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Research Article

Ordered Variational Inequalities and Ordered Complementarity Problems in Banach Lattices

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We introduce the concepts of ordered variational inequalities and ordered complementarity problems with both domain and range in Banach lattices. Then we apply the Fan-KKM theorem and KKM mappings to study the solvability of these problems.

1. Introduction

Let X be a real Banach space with its norm dual X'. Let C be a nonempty convex subset of X and $f : C \to X'$ a single-valued mapping. The variational inequality problem associated with C and f, simply denoted as VI(C, f), is to find an $x^* \in C$ such that

$$\langle f(x^*), (x - x^*)\rangle \geq 0, \quad \forall x \in C. \tag{1}$$

A nonempty convex subset K of a Banach space X is called a *convex cone* in X whenever the following two conditions hold:

(1) $K \neq \{0\}$ and $aK \subseteq K$, for any nonnegative number a;

(2) $(-K) \cap K = \{0\}$.

Let K be a convex cone in X; the complementarity problem associated with K and f, simply denoted as CP(K, f), is to find an $x^* \in K$ such that

$$\langle f(x^*), x^*\rangle = 0, \quad \langle f(x^*), x\rangle \geq 0, \quad \forall x \in K. \tag{2}$$

The variational inequality problem VI(C, f) and complementarity problem CP(K, f) have been extensively studied by many authors. This theory has been recognized as an important branch in nonlinear analysis and has been widely applied to optimization theory, game theory, economic equilibrium, mechanics, and so forth. During the last five decades, many researchers have studied the existence of solutions of these problems and their applications to applied mathematical fields from finite-dimensional Euclidean spaces to infinite-dimensional general Banach spaces (see, e.g., [1–11]).

Since most classical Banach spaces are Banach lattices equipped with some lattice orders on which the positive operators appear naturally, the domain of an ordinal variational inequality defined in (1) and the complementarity problem defined in (2) may be in a Banach lattice (in particular, a Hilbert lattice). In this case, to investigate the properties of the solution set of (1) related to the partial order may be an important topic in economics theory and other applied-mathematics fields. In 2011, Li and Yao [8] and Nishimura and Ok [12] studied the solvability and the existence of order-maximum and order-minimum solutions to general variational inequalities defined in Hilbert lattices. In 2012, Li and Ok [13] extended these results to Banach lattices as the domain of the variational inequalities.

The ranges of the pairing in the variational inequality (1) and the complementarity problem (2) are both the set of real numbers, where the inequalities in (1) and (2) are the usual inequality for real numbers, which is the ordinal order of real numbers that is a complete order. In some economic circumstances, the preferences of a certain type of outcomes may not be totally ordered; that is, it may be a partial order, in particular a lattice order. In this case, any preference optimal problem must be defined under the given partial order that describes the preferences. To further demonstrate this aspect, we consider the following example.

For any positive integer k, let $(\mathbb{R}^k; \succcurlyeq^k)$ denote the k-dimensional Hilbert lattice where $\mathbb{R}^k$ is the k-dimensional Euclidean space equipped with the coordinate partial order $\succcurlyeq^k$, which is defined as $x \succcurlyeq^k y$ whenever $x_j \geq y_j$, for $j = 1, 2, \ldots, k$, for $x = (x_1, x_1, \ldots, x_k)$, and $y = (y_1, y_1, \ldots, y_k) \in \mathbb{R}^k$. Now let $(\mathbb{R}^n; \succcurlyeq^n)$ and $(\mathbb{R}^m; \succcurlyeq^m)$ be two finite-dimensional Hilbert lattices and C a nonempty closed convex subset of $(\mathbb{R}^n; \succcurlyeq^n)$. Let $f : C \rightarrow L(\mathbb{R}^n, \mathbb{R}^m)$ be a mapping. Then a new and more general (than (1) and (2)) problem is to find an $x^* \in C$ such that

$$\begin{gathered} f(x^*)(x) \succcurlyeq^m f(x^*)(x^*), \text{ that is,} \\ f(x^*)(x - x^*) \succcurlyeq^m 0, \quad \forall x \in C, \end{gathered} \tag{3}$$

where, without causing any confusion, 0 is the origin of $\mathbb{R}^m$. Taking $m = 1$, this problem turns to be the VI(C, f) defined in (1), and hence it is an obvious generalization of (1).

Based on this motivation, we consider two Banach lattices $(X; \succcurlyeq^X)$ and $(U; \succcurlyeq^U)$, where $(X; \succcurlyeq^X)$ is considered as the domain and $(U; \succcurlyeq^U)$ as the range for the values of a mapping f that is from a subset of $(X; \succcurlyeq^X)$ to $L(X, U)$. Then we extend the variational inequality problem VI(C, f) and the complementarity problem CP(K, f) to more general cases which are called the ordered variational inequalities and ordered complementarity problems defined by (9) and (10) in Section 3.

This paper is organized as follows. In Section 2, we recall some concepts and provide some properties of Banach lattices; in Section 3, we introduce the concepts of ordered variational inequalities, ordered complementarity problems and prove some solution existence theorems; in Section 4, the properties of the solution set of ordered variational inequalities, such as the order optimal solutions, will be provided; in Section 5, we give an example as an application of the main theorem (Theorem 10) in this paper.

2. Preliminaries

In this section, we recall some definitions of Banach lattices and provide some properties that are useful in this paper. Here, we adopt the notions from [14].

Let X be a vector lattice with a partial order $\succcurlyeq$. As usual, the origin of X is denoted by 0, and $x^+ := x \vee 0$, $x^- := (-x) \vee 0$, and $|x| = x^+ \vee x^-$, for all $x \in X$. A Banach space X equipped with a lattice order $\succcurlyeq$ is called a Banach lattice, which is written as $(X; \succcurlyeq)$, if the following properties hold:

(1) $x \succcurlyeq y$ implies $x + z \succcurlyeq y + z$, for all $x, y, z \in X$;

(2) $x \succcurlyeq y$ implies $\alpha x \succcurlyeq \alpha y$, for all $x, y \in X$ and $\alpha \geq 0$;

(3) $|x| \succcurlyeq |y|$ implies $\|x\| \geq \|y\|$, for every $x, y \in X$.

It is well known that, in a Banach lattice X, the distributive properties hold: $x + (y \vee z) = (x + y) \vee (x + z)$ and $x + (y \wedge z) = (x + y) \wedge (x + z)$, for all x, y, and z in X. We could not find the extension of these distributive properties to infinite cases, so we include them below as a lemma. They will be used in the content of this paper.

Lemma 1. *Let x be an element and let A, B be subsets of a Banach lattice X satisfying that $\vee A$, $\wedge A$, $\vee B$, and $\wedge B$ exist; then the following distributive properties hold:*

(1) $\vee(x + A) = x + \vee A$ *and* $\wedge(x + A) = x + \wedge A$;

(2) $\wedge(x - A) = x - \vee A$ *and* $\vee(x - A) = x - \wedge A$;

(3) $\vee A = - \wedge (-A)$ *and* $\wedge A = - \vee (-A)$;

(4) $\vee(A + B) = \vee A + \vee B$ *and* $\wedge(A + B) = \wedge A + \wedge B$.

Proof. The proofs of Parts 1, 2, and 3 are straightforward (e.g., see page 4 in [14] for the first equality in Part 1). We only prove Part 4. For every $a \in A, b \in B$, it is clear that $\wedge A \preccurlyeq a$ and $\wedge B \preccurlyeq b$. From the order linearity of Banach lattices, it yields $\wedge A + \wedge B \preccurlyeq a + b$. It implies that $\wedge A + \wedge B$ is a lower bound of $A + B$. On the other hand, suppose that z is an arbitrary lower bound of $A + B$. Then for all $a \in A$, from Part 1, we have $z \preccurlyeq \wedge(a + B) = a + \wedge B$. Applying Part 1 again, it implies that $z \preccurlyeq \wedge(A + \wedge B) = \wedge A + \wedge B$. We obtain $\wedge(A + B) = \wedge A + \wedge B$. This completes the proof of this lemma. □

A net $\{x_\alpha\}$ in a Banach lattice $(X; \succcurlyeq)$ is said to be *decreasing* (it is denoted by $x_\alpha \downarrow$) whenever $\alpha \succcurlyeq^I \beta$ implies $x_\alpha \preccurlyeq x_\beta$, where $\succcurlyeq^I$ is the partial order on the index net. If a net $\{x_\alpha\}$ satisfies $x_\alpha \downarrow$ and $\wedge\{x_\alpha\} = x$, then we denote it by $x_\alpha \downarrow x$. The meanings $x_\alpha \uparrow$ and $x_\alpha \downarrow x$ are analogously defined. A net $\{x_\alpha\}$ in a Banach lattice $(X; \succcurlyeq)$ is said to be order convergent to a vector x, which is denoted by $x_\alpha \xrightarrow{0} x$, whenever there exists another net $\{y_\alpha\}$ with the same index net satisfying $y_\alpha \downarrow 0$ and $|x_\alpha - x| \preccurlyeq y_\alpha$, for each α. A subset A of a Banach lattice $(X; \succcurlyeq)$ is said to be *order closed* whenever for any $\{x_\alpha\} \subseteq A$ satisfying $x_\alpha \xrightarrow{0} x$ implies $x \in A$.

The positive cone of a Banach lattice $(X; \succcurlyeq)$ is denoted by X^+ which is defined as $X^+ = \{x \in X : x \succcurlyeq 0\}$. It is well known that the positive cone X^+ of a Banach lattice X is norm closed. In the next two lemmas, we show that the positive cone also has the order closeness and weak closeness.

Lemma 2. *Let $(X; \succcurlyeq)$ be an arbitrary Banach lattice. Then the positive cone X^+ is order closed.*

Proof. Let $\{x_\alpha\}$ be a net in X^+, which order converges to x. That is, $x_\alpha \xrightarrow{0} x$. So there exists another net $\{y_\alpha\}$ with the same index net satisfying $y_\alpha \downarrow 0$ and $|x_\alpha - x| \preccurlyeq y_\alpha$, for each α. Then for every α, we have

$$\begin{aligned} x_\alpha &= (x_\alpha - x)^+ + x_\alpha \wedge x \\ &\preccurlyeq |x_\alpha - x| + x_\alpha \wedge x \\ &\preccurlyeq y_\alpha + x_\alpha \wedge x \\ &\preccurlyeq y_\alpha + x. \end{aligned} \tag{4}$$

It implies that $\wedge(x_\alpha) \preccurlyeq \wedge(y_\alpha + x)$. Since $\wedge y_\alpha = 0$, from Lemma 1, we have $\wedge(y_\alpha + x) = \wedge y_\alpha + x = x$. Substituting this to the above order inequality and noticing $\{x_\alpha\} \subseteq X^+$, we get $0 \preccurlyeq \wedge(x_\alpha) \preccurlyeq \wedge(y_\alpha + x) = x$. So $x \in X^+$, and hence X^+ is order closed. □

Lemma 3. *Let $(X; \succcurlyeq)$ be a Banach lattice. Then the positive cone X^+ is weakly closed.*

Proof. It is clear that the positive cone X^+ of the Banach lattice X is convex. We have mentioned that the positive cone X^+ of the Banach lattice X is norm closed. Applying Mazur's lemma (see [12] or [15]), we have in a Banach space, a convex set is norm closed if and only if it is weakly closed. This completes the proof of this lemma. □

Let $(X; \succcurlyeq^X)$ and $(U; \succcurlyeq^U)$ be two Banach lattices. A linear operator $T : X \longrightarrow U$ is said to be *order bounded* if it maps every order bounded subset of X to order-bounded subset of U. Let $\mathscr{L}_b(X, U)$ denote the collection of all order bounded linear operators from X to U. $\mathscr{L}_b(X, U)$ is also a vector space. A natural partial order $\succcurlyeq^{\mathscr{L}}$ on $\mathscr{L}_b(X, X)$ is induced by the positive cone X^+ as follows: for any $S, T \in \mathscr{L}_b(X, U)$, we say that $S \succcurlyeq^{\mathscr{L}} T$ if and only if $S(x) \succcurlyeq^U T(x)$, for all $x \succcurlyeq^X 0$. Then $\mathscr{L}_b(X, U)$ is a partially ordered vector space with respect to $\succcurlyeq^{\mathscr{L}}$.

A linear operator $T : X \longrightarrow U$ between two Banach lattices is said to be *order continuous* if $x_\alpha \xrightarrow{0} x$ in X implies $T(x_\alpha) \xrightarrow{0} T(x)$ in U. It is known that all order-continuous linear operators between two Banach lattices are order-bounded linear operators. The following lemma is useful in the content of this paper.

Lemma 4. *Let $(X; \succcurlyeq^X)$ and $(U; \succcurlyeq^U)$ be two Banach lattices with U being Dedekind complete. Then one has*

$$\mathscr{L}_b(X, U) \subseteq L(X, U). \tag{5}$$

Proof. A linear operator $T : X \longrightarrow U$ between two Banach lattices is said to be positive whenever $T(X^+) \subseteq U^+$. It is known that any positive linear operator between two Banach lattices is an order-bounded linear operator. Let $\mathscr{L}_r(X, U)$ denote the collection of all linear operators from X to U which can be represented as a difference between two positive operators. Then as a consequence, we have $\mathscr{L}_r(X, U) \subseteq \mathscr{L}_b(X, U)$. Furthermore, if U is Dedekind complete, then $\mathscr{L}_r(X, U) = \mathscr{L}_b(X, U)$. From Theorem 4.3 in [14], we have that every positive linear operator between two Banach lattices is (strongly) continuous. It implies that if U is Dedekind complete, then we have $\mathscr{L}_b(X, U) = \mathscr{L}_r(X, U) \subseteq L(X, U)$. This lemma is proved. □

If $(X; \succcurlyeq^X)$ and $(U; \succcurlyeq^U)$ are two Banach lattices with U Dedekind complete, then for any T in $\mathscr{L}_b(X, U)$, we have $|T| \in \mathscr{L}_b(X, U)$; therefore, from Lemma 4, $|T| \in L(X, U)$ holds. Hence we can define a norm $\|\cdot\|_r$ on $\mathscr{L}_b(X, U)$ by $\|T\|_r = \||T|\|$, for all $T \in \mathscr{L}_b(X, U)$. This norm $\|\cdot\|_r$ is called the regular norm on $\mathscr{L}_b(X, U)$ that satisfies the following inequality

$$\|T\| \leq \|T\|_r, \quad \forall T \in \mathscr{L}_b(X, U). \tag{6}$$

By applying the Riesz-Kantorovich theorem, $\mathscr{L}_b(X, U)$ under the regular norm and with the partial order $\succcurlyeq^{\mathscr{L}}$ becomes a Dedekind-complete Banach lattice. In addition, for any net $\{T_\alpha\}$ in $\mathscr{L}_b(X, U)$, we have

$$T_\alpha \downarrow 0 \text{ in } \mathscr{L}_b(X, U), \quad \text{iff}, T_\alpha(x) \downarrow 0 \text{ in } U \text{ for each } x \in X. \tag{7}$$

Let $(X; \succcurlyeq)$ be a Banach lattice. The norm of X is said to be an *order-continuous norm*, if for any net $\{x_\alpha\}$ in X, $x_\alpha \xrightarrow{0} 0$ in X implies $\|x_\alpha\| \longrightarrow 0$. A Banach lattice with order-continuous norm has many useful properties. We list some below.

(1) Every Banach lattice with order-continuous norm is Dedekind complete.

(2) Every reflexive Banach lattice has order-continuous norm (Nakano theorem); therefore, every reflexive Banach lattice is Dedekind complete.

The class of Banach lattices with order-continuous norms is pretty large and includes many useful Banach spaces. For example, the classical $L_p(\mu)$, where $1 \leq p < \infty$, are Banach lattices with order-continuous norms. The following result is a consequence of order-continuous norm. We list it as a lemma which is useful in the following sections.

Lemma 5. *If the norm of a Banach lattice $(X; \succcurlyeq^X)$ is order continuous, then the σ-order convergence implies norm convergence, that is,*

$$x_n \xrightarrow{0} x \text{ implies } x_n \longrightarrow x, \quad \text{in the norm of } X. \tag{8}$$

Proof. Suppose that $\{x_n\}$ is a sequence in X satisfying $x_n \xrightarrow{0} x$. It is equivalent to $x_n - x \xrightarrow{0} 0$. Since $(X; \succcurlyeq^X)$ is a Banach lattice with order-continuous norm, it implies $\|x_n - x\| \longrightarrow 0$. This completes the proof of this lemma. □

3. The Solvability of Ordered Variational Inequalities in Banach Lattices

In this section, we introduce the concepts of ordered variational inequalities and ordered complementarity problems on suitable Banach lattices. Then we extend some already known solvability results about variational inequalities and complementarity problems (see [3–8, 10, 12, 13]) to the cases of ordered variational inequalities and ordered complementarity problems.

Definition 6. Let $(X; \succcurlyeq^X)$ and $(U; \succcurlyeq^U)$ be two Banach lattices. Let C be a nonempty convex subset of X and $f : C \longrightarrow \mathscr{L}_b(X, U)$ a mapping. The ordered variational inequality problem associated with C and f, denoted by VOI(C, f), is to find an $x^* \in C$ such that

$$f(x^*)(x - x^*) \succcurlyeq^U 0, \quad \forall x \in C, \tag{9}$$

where, as usual, 0 denotes the origin of U. If f is linear, then the problem VOI(C, f) is called a linear ordered variational inequality problem; otherwise, it is called a nonlinear ordered variational inequality problem.

Definition 7. Let $(X;\succcurlyeq^X)$ and $(U;\succcurlyeq^U)$ be two Banach lattices. Let K be a convex cone of X and $f: K \to \mathscr{L}_b(X,U)$ a mapping. The ordered complementarity problem associated with K and f, denoted by OCP(K,f), is to find an $x^* \in K$ such that

$$f(x^*)(x^*) = 0, \quad f(x^*)(x) \succcurlyeq^U 0, \quad \forall x \in K. \tag{10}$$

If f is linear, then the problem OCP(K,f) is called a linear ordered complementarity problem; otherwise, it is called a nonlinear ordered complementarity problem.

For a given Banach lattice $(X,\succcurlyeq)$, let X' denote the norm dual of X, that is, $X' = L(X,\mathbb{R})$. The order dual of $(X,\succcurlyeq)$ is denoted by $X^{\sim}$ that is defined to be $\mathscr{L}_b(X,\mathbb{R})$, where $(\mathbb{R},\geq)$ is the Banach lattice of the set of real numbers with the ordinal topology and the standard order $\geq$, which is complete. From Garrett Birkhoff Theorem, the norm dual X' of a Banach lattice $(X,\succcurlyeq)$ coincides with its order dual $X^{\sim}$, that is,

$$X' = L(X,\mathbb{R}) = \mathscr{L}_b(X,\mathbb{R}) = X^{\sim}. \tag{11}$$

In Definitions 6 and 7, if we take $(U;\succcurlyeq^U) = (\mathbb{R},\geq)$, then $\mathscr{L}_b(X,\mathbb{R}) = X'$ holds; therefore, in this case, we have

$$f(x^*)(x-x^*) = \langle f(x^*), (x-x^*)\rangle, \tag{12}$$

where $\langle\cdot,\cdot\rangle$ is the pairing between X' and X. Hence, the ordered variational inequality VOI(C,f) and the ordered complementarity problem OCP(K,f) turn to be an ordinary variational inequality VI(C,f) and an ordinal complementarity problem CP(K,f), respectively. Thus the ordered variational inequality problems and the ordered complementarity problems in Banach lattices are generalizations of the variational inequality problems and complementarity problems in Banach spaces that extend the ranges from the real numbers to more general Banach lattices.

There are close connections between variational inequality problems and complementarity problems in Banach spaces (e.g., see [3–5, 9–11]). In the next lemma, we show the similar connections between ordered variational inequality problems and ordered complementarity problems in Banach lattices.

Lemma 8. *Let $(X;\succcurlyeq^X)$ and $(U;\succcurlyeq^U)$ be two Banach lattices. Let K be a convex cone of X and $f: K \to \mathscr{L}_b(X,U)$ a mapping. Then $x^* \in K$ is a solution to VOI(K,f) if and only if x^* is a solution to OCP(K,f).*

Proof. It can be seen that x^* is a solution to OCP(K,f) that implies that x^* is a solution to VOI(K,f). Conversely, we will show that (9) implies (10). Suppose x^* is a solution to VOI(K,f) satisfying (9). In the case, if $x^* = 0$, then (10) obviously follows from (9). So we assume $x^* \neq 0$. Since K is a convex cone, then $2x^*$ and $0.5x^*$ are both in K. From (9), we have

$$f(x^*)(2x^*-x^*) \succcurlyeq^U 0, \qquad f(x^*)(0.5x^*-x^*) \succcurlyeq^U 0. \tag{13}$$

They imply

$$f(x^*)(x^*) \succcurlyeq^U 0, \qquad f(x^*)(-x^*) \succcurlyeq^U 0. \tag{14}$$

The last order inequality is equivalent to $f(x^*)(x^*) \preccurlyeq^U 0$. From the antisymmetric property of $\preccurlyeq^U$, we obtain $f(x^*)(x^*) = 0$. Since $f(x^*) \in \mathscr{L}_b(X,U)$, then from the linearity of $f(x^*)$ and by substituting $f(x^*)(x^*) = 0$ into (9), it yields $f(x^*)(x) \succcurlyeq^U 0$, for all $x \in C$. This lemma is proved. □

Definition 9. Let $(X;\succcurlyeq^X)$ and $(U;\succcurlyeq^U)$ be two Banach lattices. Let C be a nonempty convex subset of X. A mapping $f: C \to L(X,U)$ is said to be linearly order comparable on C whenever for any given $x, y \in C$ and for every $\alpha \in (0,1)$, $f(\alpha x + (1-\alpha)y)(x)$, and $f(\alpha x + (1-\alpha)y)(y)$ are $\succcurlyeq^U$-comparable in U, that is, either

$$f(\alpha x + (1-\alpha)y)(x) \succcurlyeq^U f(\alpha x + (1-\alpha)y)(y) \tag{15}$$

or

$$f(\alpha x + (1-\alpha)y)(x) \prec^U f(\alpha x + (1-\alpha)y)(y). \tag{16}$$

Now we prove the main theorem of this paper.

Theorem 10. *Let $(X;\succcurlyeq^X)$ and $(U;\succcurlyeq^U)$ be two Banach lattices. Let C be a nonempty convex closed subset of X and let $f: C \to L(X,U)$ be a linearly order comparable and continuous mapping (see Remark 11 below). If there exists a point $y_0 \in C$ such that*

$$\left\{x \in C : f(x)(y_0 - x) \succcurlyeq^U 0\right\} \text{ is compact in } X, \tag{17}$$

then the problem VOI(C,f) is solvable, that is, there exists $x^* \in C$ such that

$$f(x^*)(y-x^*) \succcurlyeq^U 0, \quad \forall y \in C. \tag{18}$$

Remark 11. That a mapping $f: C \to L(X,U)$ is continuous means that, for any sequence $\{x_n\} \subseteq C$, whenever $x_n \to x$ in X, the following conditions hold:

$$(1) \quad \|f(y)x_n - f(y)x\|_U \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \quad \text{for every fixed } y \in C; \tag{19}$$

$$(2) \quad \|f(x_n) - f(x)\|_{L(X,U)} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{20}$$

Proof of Theorem 10. In the proof and the following contents, not causing confusion, we drop the foot marks for the norms of the Banach spaces X, U, and $L(X,U)$. Define a set valued mapping $\Gamma: C \to 2^C$ as follows:

$$\Gamma(y) = \left\{x \in C : f(x)(y-x) \succcurlyeq^U 0\right\}, \quad \forall y \in C. \tag{21}$$

It is clear that $y \in \Gamma(y)$, and hence $\Gamma(y) \neq \phi$, for all $y \in C$. Next, we show that for any $y \in C, \Gamma(y)$ is a closed subset of C. To this end, take any sequence $\{x_n\} \subseteq \Gamma(y)$ satisfying $x_n \to x$ in X. Since C is closed, then $x \in C$. On the other hand, the condition $x_n \to x$ clearly implies $y - x_n \to y - x$ in X. We have

$$\begin{aligned} &\| f(x_n)(y-x_n) - f(x)(y-x) \| \\ &= \| f(x_n)(y-x_n) - f(x)(y-x_n) \\ &\quad + f(x)(y-x_n) - f(x)(y-x) \| \\ &\le \| f(x_n)(y-x_n) - f(x)(y-x_n) \| \\ &\quad + \| f(x)(y-x_n) - f(x)(y-x) \| \\ &\le \| f(x_n) - f(x) \| \| y - x_n \| + \| f(x)(x_n) - f(x)(x) \| . \end{aligned} \tag{22}$$

Since $\{y - x_n\}$ is a convergent sequence in X, then $\{\|y - x_n\|\}$ is bounded. Applying (20) and (19), inequality (22) implies

$$f(x_n)(y-x_n) \longrightarrow f(x)(y-x) \quad \text{in } U. \tag{23}$$

The assumption that $\{x_n\} \subseteq \Gamma(y)$ implies $f(x_n)(y-x_n) \succcurlyeq^U 0$, that is, $f(x_n)(y-x_n) \in U^+$ for all n. In Section 2, we recalled that for any Banach lattice U, its positive cone U^+ is $\|\cdot\|$-closed and from the limit (23), it yields that $f(x)(y-x) \in U^+$, that is, $f(x)(y-x) \succcurlyeq^U 0$. Hence $x \in \Gamma(y)$, and, therefore, $\Gamma(y)$ is closed in C.

Now we show that Γ is a KKM mapping. Assume, by way of contradiction, that Γ is not a KKM mapping, that is, there is a finite subset $\{y_i : 1 \le i \le n\}$ of C, for some positive integer $n > 1$, and a finite set of positive numbers $\{\lambda_i : 1 \le i \le n\}$ satisfying $\sum_{1 \le i \le n} \lambda_i = 1$, such that $\sum_{1 \le i \le n} \lambda_i y_i \notin \bigcup_{1 \le i \le n} \Gamma(y_i)$. Set $y = \sum_{1 \le i \le n} \lambda_i y_i$. It implies $y \notin \Gamma(y_j)$, for all j, that is,

$$f(y)(y_j - y) \not\succcurlyeq^U 0, \quad \forall 1 \le j \le n. \tag{24}$$

Notice that for any fixed $1 \le j \le n$, y can be rewritten as

$$\begin{aligned} y &= \sum_{1 \le i \le n} \lambda_i y_i = \lambda_j y_j + (1 - \lambda_j) \sum_{i \ne j} \frac{\lambda_i}{1 - \lambda_j} y_i \\ &= \lambda_j y_j + (1 - \lambda_j) z_j, \end{aligned} \tag{25}$$

where $z_j := \sum_{i \ne j} (\lambda_i / (1 - \lambda_j)) y_i$. Since C is convex, then $z_j \in C$, for any fixed $1 \le j \le n$. We have

$$\begin{aligned} f(y)(y_j - y) &= f(\lambda_j y_j + (1-\lambda_j) z_j)((1-\lambda_j)(y_j - z_j)) \\ &= (1-\lambda_j) f(\lambda_j y_j + (1-\lambda_j) z_j)(y_j - z_j). \end{aligned} \tag{26}$$

Since $f(\lambda_j y_j + (1-\lambda_j) z_j)(y_j - z_j) \not\succcurlyeq^U 0$, for all $1 \le j \le n$, from the linearly order comparable property of f, we must have

$$f(\lambda_j y_j + (1-\lambda_j) z_j)(y_j - z_j) \prec^U 0, \quad \forall 1 \le j \le n. \tag{27}$$

From (26), we obtain

$$f(y)(y_j - y) \prec^U 0, \quad \forall 1 \le j \le n. \tag{28}$$

Multiplying by λ_j the above order inequality and summing up from 1 to n, we get

$$f(y)(y - y) \prec^U 0. \tag{29}$$

It is a contradiction to the fact that $f(y) \in L(X, U)$, which must satisfy $f(y)(0) = 0$. Hence we must have $\sum_{1 \le i \le n} \lambda_i y_i \in \bigcup_{1 \le i \le n} \Gamma(y_i)$; therefore, Γ is a KKM mapping. Condition (17) implies that there exists a point $y_0 \in C$ such that $\Gamma(y_0)$ is compact. Applying the Fan-KKM Theorem, we obtain

$$\bigcap_{y \in C} \Gamma(y) \ne \emptyset. \tag{30}$$

Taking $x^* \bigcap_{y \in C} \Gamma(y)$, then $f(x^*)(y - x^*) \succcurlyeq^U 0$, for all $y \in C$. Hence x^* is a solution to VOI(C, f). This completes the proof of this theorem. □

In particular, if C is compact, as a special case of Theorem 10, we get the following corollary.

Corollary 12. *Let $(X; \succcurlyeq^X)$ and $(U; \succcurlyeq^U)$ be two Banach lattices. Let C be a nonempty convex compact subset of X and let $f : C \to L(X, U)$ be a linearly order comparable and continuous mapping. Then the problem VOI(C, f) is solvable.*

In the following result, we apply Theorem 10 to the case that U has order-continuous norm.

Corollary 13. *Let $(X; \succcurlyeq^X)$ and $(U; \succcurlyeq^U)$ be two Banach lattices with U having order-continuous norm. Let C be a nonempty convex closed subset of X and let $f : C \to \mathscr{L}_b(X, U)$ be a linearly order comparable and continuous mapping (with respect to the regular norm on $\mathscr{L}_b(X, U)$). If there exists a point $y_0 \in C$ such that*

$$\{x \in C : f(x)(y_0 - x) \succcurlyeq^U 0\} \text{ is compact,} \tag{31}$$

then the problem VOI(C, f) is solvable.

Proof. From the properties of Banach lattices with order-continuous norms, U is Dedekind complete. Then $(\mathscr{L}_b(X, Y), \succcurlyeq^{\mathscr{L}})$ under the regular norm $\|\cdot\|_r$ is a Dedekind-complete Banach lattice. From Lemma 4, we have $\mathscr{L}_b(X, U) \subseteq L(X, U)$. From (6), $\|T\| \le \|T\|_r$ holds, for all $T \in \mathscr{L}_b(X, Y)$. Since f is continuous in $\mathscr{L}_b(X, U)$ with respect to the regular norm $\|\cdot\|_r$ on $\mathscr{L}_b(X, U)$, it implies that f is continuous in $L(X, U)$ with respect to the norm $\|\cdot\|$ in $L(X, U)$. Then this corollary immediately follows from Theorem 10. □

It is well known that every reflexive Banach lattice has order-continuous norm. As reflexive Banach lattices have been widely used in many mathematics fields, we list the following result as a special case of Corollary 13.

Corollary 14. *Let $(X; \succcurlyeq^X)$ be a Banach lattice and $(U; \succcurlyeq^U)$ a reflexive Banach lattice. Let C be a nonempty convex closed subset of X and let $f : C \to \mathscr{L}_b(X, U)$ be a linearly order comparable and continuous mapping (with respect to* *the regular norm on $\mathscr{L}_b(X,U)$). If there exists a point $y_0 \in C$ such that*

$$\left\{x \in C : f(x)(y_0 - x) \succcurlyeq^U 0\right\} \text{ is compact,} \tag{32}$$

then the problem VOI(C, f) is solvable.

Taking into account Lemma 8, an easy application of Theorem 10 to ordered complementarity problem yields the following result that provides a solvability of ordered complementarity problem in Banach lattices. Similarly, the solvability results for problem VOI(C, f) provided in Corollaries 13 and 14 can be extended to solvability of ordered complementarity problems.

Corollary 15. *Let $(X; \succcurlyeq^X)$ and $(U; \succcurlyeq^U)$ be two Banach lattices. Let K be a nonempty convex closed cone of X, and let $f : K \to L(X,U)$ be a linearly order comparable and continuous mapping. If there exists a point $y_0 \in K$ such that*

$$\left\{x \in K : f(x)(y_0 - x) \succcurlyeq^U 0\right\} \text{ is compact in } X, \tag{33}$$

then the problem OCP(K, f) is solvable; that is, there exists $x^* \in K$ such that

$$f(x^*)(x^*) = 0, \qquad f(x^*)(y)) \succcurlyeq^U 0, \quad \forall y \in K. \tag{34}$$

Recall that for a given Banach lattice $(U; \succcurlyeq^U)$, the partial order $\succcurlyeq^{U^\sim}$ in its order dual $(U^\sim; \succcurlyeq^{U^\sim})$ is induced by its positive cone U^+, that is, for any $w \in U^\sim, w \succcurlyeq^{U^\sim} 0$, if and only if $\langle w, u\rangle \geq 0$, for all $u \in U^+$. In particular, if U is reflexive, applying the Garrett Birkhoff Theorem, we have the following result. It describes the connections between ordinal variational inequalities and ordered variational inequalities.

Proposition 16. *Let $(X; \succcurlyeq^X)$ be a Banach lattice and $(U; \succcurlyeq^U)$ a reflexive Banach lattice. Let C be a nonempty convex closed subset of X, and let $f : C \to L(X,U)$ be a mapping. Then $x^* \in C$ is a solution to the problem VOI(C, f) if and only if the following inequality holds:*

$$\left\langle w, f(x^*)(x - x^*)\right\rangle \geq 0, \quad \forall w \in \left(U^\sim\right)^+. \tag{35}$$

4. The Existence of Order-Optimal Solutions of Ordered Variational Inequalities

As mentioned in the introduction, Li and Yao [8], Nishimura and Ok [12], and Li and Ok [13] have studied the existence of order-maximum and order-minimum solutions to general variational inequalities defined in Banach lattices with real values. After we studied the solvability of ordered variational inequalities in Section 3, in this section, we investigate the existence of order-optimal solutions and the convexity of the solution set of ordered variational inequalities defined in Banach lattices.

Definition 17. A linear operator T from a Banach lattice $(X; \succcurlyeq^X)$ to a Banach lattice $(U; \succcurlyeq^U)$ is said to be

(1) a positive operator whenever it maps positive element to positive element, that is, whenever $x \succcurlyeq^X 0$ implies $T_x \succcurlyeq^U 0$;

(2) a negative operator whenever it maps positive element to negative element, that is, whenever $x \succcurlyeq^X 0$ implies $T_x \preccurlyeq^U 0$.

The collection of all positive (negative) operators between $(X; \succcurlyeq^X)$ and $(U; \succcurlyeq^U)$ is denoted by $\mathscr{L}_p(X,U)(\mathscr{L}_n(X,U))$. It is clear that a linear operator T is negative if and only if $-T$ is positive. It is well known that every positive operator between two Banach lattices $(X; \succcurlyeq^X)$ and $(U; \succcurlyeq^U)$ is an order-bounded linear operator, so is every negative operator; therefore we have

$$-\mathscr{L}_n(X,U) = \mathscr{L}_p(X,U) \subseteq \mathscr{L}_b(X,U). \tag{36}$$

The theory of positive operators between two Banach lattices has become a major theme in the field of Banach lattices. It has been widely applied to many fields. In this section, we apply it to study the existence of order-optimal solutions and the order-preserving properties of solutions to ordered variational inequalities defined in Banach lattices.

The following results are easy consequences of positive and negative operators. We state them as a lemma without proof.

Lemma 18. *Let $(X; \succcurlyeq^X)$ and $(U; \succcurlyeq^U)$ be two Banach lattices and C a nonempty convex subset of X. Let $f : C \to \mathscr{L}_b(X,U)$ be a mapping. Let S denote the set of solutions to VOI(C, f). Then S has the following order preserving properties.*

(1) *If $f(C) \subseteq \mathscr{L}_p(X,U)$, $\wedge C$ exists, and $\wedge C \in C$, then $S \neq \phi$ and $\wedge C \in S$.*

(2) *If $f(C) \subseteq \mathscr{L}_n(X,U)$, $\vee C$ exists, and $\vee C \in C$, then $S \neq \phi$ and $\vee C \in S$.*

Definition 19. Let $(X; \succcurlyeq^X)$ and $(U; \succcurlyeq^U)$ be two Banach lattices and A a nonempty subset of X. Let $f : C \to \mathscr{L}_b(X,U)$ be a mapping. f is said to be

(1) totally order increasing on A whenever $x \succcurlyeq^X z$ implies $f(x)(y) \succcurlyeq^U f(z)(y)$, for all $y \in X$;

(2) totally order decreasing on A whenever $x \succcurlyeq^X z$ implies $f(x)(y) \preccurlyeq^U f(z)(y)$, for all $y \in X$.

Noticing that $f(x) \succcurlyeq^{\mathscr{L}} f(z)$ if and only if $f(x)(y) \succcurlyeq^U f(z)(y)$, for all $y \in X^+$, we immediately obtain that a mapping f is totally order increasing (decreasing) on A implying that f is $\succcurlyeq^{\mathscr{L}}$-increasing (decreasing) on A.

Lemma 20. *Let $(X; \succcurlyeq^X)$ and $(U; \succcurlyeq^U)$ be two Banach lattices and C a nonempty convex subset of X. Let $f : C \to \mathscr{L}_b(X,U)$ be a mapping. Let S denote the set of solutions to VOI(C, f). Then S has the following order preserving properties.*

(1) *If $f : C \to \mathscr{L}_p(X,U)$ is totally order decreasing on C, then $x^* \in S, x^* \succcurlyeq^X z$, and $z \in C$ imply $z \in S$.*

(2) *If $f : C \to \mathscr{L}_n(X,U)$ is totally order increasing on C, then $x^* \in S, x^* \preccurlyeq^X z$, and $z \in C$ imply $z \in S$.*

Proof. At first we prove Part 1. Suppose that f is totally order decreasing on C. For any $x^* \in S$ and for any $z \in C$ satisfying $x^* \succcurlyeq^X z$, we have

$$\begin{aligned} & f(z)(y-z) \\ & \quad \succcurlyeq^U f(x^*)(y-z) && \text{(totally order decreasing)} \\ & \quad \succcurlyeq^U f(x^*)(y-x^*) && \left(f(x^*) \in \mathscr{L}_p(X,U)\right) \\ & \quad \succcurlyeq^U 0, \quad \forall y \in C. && (x^* \in S). \end{aligned} \tag{37}$$

It implies $z \in S$. To show Part 2, suppose that f is totally order increasing on C. For any $x^* \in S$ and for any $z \in C$ satisfying $x^* \preccurlyeq^X z$, we have

$$\begin{aligned} & f(z)(y-z) \\ & \quad \succcurlyeq^U f(x^*)(y-z) && \text{(totally order increasing)} \\ & \quad \succcurlyeq^U f(x^*)(y-x^*) && (f(x^*) \in \mathscr{L}_n(X,U)) \\ & \quad \succcurlyeq^U, \quad \forall y \in C. && (x^* \in S). \end{aligned} \tag{38}$$

It implies $z \in S$. This proves the lemma. □

As an immediate consequence, we have the following result.

Corollary 21. *Let $(X; \succcurlyeq^X)$ and $(U; \succcurlyeq^U)$ be two Banach lattices and C a nonempty convex Dedekind-complete subset of X. Let $f : C \to \mathscr{L}_b(X,U)$ be a mapping. Let S denote the set of solutions to VOI(C, f). Then S has the following order preserving properties.*

(1) *If $f : C \to \mathscr{L}_p(X,U)$ is totally order decreasing on C and $\wedge S$ exists, then $\wedge S \in S$.*

(2) *If $f : C \to \mathscr{L}_n(X,U)$ is totally order increasing on C and $\vee S$ exists, then $\vee S \in S$.*

Proof. Since C is Dedekind complete, if $\wedge S(\vee S)$ exists, then $\wedge S(\vee S) \in C$. The rest of the proof immediately follows from Lemma 20. □

Lemma 20 and Corollary 21 do not claim the convexity of the solution set S of the problem VOI(C, f). However, there are some conditions on the mapping f to guarantee the convexity of S. To this end, we need some concepts.

Definition 22. Let $(X; \succcurlyeq^X)$ and $(U; \succcurlyeq^U)$ be two Banach lattices. Let C be a nonempty convex subset of X and $f : C \to \mathscr{L}_b(X,U)$ a mapping. f is said to be

(1) order monotone if $(f(y) - f(x))(y - x) \succcurlyeq^U 0$, for any $x, y \in C$;

(2) order pseudomonotone if $f(x)(y - x) \succcurlyeq^U 0$ implies $f(y)(y - x) \succcurlyeq^U 0$, for any $x, y \in C$.

From the above definition, it is clear that every order-monotone mapping is order-pseudomonotone. It is well-known that, in the special case $(\mathbb{R}, \geq)$, there are examples of pseudomonotone mappings which are not monotonic.

Definition 23. Let $(X; \succcurlyeq^X)$ and $(U; \succcurlyeq^U)$ be two Banach lattices. Let C be a nonempty convex subset of X and $f : C \to L(X,U)$ a mapping. f is said to be hemicontinuous on C whenever, for any $x, y \in C$, the following limit holds:

$$\begin{gathered} f(tx + (1-t)y)(y-x) \longrightarrow f(y)(y-x) \\ \text{in } U \text{ as } t \downarrow 0, \quad t \in (0,1]. \end{gathered} \tag{39}$$

Lemma 24. *Let $(X; \succcurlyeq^X)$ and $(U; \succcurlyeq^U)$ be two Banach lattices. Let C be a nonempty convex order-closed subset of X and $f : C \to L(X,U)$ an order-pseudomonotone and hemicontinuous mapping. Then x^* is a solution to the problem VOI(K, f), for some $x^* \in C$, if and only if the following order inequality holds:*

$$f(x)(x - x^*) \succcurlyeq^U 0, \quad \forall x \in C. \tag{40}$$

Proof. If x^* is a solution to the problem VOI(K, f), then

$$f(x^*)(x - x^*) \succcurlyeq^U 0, \quad \forall x \in C. \tag{41}$$

The order inequality (40) immediately follows from the order pseudomonotony of f. Conversely, for any $y \in C$ and $t \in (0,1]$, we define $y_t = (1-t)x^* + ty$. Since C is convex, then $y_t \in C$ for all $t \in (0,1]$. From (40), we have

$$f(y_t)(y_t - x^*) \succcurlyeq^U 0, \quad \forall t \in (0,1]. \tag{42}$$

Noticing that $y_t - x^* = t(y - x^*)$, for $t > 0$, and $f(y_t) \in L(X,U)$, the above order inequality implies

$$f(y_t)(y - x^*) \succcurlyeq^U 0, \quad \forall t \in (0,1]. \tag{43}$$

Since f is hemicontinuous on C, we have

$$\begin{gathered} f((1-t)x^* + ty)(y - x^*) \longrightarrow f(x^*)(y - x^*) \\ \text{in } U \text{ as } t \downarrow 0. \end{gathered} \tag{44}$$

From (42), $f((1-t)x^* + ty)(y - x^*) \in U^+$ and since U^+ is norm closed, (44) implies $f(x^*)(y - x^*) \succcurlyeq^U 0$. This completes the proof of this lemma. □

Theorem 25. *Let $(X; \succcurlyeq^X)$ and $(U; \succcurlyeq^U)$ be two Banach lattices both with order-continuous norms. Let C be a nonempty convex, closed, Dedekind-complete subset of X, and let $f : C \to L(X,U)$ be a linearly order comparable and continuous mapping. Suppose that there exists a point $y_0 \in C$ such that*

$$\left\{x \in C : f(x)(y_0 - x) \succcurlyeq^U 0\right\} \text{ is compact in } X. \tag{45}$$

If f is also order-pseudomonotone and hemicontinuous mapping, then the solution set S to the problem VOI(C, f) is a nonempty closed convex subset of C.

Proof. From Theorem 10, we have $S \neq \phi$. We need to show that S is convex. To this end, for any $x_1, x_2 \in S$, from Lemma 24, we have

$$f(y)(y - x_i) \succcurlyeq^U 0, \quad \forall y \in C, \text{ for } i = 1, 2. \tag{46}$$

Then for any $\alpha \in [0,1]$, it implies

$$f(y)(y-(\alpha x_1+(1-\alpha)x_2)) = \alpha f(y)(y-x_1)+(1-\alpha)f(y)(y-x_2) \succcurlyeq^U 0, \quad \forall y \in C. \tag{47}$$

From Lemma 24 again, we obtain

$$f((\alpha x_1+(1-\alpha)x_2))(y-(\alpha x_1+(1-\alpha)x_2)) \succcurlyeq^U 0, \quad \forall y \in C. \tag{48}$$

It implies that $\alpha x_1+(1-\alpha)x_2 \in S$, and hence S is convex. The closeness of C follows from the closeness of the positive cone of a Banach lattice. The proof is finished. □

5. An Application

In this section, we give an example of ordered variational inequality problem in finite-dimensional cases as an application of Theorem 10, which can be considered as an application to economics theory.

Example 26. In an economy, we consider two finite-dimensional Hilbert lattices $(\mathbb{R}^n;\succcurlyeq^n)$, $(\mathbb{R}^m;\succcurlyeq^m)$ equipped with the coordinate partial orders. Let C be a bounded closed convex subset of $(\mathbb{R}^n;\succcurlyeq^n)$, which is the capital resources set and $(\mathbb{R}^m;\succcurlyeq^m)$ is the outcome set. Suppose that a mapping $f: C \to L(\mathbb{R}^n,\mathbb{R}^m)$ is the plan-product function in this economy which assigns every point in C to a producing plan. This plan is a production function with C as the capital resources and with $(\mathbb{R}^m;\succcurlyeq^m)$ as the outcome space. Suppose that there is a fixed $n\times m$ matrix $\phi=(a_{ij})_{0\le i\le n, 0\le j\le m}$ satisfying $\sum_{1\le i\le n} a_{ij} \ge 0$, for $j=1,2,\ldots,m$, and the plan-product f is a linearly weighted distribution defined by

$$f(x)=\left(x^T,x^T,\ldots,x^T\right)\phi, \quad \forall x \in C, \tag{49}$$

where $(x^T,x^T,\ldots,x^T)$ is an $n\times n$ square with every column as x^T. Then there exists $x^* \in C$ such that the economy takes the least production at the capital x^* under the producing plan $f(x^*)$:

$$f(x^*)(y) \succcurlyeq^m f(x^*)(x^*), \quad \forall y \in C. \tag{50}$$

That is, the problem $\mathrm{VOI}(C,f)$ is solvable. Furthermore, there is $z^* \in C$ such that the economy takes the most production at the capital z^* under the producing plan $f(z^*)$:

$$f(z^*)(z^*) \succcurlyeq^m f(z^*)(y), \quad \forall y \in C. \tag{51}$$

Remark 27. The capitals x^* and z^* obtained in the previous example can be, respectively, viewed as the worst-case scenario and the best-case scenario in this economy.

Proof. At first, we show that $f: C \to L(\mathbb{R}^n,\mathbb{R}^m)$ is linearly order comparable on C. To this end, for any given $x,y \in C$ and for every $\alpha \in (0,1)$, we have

$$\begin{aligned} f(\alpha x+(1-\alpha)y)(x) &= x\left((\alpha x+(1-\alpha)y)^T,\ldots,(\alpha x+(1-\alpha)y)^T\right)\phi, \\ f(\alpha x+(1-\alpha)y)(y) &= y\left((\alpha x+(1-\alpha)y)^T,\ldots,(\alpha x+(1-\alpha)y)^T\right)\phi. \end{aligned} \tag{52}$$

Calculating gets

$$\begin{aligned} f(\alpha x+(1-\alpha)y)(x) &= \left(\alpha\|x\|^2+(1-\alpha)\langle x,y\rangle, \|x\|^2 \right. \\ &\quad \left. +(1-\alpha)\langle x,y\rangle,\ldots,\alpha\|x\|^2+(1-\alpha)\langle x,y\rangle\right)\phi, \\ f(\alpha x+(1-\alpha)y)(y) &= \left(\alpha\langle x,y\rangle+(1-\alpha)\|y\|^2, \alpha\langle x,y\rangle \right. \\ &\quad \left. +(1-\alpha)\|y\|^2,\ldots,\alpha\langle x,y\rangle+(1-\alpha)\|y\|^2\right)\phi. \end{aligned} \tag{53}$$

We obtain

$$\begin{aligned} f(\alpha x+(1-\alpha)y)(x) &= \left(\alpha\|x\|^2+(1-\alpha)\langle x,y\rangle\right) \\ &\quad \times\left(\sum_{1\le i\le n} a_{i1}, \sum_{1\le i\le n} a_{i2},\ldots,\sum_{1\le i\le n} a_{im}\right), \\ f(\alpha x+(1-\alpha)y)(y) &= \left(\alpha\langle x,y\rangle+(1-\alpha)\|y\|^2\right) \\ &\quad \times\left(\sum_{1\le i\le n} a_{i1}, \sum_{1\le i\le n} a_{i2},\ldots,\sum_{1\le i\le n} a_{im}\right). \end{aligned} \tag{54}$$

Since $\alpha\|x\|^2+(1-\alpha)\langle x,y\rangle$ and $\alpha\langle x,y\rangle+(1-\alpha)\|y\|^2$ are both real numbers and $\sum_{1\le i\le n} a_{ij} \ge 0$, for $j=1,2,\ldots,m$, it yields that $f(\alpha x+(1-\alpha)y)(x)$ and $f(\alpha x+(1-\alpha)y)(y)$ are $\succcurlyeq^m$-comparable in $(\mathbb{R}^m;\succcurlyeq^m)$. Hence f is linearly order comparable on C. To apply Theorem 10, we have to check that $f(x) \in L(X,U)$ and f satisfies Conditions (19) and (20). For this purpose, suppose that $\{x^k\} \subseteq C$ satisfying $x^k \to x$ in $\mathbb{R}^n$. Since C is closed, it implies $x \in C$. For every fixed

$z \in C, f(z) = (z^T, z^T, \ldots, z^T)\phi$ is an $n \times m$ matrix. Then immediately following from $x^k \to x$ in $\mathbb{R}^n$, we have

$$\begin{aligned} &\left\| f(z)\,x^k - f(z)\,x \right\|_{\mathbb{R}m} \\ &\quad = \left\| \left(x^k - x\right)\left(z^T, z^T, \ldots, z^T\right)\phi \right\|_{\mathbb{R}m} \\ &\quad \le \left\| \left(x^k - x\right) \right\|_{\mathbb{R}m} \left\| \left(z^T, z^T, \ldots, z^T\right)\phi \right\|_{L(\mathbb{R}n,\mathbb{R}m)} \\ &\quad \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \end{aligned} \tag{55}$$

Hence, f satisfies Condition (19) in Theorem 10. Similarly, $f(x^k)$ and $f(x)$ are also $n \times m$ matrices and from $x^k \to x$ in $\mathbb{R}^n$, we obtain

$$\begin{aligned} &\left\| f\left(x^k\right) - f(x) \right\|_{L(\mathbb{R}n,\mathbb{R}m)} \\ &\quad = \left\| \left(\left(x^k - x\right)^T, \left(x^k - x\right)^T, \ldots, \left(x^k - x\right)^T \right)\phi \right\|_{L(\mathbb{R}n,\mathbb{R}m)} \\ &\quad \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \end{aligned} \tag{56}$$

This is Condition (20) in Theorem 10. Since C is compact, all conditions for f in Theorem 10 are satisfied. Hence the problem VOI(C, f) is solvable, that is, there exists $x^* \in C$ such that

$$f\left(x^*\right)\left(y - x^*\right) \succcurlyeq^m 0, \quad \forall y \in C. \tag{57}$$

The proof of the second part can be reduced to Part 1 by considering a new function $g = -f$. This completes the proof of this example. □

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