1} \cdot \frac{[k]_q^\alpha}{[k+1]_q^\alpha} \le \frac{q}{q^{k+1}-1} \le \frac{q}{q^2-1} < 1 \quad \text{if } q \ge 2. \quad (29)$$

Consequently,

$$\sum_{k=1}^{\infty} (-1)^k a_k = -(a_1-a_2)-(a_3-a_4)-\cdots-(a_{2k-1}-a_{2k})-\cdots < 0. \quad (30)$$

The grouping of terms is justified, because the series is absolutely convergent.

As a result, one concludes that $\lim_{n\to\infty} \sum_{k=1}^{\infty} c_{kn}(q;x) \neq 0$ when $x \notin \mathbb{J}_q$, and the proof is complete. □

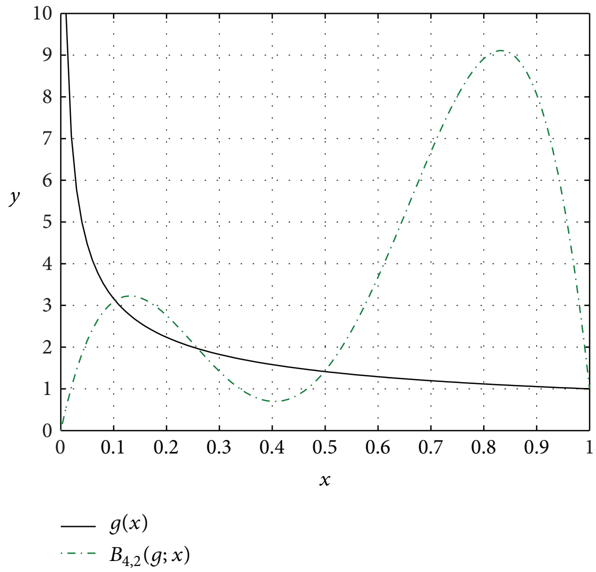

Figure 1: Graphs of $y = g(x)$ and $y = B_{n,2}(g;x), n = 4$.

3. Numerical Examples

In this section, we provide the results of some numerical experiments to exemplify the validity of the theoretical results through a few test examples performed using high-precision computations with Maple 8. All of these computations are performed with 1000 digits so as to minimize the round-off errors, and 3 or 8 digits are used in showing the results in the tables.

Example 7. Let $g(x)$ be defined by

$$g(x) = \begin{cases} \dfrac{1}{\sqrt{x}} & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0. \end{cases} \quad (31)$$

The graphs of $y = g(x)$ and $y = B_{n,q}(g;x)$ for $q = 2, n = 4$ are exhibited in Figure 1. Similarly, Figure 2 represents the graphs of $y = g(x)$ and $y = B_{n,q}(g;x)$ for $q = 2, n = 5, 6$ over the subintervals $[0, 0.3]$, $[0.3, 0.5]$, and $[0.5, 1]$, respectively.

In addition, in Table 1, the values of the *error function* $E(n,q,x) := B_{n,q}(g;x) - g(x)$ with $q = 2$ at some points $x \in [0,1]$ are presented. These points are taken both in $\mathbb{J}_q$ and in $[0,1] \setminus \mathbb{J}_q$. It can be observed from the table that for $j < n$, the values of the error function at the points of $\mathbb{J}_q$ are close to 0, while, at the points in between, they become rather large in magnitude, as it can be noticed from the upper part of Table 1. On the other hand, if $x \le q^{-j}$ with $j > n$, then $|E(n,q,x)| \ge g(x) - g(1/[n]_q) = 1/\sqrt{x} - \sqrt{[n]_q}$. The bottom part of the table refers to this case. Finally, the middle part of the table shows the transition between the two cases.

In general, if a function $f : [0,1] \to \mathbb{R}$ does not satisfy the conditions of Theorem 3, then neither the approximation

TABLE 1: The values of $E(n,q,x) = B_{n,q}(g;x) - g(x)$ for $q = 2$ at some points $x \in [0,1]$.

x	$E(5,q,x)$	$E(10,q,x)$	$E(20,q,x)$	$E(30,q,x)$	$E(50,q,x)$
$(q+1)/2q$	$-106.$	4.50×10^{12}	3.61×10^{56}	3.68×10^{130}	7.77×10^{368}
$1/q$	9.7×10^{-3}	2.78×10^{-3}	2.79×10^{-7}	2.73×10^{-10}	2.6×10^{-16}
$(q+1)/2q^2$	6.25	-7.34×10^{9}	-5.73×10^{50}	-5.71×10^{121}	-1.15×10^{354}
$1/q^2$	4.28×10^{-2}	1.14×10^{-3}	1.19×10^{-6}	1.16×10^{-9}	1.1×10^{-15}
$(q+1)/2q^3$	-1.0	3.13×10^{7}	2.37×10^{45}	2.3×10^{113}	4.43×10^{339}
$1/q^3$	0.154	1.55×10^{-3}	3.91×10^{-6}	3.82×10^{-9}	3.64×10^{-15}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$1/q^{19}$	$-724.$	$E(5,q,x) + 6.20 \times 10^{-2}$	$109.$	7.32×10^{-2}	6.98×10^{-8}
$(q+1)/2q^{20}$	$-836.$	$E(5,q,x) + 4.65 \times 10^{-2}$	-38.6	-1.48×10^{18}	-1.27×10^{142}
$1/q^{20}$	-1.02×10^{3}	$E(5,q,x) + 3.10 \times 10^{-2}$	$-355.$	0.207	1.98×10^{-7}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$1/q^{49}$	-2.37×10^{7}	$E(5,q,x) + 5.78 \times 10^{-11}$	$E(5,q,x) + 1.91 \times 10^{-6}$	$E(5,q,x) + 6.25 \times 10^{-2}$	3.58×10^{6}
$(q+1)/2q^{50}$	-2.74×10^{7}	$E(5,q,x) + 4.33 \times 10^{-11}$	$E(5,q,x) + 1.43 \times 10^{-6}$	$E(5,q,x) + 4.69 \times 10^{-2}$	-1.26×10^{6}
$1/q^{50}$	-3.36×10^{7}	$E(5,q,x) + 2.89 \times 10^{-11}$	$E(5,q,x) + 9.54 \times 10^{-7}$	$E(5,q,x) + 3.12 \times 10^{-2}$	-1.16×10^{7}

TABLE 2: The values of $E(n,q,x) = B_{n,q}(h;x) - h(x)$ for $q = 2$ at some points $x \in [0,1]$.

x	$E(5,q,x)$	$E(10,q,x)$	$E(20,q,x)$	$E(30,q,x)$	$E(50,q,x)$
$(q+1)/2q$	$142.$	3.92×10^{4}	2.32×10^{9}	1.37×10^{14}	4.79×10^{23}
$1/q$	56.2	2.04×10^{3}	2.10×10^{6}	2.15×10^{9}	2.25×10^{15}
$(q+1)/2q^2$	18.2	$187.$	1.11×10^{4}	6.39×10^{5}	2.13×10^{9}
$1/q^2$	5.2675781	$E(5,q,x) + 0.70900154$	$E(5,q,x) + 0.73239899$	$E(5,q,x) + 0.73242185$	$E(5,q,x) + 0.73242187$
$(q+1)/2q^3$	1.81	0.486	2.75×10^{-2}	1.55×10^{-3}	4.91×10^{-6}
$1/q^3$	0.384	1.36×10^{-2}	1.34×10^{-5}	1.3×10^{-8}	1.24×10^{-14}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$1/q^{19}$	2.38×10^{-20}	6.98×10^{-46}	4.68×10^{-97}	3.13×10^{-148}	1.40×10^{-250}
$(q+1)/2q^{20}$	7.53×10^{-21}	5.24×10^{-47}	1.98×10^{-99}	7.45×10^{-152}	1.05×10^{-256}
$1/q^{20}$	1.49×10^{-21}	1.36×10^{-48}	8.93×10^{-103}	5.83×10^{-157}	2.48×10^{-265}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$1/q^{49}$	1.79×10^{-56}	3.68×10^{-127}	1.21×10^{-268}	3.97×10^{-410}	4.28×10^{-693}
$(q+1)/2q^{50}$	5.66×10^{-57}	2.76×10^{-128}	5.12×10^{-271}	9.46×10^{-414}	3.23×10^{-699}
$1/q^{50}$	1.12×10^{-57}	7.19×10^{-130}	2.31×10^{-274}	7.4×10^{-419}	7.60×10^{-708}

on $\mathbb{J}_q$ nor the divergence of $\{B_{n,q}(f;x)\}$ for all $x \in \mathbb{R} \setminus \mathbb{J}_q$ is guaranteed. This fact is illustrated by the following example.

Example 8. Consider the function h defined by

$$h(x) = \begin{cases} \dfrac{q}{1-qx} - \dfrac{q^2}{q-1} & \text{if } x \in \left[q^{-2}, q^{-1}\right) \\ 0 & \text{otherwise.} \end{cases} \tag{32}$$

Then, for n large enough,

$$\begin{aligned} B_{n,q}(h;x) &= h\left(\frac{[n-1]_q}{[n]_q}\right) p_{n,n-1}(q;x) \\ &= \frac{q(q^n-1)-q^2}{q-1} \cdot \frac{q^n-1}{q-1} x^{n-1}(1-x) \\ &\sim \frac{q^3(1-x)}{(q-1)^2} \cdot \left(q^2 x\right)^{n-1}, \quad n \longrightarrow \infty. \end{aligned} \tag{33}$$

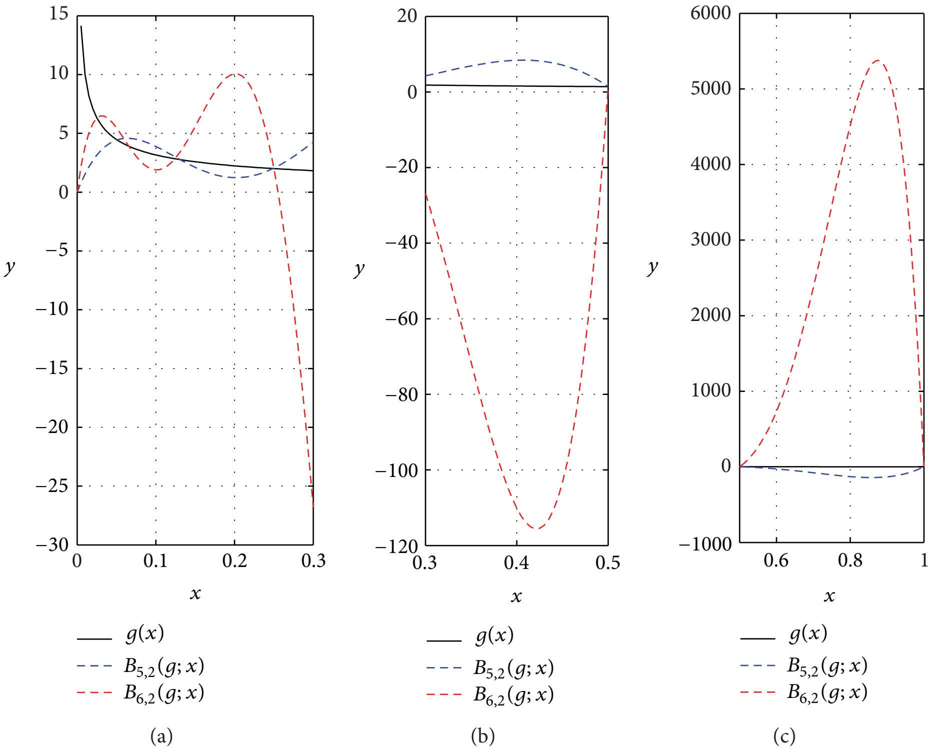

FIGURE 2: Graphs of $y = g(x)$ and $y = B_{n,2}(g;x)$, $n = 5, 6$.

Therefore,

$$\lim_{n\to\infty} B_{n,q}(h;x) = \begin{cases} 0 & \text{if } |x| < q^{-2}, \\ \infty & \text{if } |x| > q^{-2}, x \neq 1, \\ \dfrac{q(q+1)}{q-1} & \text{if } x = q^{-2}, \\ 0 & \text{if } x = 1. \end{cases} \tag{34}$$

Consequently, for the error function $E(n,q,x)$, one has

$$\lim_{n\to\infty} E(n,q,x) = \begin{cases} 0 & \text{if } x \in \left[0, q^{-2}\right), \\ +\infty & \text{if } x \in \left(q^{-2}, 1\right), \\ \dfrac{q(q+1)}{q-1} & \text{if } x = q^{-2}, \\ 0 & \text{if } x = 1. \end{cases} \tag{35}$$

If $x = -q^{-2}$, the limit does not exist. Figure 3 exhibits the graphs of $y = h(x)$ and $y = B_{n,q}(h;x)$ for $q = 2$, $n = 4$, while Figure 4 provides those of $y = h(x)$ and $y = B_{n,q}(h;x)$ for $q = 2$, $n = 5, 6$ over the subintervals $[0, 0.6]$ and $[0.6, 1]$. In addition, in Table 2, the values of the error function $E(n,q,x)$ with $q = 2$ at some points $x \in [0,1]$ which are taken both in $\mathbb{J}_q$ and in $[0,1] \setminus \mathbb{J}_q$ are presented.

It can be observed from the first three rows that as n increases, the values of the error function become very large—a trend which reflects its behavior on $(q^{-2}, 1)$. At the point $x = q^{-2}$, the values approach 6 from below, whereas for the remaining part of the table, the values of error function come close to zero as n increases, which is in full agreement with the limit given by (35).

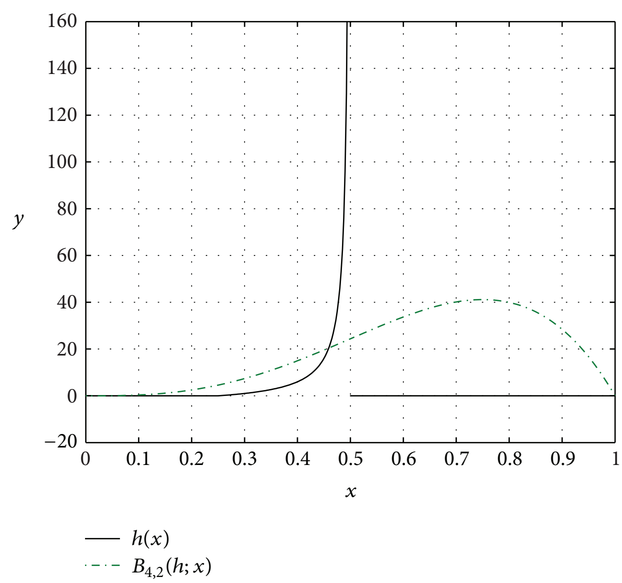

FIGURE 3: Graphs of $y = h(x)$ and $y = B_{n,2}(h;x)$, $n = 4$.

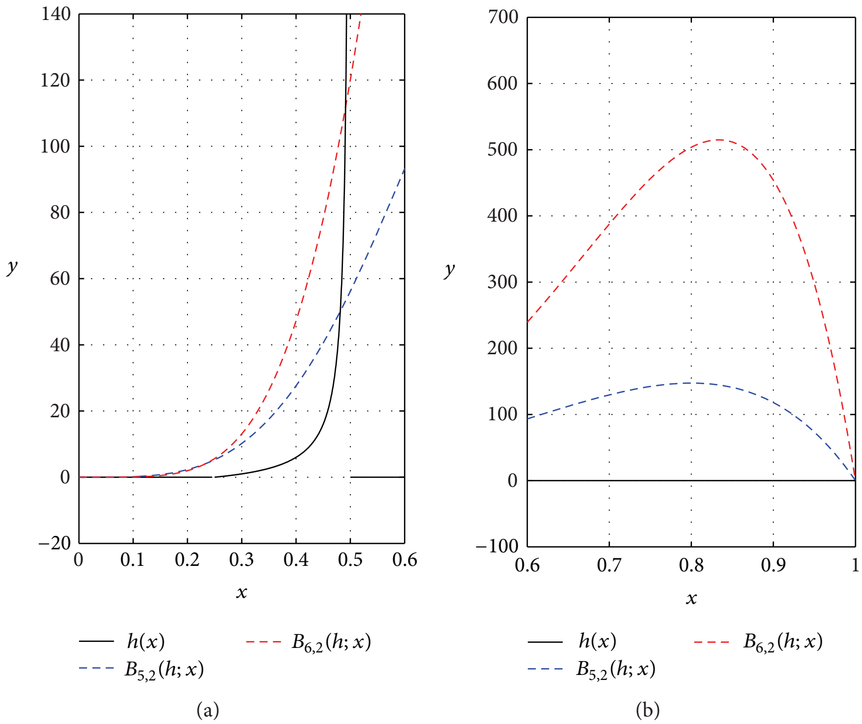

FIGURE 4: Graphs of $y = h(x)$ and $y = B_{n,2}(h; x)$, $n = 5, 6$.

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